An Extended Single-valued Neutrosophic Normalized Weighted Bonferroni Mean Einstein Aggregation Operator

Lihua Yang, Baolin Li

***Abstract*—In order to deal with the indeterminate information and inconsistent information existing common in real decision making problems, an extended Single-valued Neutrosophic Normalized Weighted Bonferroni Mean (SVNNWBM) aggregation operator based on Einstein operations is proposed under single-valued neutrosophic environment. The novel SVNNWBM aggregation operator can not only take into account the interrelationship of the input arguments, but also overcome the shortcomings of unreasonable operations in some cases. Meanwhile, some new operational laws based on Einstein are defined, and some desirable properties of the proposed operator are analyzed. In addition, a decision making method based on the novel operator is given. Finally, to solve multi-criteria decision-making (MCDM) problems, an illustrative example based on the SVNNWBM operator and cosine similarity measure is shown to verify the effectiveness and practicality of the proposed method.**

***Index Terms*—multi-criteria decision-making, Einstein, weighted Bonferroni Mean aggregation operator, single-valued neutrosophic sets**

I. INTRODUCTION

Fuzzy sets (FSs) [1], intuitionistic fuzzy sets (IFSs) [2-3] and hesitant fuzzy sets (HFSs) [4-5] have been applied successfully in many fields. However, they cannot manage the inconsistent information and the indeterminate information. Therefore, neutrosophic sets (NSs) firstly proposed by Smarandache [6] has become a hot spot for fuzzy information in recent years. In neutrosophic set, an indeterminacy-membership is added explicitly, and truth-membership, indeterminacy-membership and false-membership are independent.

At present, NS has been made many attentions, and there are some achievements on it. A single-valued neutrosophic set (SVNS) was proposed [7], and some properties of SVNS were also analyzed. Ye [8] defined the simplified neutrosophic sets (SNSs), and generated a MCDM method using a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. Wang[9] introduced the concept of interval neutrosophic set (INS) and gave the corresponding operators of INS. Ye [10] utilized the correlation coefficient and weighted correlation coefficient of SVNS to rank the alternatives. Majumdar [11] defined similarity measures between two SVNSs and proposed a measure of entropy. Wang [12] proposed some new distance measures for dual hesitant fuzzy sets, and study the properties of the measures. Aydin [13] applied fuzzy multi-criteria decision making approach to evaluate of E-commerce website quality.

Information aggregation is of great important to MCDM problems, so different aggregation operators have been developed. Yager [14] and Xu [15] proposed weighted arithmetic average operator and weighted geometric average operator. The neutrosophic number aggregation operators based on the algebraic operational rules [16] are proposed. Wang and Liu [17] proposed intuitionistic fuzzy geometric aggregation operators. Liu [18] proposed some Hamacher aggregation operators for the interval-valued intuitionistic fuzzy numbers. Yang [19] proposed power aggregation operators for single-valued neutrosophic sets.

Up to date, there are a few researches about aggregation operators under the single-valued neutrosophic environment. Bonferroni mean (BM) is an important aggregation operator to combine all the input individual arguments into a single representative value, which can capture the interrelationship between the individual data and the other data. Xia [20] presented the weighted intuitionistic fuzzy Bonferroni geometric mean. Liu [21] proposed a single-valued neutrosophic normalized weighted Bonferroni mean operator based on algebraic t-norm and t-conorm. Li [22] defined the multi-valued neutrosophic linguistic normalized weighted Bonferroni mean Hamacher operator.

However, the operations of SVNNs based on operational rules may be irrational in some cases [23]. Algebraic and Einstein t-norm and t-conorm are important in the building of operational rules and aggregation operators, but there is little research on aggregation operator using Einstein operations in the situation of single-valued neutrosophic set. Hence, some novel operations of SVNNs based on Einstein operational laws are proposed in this paper, and an extended Single-valued Neutrosophic Normalized Weighted Bonferroni

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Mean (SVNNWBM) aggregation operator based on Einstein operations is proposed under single-valued neutrosophic environment.

Therefore, as a complement to the existing works, this paper aims to develop Normalized Weighted Bonferroni Mean based on the new operations under single-valued neutrosophic environment. At the same time, we will analyze its desirable properties and apply it to MCDM problems.

The rest of the paper is presented as follows. In section Ⅱ, some definitions and operations are introduced, and some novel operational laws of SVNNs based on Einstein operations are proposed. In section Ⅲ, single-valued neutrosophic normalized weighted Bonferroni Mean aggregation operator based on Einstein operations is proposed, and some desirable properties are analyzed. The new method for multi-criteria decision making based on the proposed operator is presented in section Ⅳ. An illustrative example is also shown to verify the effectiveness of the new method in section Ⅳ. Finally, the main conclusions of this paper are summarized in section Ⅴ.

II. NOVEL OPERATIONS

A single-valued neutrosophic set (SVNS) is an extension of NSs, and some properties of SVNS are also proposed.

Let X be a universe of discourse, with a generic element in X denoted by x. A SVNS A in X is

$$A=\{\langle x,T_A(x),I_A(x),F_A(x)\rangle | x\in X\},$$

Where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, we have $T_A(x)$, $I_A(x)$, and $F_A(x)\in[0,1]$, and $0\le T_A(x)+I_A(x)+F_A(x)\le 3$.

For simplicity, we can use $x=(T_x,I_x,F_x)$ to represent an element in SVNS. In particular, if X has only one element, and we call it a single-valued neutrosophic number (SVNN). The set of all SVNNs is represented as SVNNS.

Ye [8] proposed the following operations under neutrosophic environment.

Let $A=(T_1,I_1,F_1)$ and $B=(T_2,I_2,F_2)$ be two SVNNs, then operational relations are defined as follows:

(1) $A\oplus B=<T_1+T_2-T_1T_2, I_1+I_2-I_1I_2, F_1+F_2-F_1F_2>$

(2) $A\otimes B=<T_1T_2, I_1I_2, F_1F_2>$

(3) $\lambda A=<1-(1-T_1)^\lambda, 1-(1-I_1)^\lambda, 1-(1-F_1)^\lambda> \quad \lambda>0$

(4) $A^\lambda=<T_1^{\ \lambda}, I_1^{\ \lambda}, F_1^{\ \lambda}> \quad \lambda>0$

However, in some situations, operations above provided by Ye [8] might be impractical. This is discussed in the following example.

Let $A=<0.6,0.5,0.5>$ and $B=<1,0,0>$ be two SVNNs. Obviously, $B=(1,0,0)$ is the larger value of these SVNNs. It is well known that the sum of any number and the maximum number is equal to the maximum value. However, $A\oplus B=<1,0.5,0.5>\neq B$, which is based on the above definition. Hence, the operations above are incorrect. Similarly, $A=<0.6,0.5,0.5>$ is the smaller value of these SVNNs. Theoretically, the multiple of any number and the minimum number is equal to the any one. However, according to operations above, $A\otimes B=<0.6,0,0>\neq A$. Thus, the operations defined above are irrational.

Therefore, some new operations to avoid shortcomings above are proposed in this paper.

Let $A=(T_1,I_1,F_1)$ and $B=(T_2,I_2,F_2)$ be two SVNNs, then the new operational relations based on Einstein operations are proposed as follows:

(1) $A\oplus B=<\frac{T_1+T_2}{1+T_1T_2}, \frac{I_1I_2}{1+(1-I_1)(1-I_2)}, \frac{F_1F_2}{1+(1-F_1)(1-F_2)}>$

(2) $A\otimes B=<\frac{T_1T_2}{1+(1-T_1)(1-T_2)}, \frac{I_1+I_2}{1+I_1I_2}, \frac{F_1+F_2}{1+F_1F_2}>$

(3) $\lambda A=<\frac{(1+T_1)^\lambda-(1-T_1)^\lambda}{(1+T_1)^\lambda+(1-T_1)^\lambda}, \frac{2(I_1)^\lambda}{(2-I_1)^\lambda+(I_1)^\lambda}, \frac{2(F_1)^\lambda}{(2-F_1)^\lambda+(F_1)^\lambda}> \quad \lambda>0$

(4) $A^\lambda=<\frac{2(T_1)^\lambda}{(2-T_1)^\lambda+(T_1)^\lambda}, \frac{(1+I_1)^\lambda-(1-I_1)^\lambda}{(1+I_1)^\lambda+(1-I_1)^\lambda}, \frac{(1+F_1)^\lambda-(1-F_1)^\lambda}{(1+F_1)^\lambda+(1-F_1)^\lambda}> \quad \lambda>0$

The corresponding operational relations can also be obtained.

Let $A=(T_1,I_1,F_1)$, $B=(T_2,I_2,F_2)$, and $C=(T_3,I_3,F_3)$ be three SVNNs, then

(1) $A\oplus B=B\oplus A$

(2) $A\otimes B=B\otimes A$

(3) $\lambda(A\oplus B)=\lambda A\oplus\lambda B, \lambda>0$

(4) $(A\otimes B)^\lambda=A^\lambda\otimes B^\lambda, \lambda>0$

(5) $\lambda_1 A\oplus\lambda_2 A=(\lambda_1+\lambda_2)A, \lambda_1>0, \lambda_2>0$

(6) $A^{\lambda_1}\otimes A^{\lambda_2}=A^{\lambda_1+\lambda_2}, \lambda_1>0, \lambda_2>0$

(7) $(A\oplus B)\oplus C=A\oplus(B\oplus C)$

(8) $(A\otimes B)\otimes C=A\otimes(B\otimes C)$

Then, we will prove the operational relations above. Obviously, (1), (2), (7), and (8) can be easily obtained.

Let us prove (5).

Proof (5) Since $\lambda_1>0, \lambda_2>0$

$$\lambda_1 A\oplus\lambda_2 A=\left\langle\frac{(1+T_1)^{\lambda_1}-(1-T_1)^{\lambda_1}}{(1+T_1)^{\lambda_1}+(1-T_1)^{\lambda_1}}, \frac{2(I_1)^{\lambda_1}}{(2-I_1)^{\lambda_1}+(I_1)^{\lambda_1}}, \frac{2(F_1)^{\lambda_1}}{(2-F_1)^{\lambda_1}+(F_1)^{\lambda_1}}\right\rangle$$

$$\oplus\left\langle\frac{(1+T_1)^{\lambda_2}-(1-T_1)^{\lambda_2}}{(1+T_1)^{\lambda_2}+(1-T_1)^{\lambda_2}}, \frac{2(I_1)^{\lambda_2}}{(2-I_1)^{\lambda_2}+(I_1)^{\lambda_2}}, \frac{2(F_1)^{\lambda_2}}{(2-F_1)^{\lambda_2}+(F_1)^{\lambda_2}}\right\rangle$$

$$=\left\langle\frac{\frac{(1+T_1)^{\lambda_1}-(1-T_1)^{\lambda_1}}{(1+T_1)^{\lambda_1}+(1-T_1)^{\lambda_1}}+\frac{(1+T_1)^{\lambda_2}-(1-T_1)^{\lambda_2}}{(1+T_1)^{\lambda_2}+(1-T_1)^{\lambda_2}}}{1+\frac{(1+T_1)^{\lambda_1}-(1-T_1)^{\lambda_1}}{(1+T_1)^{\lambda_1}+(1-T_1)^{\lambda_1}}\cdot\frac{(1+T_1)^{\lambda_2}-(1-T_1)^{\lambda_2}}{(1+T_1)^{\lambda_2}+(1-T_1)^{\lambda_2}}},\right.$$

$$\frac{\frac{2(I_1)^{\lambda_1}}{(2-I_1)^{\lambda_1}+(I_1)^{\lambda_1}}\cdot\frac{2(I_1)^{\lambda_2}}{(2-I_1)^{\lambda_2}+(I_1)^{\lambda_2}}}{1+(1-\frac{2(I_1)^{\lambda_1}}{(2-I_1)^{\lambda_1}+(I_1)^{\lambda_1}})(1-\frac{2(I_1)^{\lambda_2}}{(2-I_1)^{\lambda_2}+(I_1)^{\lambda_2}})},$$

$$\left.\frac{\frac{2(F_1)^{\lambda_1}}{(2-F_1)^{\lambda_1}+(F_1)^{\lambda_1}}\cdot\frac{2(F_1)^{\lambda_2}}{(2-F_1)^{\lambda_2}+(F_1)^{\lambda_2}}}{1+(1-\frac{2(F_1)^{\lambda_1}}{(2-F_1)^{\lambda_1}+(F_1)^{\lambda_1}})(1-\frac{2(F_1)^{\lambda_2}}{(2-F_1)^{\lambda_2}+(F_1)^{\lambda_2}})}\right\rangle$$

$$=\left\langle\frac{((1+T_1)^{\lambda_1}-(1-T_1)^{\lambda_1})\cdot((1+T_1)^{\lambda_2}+(1-T_1)^{\lambda_2})+((1+T_1)^{\lambda_2}-(1-T_1)^{\lambda_2})\cdot((1+T_1)^{\lambda_1}+(1-T_1)^{\lambda_1})}{((1+T_1)^{\lambda_1}+(1-T_1)^{\lambda_1})\cdot((1+T_1)^{\lambda_2}+(1-T_1)^{\lambda_2})+((1+T_1)^{\lambda_1}-(1-T_1)^{\lambda_1})\cdot((1+T_1)^{\lambda_2}-(1-T_1)^{\lambda_2})},\right.$$

$$\frac{4\cdot I_1^{\ \lambda_1+\lambda_2}}{((2-I_1)^{\lambda_1}+(I_1)^{\lambda_1})((2-I_1)^{\lambda_2}+(I_1)^{\lambda_2})+((2-I_1)^{\lambda_1}-(I_1)^{\lambda_1})((2-I_1)^{\lambda_2}-(I_1)^{\lambda_2})},$$

$$\left.\frac{4\cdot F_1^{\ \lambda_1+\lambda_2}}{((2-F_1)^{\lambda_1}+(F_1)^{\lambda_1})\cdot((2-F_1)^{\lambda_2}+(F_1)^{\lambda_2})+((2-F_1)^{\lambda_1}-(F_1)^{\lambda_1})\cdot((2-F_1)^{\lambda_2}-(F_1)^{\lambda_2})}\right\rangle$$

$$=\left\langle\frac{(1+T_1)^{\lambda_1+\lambda_2}-(1-T_1)^{\lambda_1+\lambda_2}}{(1+T_1)^{\lambda_1+\lambda_2}+(1-T_1)^{\lambda_1+\lambda_2}}, \frac{2\cdot I_1^{\ \lambda_1+\lambda_2}}{(2-I_1)^{\lambda_1+\lambda_2}+I_1^{\ \lambda_1+\lambda_2}}, \frac{2\cdot F_1^{\ \lambda_1+\lambda_2}}{(2-F_1)^{\lambda_1+\lambda_2}+F_1^{\ \lambda_1+\lambda_2}}\right\rangle$$

$$=(\lambda_1+\lambda_2)A$$

The proof ends.

Similarly, (3), (4) and (6) can be true.

Then, the novel operational relations can be established.

Similarly, let $A=<0.6,0.5,0.5>$ and $B=<1,0,0>$ be two SVNNs. According to the novel operations, $A\oplus B=<1,0,0>=B$. Note that the operations coincide with the sum of any number and the maximum number is equal to the maximum value.

Meanwhile, $A \otimes B = <0.6, 0.5, 0.5> = A$. Note that the operations coincide with the multiple of any number and the minimum number is equal to the any one.

Therefore, the novel operations proposed in this paper can avoid above disadvantages.

III. Novel Single-valued Neutrosophic Weighted Bonferroni Mean Aggregation Operator

In this section, we shall extend the NWBM operator to the single-valued neutrosophic environment where the input arguments are SVNNs, and some properties are also analyzed.

A. SVNNWBM operator

The novel SVNNWBM aggregation operator on the basis of new operations above is proposed.

Let $p, q \geq 0$, and $x_i = \langle T_i, I_i, F_i \rangle (i = 1, 2, \cdots n)$ be a collection of the SVNNs, the single-valued neutrosophic normalized weighted Bonferroni mean (SVNNWBM) operator is defined as follows:

$$SVNNWBM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \frac{w_i w_j}{1-w_i} (x_i^p \otimes x_j^q) \right)^{\frac{1}{p+q}}$$

$$= \left\langle \frac{2\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_iw_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_iw_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_iw_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, \right.$$

$$\frac{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_iw_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} - \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_iw_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_iw_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_iw_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}},$$

$$\left. \frac{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_iw_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} - \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_iw_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_iw_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_iw_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_iw_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \right\rangle$$

Where $w = (w_1, w_2, \cdots, w_n)$ is the weight vector of x_i $(i = 1, 2, \cdots, n)$, satisfying $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. The weight vector can be given by decision-makers in real problem.

Then the aggregated value using the SVNNWBM operator is defined as follows:

$$SVNNWBM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \frac{w_i w_j}{1-w_i} (x_i^p \otimes x_j^q) \right)^{\frac{1}{p+q}}$$

According to the new operational rules based on Einstein operations, we have

$$x_i^p = \left\langle \frac{2(T_i)^p}{(2-T_i)^p + (T_i)^p}, \frac{(1+I_i)^p - (1-I_i)^p}{(1+I_i)^p + (1-I_i)^p}, \frac{(1+F_i)^p - (1-F_i)^p}{(1+F_i)^p + (1-F_i)^p} \right\rangle$$

$$x_j^q = \left\langle \frac{2(T_j)^q}{(2-T_j)^q + (T_j)^q}, \frac{(1+I_j)^q - (1-I_j)^q}{(1+I_j)^q + (1-I_j)^q}, \frac{(1+F_j)^q - (1-F_j)^q}{(1+F_j)^q + (1-F_j)^q} \right\rangle$$

$$x_i^p \otimes x_j^q$$

$$= \left\langle \frac{\frac{2(T_i)^p}{(2-T_i)^p + (T_i)^p} \cdot \frac{2(T_j)^q}{(2-T_j)^q + (T_j)^q}}{1 + \left(1 - \frac{2(T_i)^p}{(2-T_i)^p + (T_i)^p}\right)\left(1 - \frac{2(T_j)^q}{(2-T_j)^q + (T_j)^q}\right)}, \right.$$

$$\left. \frac{\frac{(1+I_i)^p - (1-I_i)^p}{(1+I_i)^p + (1-I_i)^p} + \frac{(1+I_j)^q - (1-I_j)^q}{(1+I_j)^q + (1-I_j)^q}}{1 + \frac{(1+I_i)^p - (1-I_i)^p}{(1+I_i)^p + (1-I_i)^p} \cdot \frac{(1+I_j)^q - (1-I_j)^q}{(1+I_j)^q + (1-I_j)^q}}, \frac{\frac{(1+F_i)^p - (1-F_i)^p}{(1+F_i)^p + (1-F_i)^p} + \frac{(1+F_j)^q - (1-F_j)^q}{(1+F_j)^q + (1-F_j)^q}}{1 + \frac{(1+F_i)^p - (1-F_i)^p}{(1+F_i)^p + (1-F_i)^p} \cdot \frac{(1+F_j)^q - (1-F_j)^q}{(1+F_j)^q + (1-F_j)^q}} \right\rangle$$

$$= \left\langle \frac{2T_i^p T_j^q}{(2-T_i)^p (2-T_j)^q + T_i^p T_j^q}, \right.$$

$$\left. \frac{(1+I_i)^p (1+I_j)^q - (1-I_i)^p (1-I_j)^q}{(1+I_i)^p (1+I_j)^q + (1-I_i)^p (1-I_j)^q}, \frac{(1+F_i)^p (1+F_j)^q - (1-F_i)^p (1-F_j)^q}{(1+F_i)^p (1+F_j)^q + (1-F_i)^p (1-F_j)^q} \right\rangle$$

Then,

$$\frac{w_i w_j}{1-w_i}(x_i^p \otimes x_j^q)$$

$$= \left\langle \frac{(1+\frac{2T_i^pT_j^q}{(2-T_i)^p(2-T_j)^q+T_i^pT_j^q})^{\frac{w_iw_j}{1-w_i}}-(1-\frac{2T_i^pT_j^q}{(2-T_i)^p(2-T_j)^q+T_i^pT_j^q})^{\frac{w_iw_j}{1-w_i}}}{(1+\frac{2T_i^pT_j^q}{(2-T_i)^p(2-T_j)^q+T_i^pT_j^q})^{\frac{w_iw_j}{1-w_i}}+(1-\frac{2T_i^pT_j^q}{(2-T_i)^p(2-T_j)^q+T_i^pT_j^q})^{\frac{w_iw_j}{1-w_i}}}, \right.$$

$$\frac{2(\frac{(1+I_i)^p(1+I_j)^q-(1-I_i)^p(1-I_j)^q}{(1+I_i)^p(1+I_j)^q+(1-I_i)^p(1-I_j)^q})^{\frac{w_iw_j}{1-w_i}}}{(2-\frac{(1+I_i)^p(1+I_j)^q-(1-I_i)^p(1-I_j)^q}{(1+I_i)^p(1+I_j)^q+(1-I_i)^p(1-I_j)^q})^{\frac{w_iw_j}{1-w_i}}+(\frac{(1+I_i)^p(1+I_j)^q-(1-I_i)^p(1-I_j)^q}{(1+I_i)^p(1+I_j)^q+(1-I_i)^p(1-I_j)^q})^{\frac{w_iw_j}{1-w_i}}},$$

$$\left. \frac{2(\frac{(1+F_i)^p(1+F_j)^q-(1-F_i)^p(1-F_j)^q}{(1+F_i)^p(1+F_j)^q+(1-F_i)^p(1-F_j)^q})^{\frac{w_iw_j}{1-w_i}}}{(2-\frac{(1+F_i)^p(1+F_j)^q-(1-F_i)^p(1-F_j)^q}{(1+F_i)^p(1+F_j)^q+(1-F_i)^p(1-F_j)^q})^{\frac{w_iw_j}{1-w_i}}+(\frac{(1+F_i)^p(1+F_j)^q-(1-F_i)^p(1-F_j)^q}{(1+F_i)^p(1+F_j)^q+(1-F_i)^p(1-F_j)^q})^{\frac{w_iw_j}{1-w_i}}} \right\rangle$$

$$= \left\langle \frac{((2-T_i)^p(2-T_j)^q+3T_i^pT_j^q)^{\frac{w_iw_j}{1-w_i}}-((2-T_i)^p(2-T_j)^q-T_i^pT_j^q)^{\frac{w_iw_j}{1-w_i}}}{((2-T_i)^p(2-T_j)^q+3T_i^pT_j^q)^{\frac{w_iw_j}{1-w_i}}+((2-T_i)^p(2-T_j)^q-T_i^pT_j^q)^{\frac{w_iw_j}{1-w_i}}}, \right.$$

$$\frac{2((1+I_i)^p(1+I_j)^q-(1-I_i)^p(1-I_j)^q)^{\frac{w_iw_j}{1-w_i}}}{((1+I_i)^p(1+I_j)^q+3(1-I_i)^p(1-I_j)^q)^{\frac{w_iw_j}{1-w_i}}+((1+I_i)^p(1+I_j)^q-(1-I_i)^p(1-I_j)^q)^{\frac{w_iw_j}{1-w_i}}},$$

$$\left. \frac{2((1+F_i)^p(1+F_j)^q-(1-F_i)^p(1-F_j)^q)^{\frac{w_iw_j}{1-w_i}}}{((1+F_i)^p(1+F_j)^q+3(1-F_i)^p(1-F_j)^q)^{\frac{w_iw_j}{1-w_i}}+((1+F_i)^p(1+F_j)^q-(1-F_i)^p(1-F_j)^q)^{\frac{w_iw_j}{1-w_i}}} \right\rangle$$

$$= \left\langle \frac{u_{ij}^{\frac{w_iw_j}{1-w_i}}-v_{ij}^{\frac{w_iw_j}{1-w_i}}}{u_{ij}^{\frac{w_iw_j}{1-w_i}}+v_{ij}^{\frac{w_iw_j}{1-w_i}}}, \frac{2t_{ij}^{\frac{w_iw_j}{1-w_i}}}{x_{ij}^{\frac{w_iw_j}{1-w_i}}+t_{ij}^{\frac{w_iw_j}{1-w_i}}}, \frac{2y_{ij}^{\frac{w_iw_j}{1-w_i}}}{z_{ij}^{\frac{w_iw_j}{1-w_i}}+y_{ij}^{\frac{w_iw_j}{1-w_i}}} \right\rangle$$

Where

$$u_{ij} = (2-T_i)^p(2-T_j)^q+3T_i^pT_j^q,$$
$$v_{ij} = (2-T_i)^p(2-T_j)^q-T_i^pT_j^q,$$
$$t_{ij} = (1+I_i)^p(1+I_j)^q-(1-I_i)^p(1-I_j)^q,$$
$$x_{ij} = (1+I_i)^p(1+I_j)^q+3(1-I_i)^p(1-I_j)^q,$$
$$y_{ij} = (1+F_i)^p(1+F_j)^q-(1-F_i)^p(1-F_j)^q,$$
$$z_{ij} = (1+F_i)^p(1+F_j)^q+3(1-F_i)^p(1-F_j)^q.$$

Thus

$$\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \frac{w_iw_j}{1-w_i}(x_i^p \otimes x_j^q) = \left\langle \frac{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_iw_j}{1-w_i}}-\prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_iw_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_iw_j}{1-w_i}}+\prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_iw_j}{1-w_i}}}, \right.$$

$$\left. \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_iw_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_iw_j}{1-w_i}}+\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_iw_j}{1-w_i}}}, \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_iw_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_iw_j}{1-w_i}}+\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_iw_j}{1-w_i}}} \right\rangle$$

Therefore,

$$SVNNWBM^{p,q}(x_1, x_2, \cdots, x_n) = \left(\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n} \frac{w_i w_j}{1-w_i} (x_i^p \otimes x_j^q) \right)^{\frac{1}{p+q}}$$

$$= \left\langle \frac{2\left(\frac{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}}}{\left(2 - \frac{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}} + \left(\frac{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}}}, \right.$$

$$\frac{\left(1 + \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}} - \left(1 - \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}}}{\left(1 + \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}} + \left(1 - \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}}},$$

$$\left. \frac{\left(1 + \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}} - \left(1 - \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}}}{\left(1 + \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}} + \left(1 - \frac{2\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}{\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} + \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}}\right)^{\frac{1}{p+q}}} \right\rangle$$

$$== \left\langle \frac{2\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} u_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} v_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}, \right.$$

$$\frac{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} - \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} x_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} t_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}},$$

$$\left. \frac{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} - \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} + 3\prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}} + \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} z_{ij}^{\frac{w_i w_j}{1-w_i}} - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} y_{ij}^{\frac{w_i w_j}{1-w_i}}\right)^{\frac{1}{p+q}}} \right\rangle$$

The proof is completed.

Therefore, the calculating formula of novel SVNNWBM aggregation operator can be obtained.

A. Properties of SVNNWBM operator

We can prove that the novel SVNNWBM operator has the following properties, such as idempotency, commutativity, boundedness , monotonicity, and reducibility.

(1) Idempotency: Let $A_j = \langle T_{A_j}, I_{A_j}, F_{A_j}\rangle (j=1,2,\cdots,n)$ be a collection of SVNNs, and $A = \langle T_A, I_A, F_A\rangle$ be a SVNN. If $A_j = A$ $(j=1,2,\cdots,n)$, then $SVNNWBM^{p,q}(A_1, A_2, \cdots, A_n) = A$.

(2) Commutativity: Let $A_j = \langle T_{A_j}, I_{A_j}, F_{A_j}\rangle (j=1,2,\cdots,n)$ be a collection of SVNNs, if $A_j^*(j=1,2,\cdots,n)$ is any permutation of A_j $(j=1,2,\cdots,n)$, then

$$SVNNWBM^{p,q}(A_1, A_2, \cdots, A_n) = SVNNWBM^{p,q}(A_1^*, A_2^*, \cdots, A_n^*).$$

(3) Boundedness: Let $A_j = \langle T_{A_j}, I_{A_j}, F_{A_j}\rangle (j=1,2,\cdots,n)$ be a collections of SVNNs. and $A^- = \langle T_{A^-}, I_{A^-}, F_{A^-}\rangle$, $A^+ = \langle T_{A^+}, I_{A^+}, F_{A^+}\rangle$ If for all j, $T_{A^-} \le T_{A_j} \le T_{A^+}$, $I_{A^+} \le I_{A_j} \le I_{A^-}$, and $F_{A^+} \le F_{A_j} \le F_{A^-}$, then

$$A^- \le SVNNWBM^{p,q}(A_1, A_2, \cdots, A_n) \le A^+$$

$$A^- = SVNNWBM^{p,q}(A^-, A^-, \cdots, A^-) \le SVNNWBM^{p,q}(A_1, A_2, \cdots, A_n)$$

$$\le SVNNWBM^{p,q}(A^+, A^+, \cdots, A^+) = A^+$$

(4) Monotonicity:

Let $A_j = \langle T_{A_j}, I_{A_j}, F_{A_j}\rangle$ and $A_j' = \langle T_{A_j'}, I_{A_j'}, F_{A_j'}\rangle$ be two collections of SVNNs, if $A_j \le A_j'$ for all $(j = 1,2,\cdots,n)$, then

$$SVNNWBM^{p,q}(A_1, A_2, \cdots, A_n) \le SVNNWBM^{p,q}(A_1', A_2', \cdots, A_n')$$

(5) Reducibility:

Let $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ then

$$SVNNWBM^{p,q}(A_1, A_2, \cdots, A_n) = SVNBM^{p,q}(A_1, A_2, \cdots, A_n)$$

IV. An Illustrative Example

A. Multi-criteria decision-making method

Considering the MCDM problems based on novel SVNNWBM operator.

Suppose $A = \{A_1, A_2, \cdots, A_m\}$ be m alternatives, and $C = \{C_1, C_2, \cdots, C_n\}$ be n criteria. Let $w = \{w_1, w_2, \cdots, w_n\}$ be the corresponding weights of criteria, where $w_j \ge 0 (j=1,2,\cdots,n)$, and $\sum_{j=1}^{n} w_j = 1$ .The evaluation value of the attribute c_j $(j=1,2,\cdots,n)$ with respect to the alternative A_i $(i=1,2,\cdots,m)$ is given by decision maker.

The evaluation values are in the form of SVNNs.

An SVNN is denoted by $\alpha_{ij} = \langle t_{ij}, i_{ij}, f_{ij}\rangle, (i=1,2,\cdots,m; j=1,2,\cdots,n)$. Therefore, the decision matrix can be given as follows:

$$D = (\alpha_{ij})_{m\times n} = \begin{bmatrix} \langle t_{11}, i_{11}, f_{11}\rangle & \langle t_{12}, i_{12}, f_{12}\rangle & \cdots & \langle t_{1n}, i_{1n}, f_{1n}\rangle \\ \langle t_{21}, i_{21}, f_{21}\rangle & \langle t_{22}, i_{22}, f_{22}\rangle & \cdots & \langle t_{2n}, i_{2n}, f_{2n}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle t_{m1}, i_{m1}, f_{m1}\rangle & \langle t_{m2}, i_{m2}, f_{m2}\rangle & \cdots & \langle t_{mn}, i_{mn}, f_{mn}\rangle \end{bmatrix}$$

In the following, a new method to rank alternatives and select the best alternative is given.

Step1: Calculate the comprehensive evaluation value of each alternative. Utilize the SVNNWBM operator to aggregate the information of each alternative, then the comprehensive evaluation value

$$\alpha_i =< t_i, i_i, f_i >= SVNNWBM^{p,q}(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in})$$

$$SVNNWBM^{p,q}(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}) = \left(\bigoplus_{\substack{j,k=1,\\ j\ne k}}^{n} \frac{w_j w_k}{1-w_j} (\alpha_{ij}^p \otimes \alpha_{ik}^q) \right)^{\frac{1}{p+q}}$$

$$== \left\langle \frac{2(\prod_{\substack{j,k=1,\\ j\ne k}}^{n} u_{jk}^{\frac{w_j w_k}{1-w_j}} - \prod_{\substack{j,k=1,\\ j\ne k}}^{n} v_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}}}{(\prod_{\substack{j,k=1,\\ j\ne k}}^{n} u_{jk}^{\frac{w_j w_k}{1-w_j}} + 3\prod_{\substack{j,k=1,\\ j\ne k}}^{n} v_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}} + (\prod_{\substack{j,k=1,\\ j\ne k}}^{n} u_{jk}^{\frac{w_j w_k}{1-w_j}} - \prod_{\substack{j,k=1,\\ j\ne k}}^{n} v_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}}}, \right.$$

$$\frac{(\prod_{\substack{j,k=1,\\ j\ne k}}^{n} x_{jk}^{\frac{w_j w_k}{1-w_j}} + 3\prod_{\substack{j,k=1,\\ j\ne k}}^{n} t_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}} - (\prod_{\substack{j,k=1,\\ j\ne k}}^{n} x_{jk}^{\frac{w_j w_k}{1-w_j}} - \prod_{\substack{j,k=1,\\ j\ne k}}^{n} t_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}}}{(\prod_{\substack{j,k=1,\\ j\ne k}}^{n} x_{jk}^{\frac{w_j w_k}{1-w_j}} + 3\prod_{\substack{j,k=1,\\ j\ne k}}^{n} t_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}} + (\prod_{\substack{j,k=1,\\ j\ne k}}^{n} x_{jk}^{\frac{w_j w_k}{1-w_j}} - \prod_{\substack{j,k=1,\\ j\ne k}}^{n} t_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}}},$$

$$\left. \frac{(\prod_{\substack{j,k=1,\\ j\ne k}}^{n} z_{jk}^{\frac{w_j w_k}{1-w_j}} + 3\prod_{\substack{j,k=1,\\ j\ne k}}^{n} y_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}} - (\prod_{\substack{j,k=1,\\ j\ne k}}^{n} z_{jk}^{\frac{w_j w_k}{1-w_j}} - \prod_{\substack{j,k=1,\\ j\ne k}}^{n} y_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}}}{(\prod_{\substack{j,k=1,\\ j\ne k}}^{n} z_{jk}^{\frac{w_j w_k}{1-w_j}} + 3\prod_{\substack{j,k=1,\\ j\ne k}}^{n} y_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}} + (\prod_{\substack{j,k=1,\\ j\ne k}}^{n} z_{jk}^{\frac{w_j w_k}{1-w_j}} - \prod_{\substack{j,k=1,\\ j\ne k}}^{n} y_{jk}^{\frac{w_j w_k}{1-w_j}})^{\frac{1}{p+q}}} \right\rangle$$

$(i = 1, 2, \cdots, m)$ And

Where $p, q \ge 0$.

$$u_{jk} = (2 - t_{ij})^p (2 - t_{ik})^q + 3t_{ij}^p t_{ik}^q,$$
$$v_{jk} = (2 - t_{ij})^p (2 - t_{ik})^q - t_{ij}^p t_{ik}^q,$$
$$t_{jk} = (1 + i_{ij})^p (1 + i_{ik})^q - (1 - i_{ij})^p (1 - i_{ik})^q,$$
$$x_{jk} = (1 + i_{ij})^p (1 + i_{ik})^q + 3(1 - i_{ij})^p (1 - i_{ik})^q,$$
$$y_{jk} = (1 + f_{ij})^p (1 + f_{ik})^q - (1 - f_{ij})^p (1 - f_{ik})^q,$$
$$z_{jk} = (1 + f_{ij})^p (1 + f_{ik})^q + 3(1 - f_{ij})^p (1 - f_{ik})^q.$$

Step2: Calculate the cosine similarity measure [10] of each alternative and the ideal alternative $(1,0,0)$.

$$S(\alpha_i) = \frac{t_i}{\sqrt{t_i^2 + i_i^2 + f_i^2}} \ (i = 1, 2, \cdots, m)$$

Step3: Give the ranking order of all alternatives according to the calculated results $S(\alpha_i)$. The alternative will be better if the similarity measure value is bigger. Thus, the alternatives can be ordered and the best alternative can be identified.

Step4: Obtain the best alternative and the worst alternative.

A. An example

In order to validate the effectiveness and feasibility of the proposed decision-making method, we consider the same decision-making problem adapted from Ye [8].

The example is about an investment company with four possible alternatives for investing on the basis of three criteria expressed by $C_j (j = 1,2,3)$. The corresponding weight is denoted by $W = (0.35, 0.25, 0.4)$. Where A_1 is a car company, A_2 is a food company, A_3 is a computer company, A_4 is an arms company. And C_1 is the risk analysis, C_2 is the growth analysis, C_3 is the environmental impact analysis.

The single-valued neutrosophic decision matrix D can be obtained based on expert's experience.

$$D = \begin{bmatrix} \langle 0.4,0.2,0.3\rangle & \langle 0.4,0.2,0.3\rangle & \langle 0.2,0.2,0.5\rangle \\ \langle 0.6,0.1,0.2\rangle & \langle 0.6,0.1,0.2\rangle & \langle 0.5,0.2,0.2\rangle \\ \langle 0.3,0.2,0.3\rangle & \langle 0.5,0.2,0.3\rangle & \langle 0.5,0.3,0.2\rangle \\ \langle 0.7,0.0,0.1\rangle & \langle 0.6,0.1,0.2\rangle & \langle 0.4,0.3,0.2\rangle \end{bmatrix}$$

To solve this problem, the proposed method is applied according to the following computational steps:

Step1: Utilize the SVNNWBM operator to calculate the comprehensive evaluation value of each alternative. In general, suppose $p = q = 1$, then

$$u_{jk} = (2 - t_{ij})(2 - t_{ik}) + 3t_{ij}t_{ik},$$
$$v_{jk} = (2 - t_{ij})(2 - t_{ik}) - t_{ij}t_{ik},$$
$$t_{jk} = (1 + i_{ij})(1 + i_{ik}) - (1 - i_{ij})(1 - i_{ik}),$$
$$x_{jk} = (1 + i_{ij})(1 + i_{ik}) + 3(1 - i_{ij})(1 - i_{ik}),$$
$$y_{jk} = (1 + f_{ij})(1 + f_{ik}) - (1 - f_{ij})(1 - f_{ik}),$$
$$z_{jk} = (1 + f_{ij})(1 + f_{ik}) + 3(1 - f_{ij})(1 - f_{ik}).$$

For i=1,

$u_{12} = u_{21} = 3.04, u_{13} = u_{31} = 3.12, u_{23} = u_{32} = 3.12,$
$v_{12} = v_{21} = 2.4, v_{13} = v_{31} = 2.8, v_{23} = v_{32} = 2.8,$
$t_{12} = t_{21} = 0.8, t_{13} = t_{31} = 0.8, t_{23} = t_{32} = 0.8,$
$x_{12} = x_{21} = 3.36, x_{13} = x_{31} = 3.36, x_{23} = x_{32} = 3.36,$
$y_{12} = y_{21} = 1.2, y_{13} = y_{31} = 1.6, y_{23} = y_{32} = 1.6,$
$z_{12} = z_{21} = 3.16, z_{13} = z_{31} = 3, z_{23} = z_{32} = 3,$

Then

$\alpha_1 = SVNNWBM^{p,q}(\alpha_{11}, \alpha_{12}, \alpha_{13}) =< 0.3201, 0.2, 0.3749 >$

Similarly, for i=2

$u_{12} = u_{21} = 3.04, u_{13} = u_{31} = 3, u_{23} = u_{32} = 3,$
$v_{12} = v_{21} = 1.6, v_{13} = v_{31} = 1.8, v_{23} = v_{32} = 1.8,$
$t_{12} = t_{21} = 0.4, t_{13} = t_{31} = 0.6, t_{23} = t_{32} = 0.6,$
$x_{12} = x_{21} = 3.64, x_{13} = x_{31} = 3.48, x_{23} = x_{32} = 3.48,$
$y_{12} = y_{21} = 0.8, y_{13} = y_{31} = 0.8, y_{23} = y_{32} = 0.8,$
$z_{12} = z_{21} = 3.36, z_{13} = z_{31} = 3.36, z_{23} = z_{32} = 3.36,$

Then

$\alpha_2 = SVNNWBM^{p,q}(\alpha_{21}, \alpha_{22}, \alpha_{23}) =< 0.5623, 0.1359, 0.2 >$

For i=3

$u_{12} = u_{21} = 3, u_{13} = u_{31} = 3, u_{23} = u_{32} = 3,$
$v_{12} = v_{21} = 2.4, v_{13} = v_{31} = 2.4, v_{23} = v_{32} = 2,$
$t_{12} = t_{21} = 0.8, t_{13} = t_{31} = 1, t_{23} = t_{32} = 1,$
$x_{12} = x_{21} = 3.36, x_{13} = x_{31} = 3.24, x_{23} = x_{32} = 3.24,$
$y_{12} = y_{21} = 1.2, y_{13} = y_{31} = 1, y_{23} = y_{32} = 1,$
$z_{12} = z_{21} = 3.16, z_{13} = z_{31} = 3.24, z_{23} = z_{32} = 3.24,$

Then

$\alpha_3 = SVNNWBM^{p,q}(\alpha_{31}, \alpha_{32}, \alpha_{33}) =< 0.4280, 0.2369, 0.2622 >$

For i=4

$u_{12} = u_{21} = 3.08, u_{13} = u_{31} = 2.92, u_{23} = u_{32} = 2.96,$
$v_{12} = v_{21} = 1.4, v_{13} = v_{31} = 1.8, v_{23} = v_{32} = 2,$
$t_{12} = t_{21} = 0.2, t_{13} = t_{31} = 0.6, t_{23} = t_{32} = 0.8,$
$x_{12} = x_{21} = 3.8, x_{13} = x_{31} = 3.4, x_{23} = x_{32} = 3.32,$
$y_{12} = y_{21} = 0.6, y_{13} = y_{31} = 0.6, y_{23} = y_{32} = 0.8,$
$z_{12} = z_{21} = 3.48, z_{13} = z_{31} = 3.48, z_{23} = z_{32} = 3.36,$

Then

$\alpha_4 = SVNNWBM^{p,q}(\alpha_{41}, \alpha_{42}, \alpha_{43}) =< 0.5578, 0.1271, 0.1639 >$

Therefore, we can get the comprehensive evaluation value of each alternative.

$\alpha_1 =< 0.3201, 0.2, 0.3749 >, \ \alpha_2 =< 0.5623, 0.1359, 0.2 >,$
$\alpha_3 =< 0.4280, 0.2369, 0.2622 >, \ \alpha_4 =< 0.5578, 0.1271, 0.1639 >,$

Step2: Calculate cosine similarity measure $S(\alpha_i, \alpha^*)(i = 1, 2, 3, 4)$. The ideal alternative is defined as $\alpha^* = \langle 1,0,0\rangle$, the bigger the similarity measure value is, the better the alternative is.

$S(\alpha_1, \alpha^*) = 0.6017, S(\alpha_2, \alpha^*) = 0.9187,$
$S(\alpha_3, \alpha^*) = 0.7711, S(\alpha_4, \alpha^*) = 0.9373,$

Step3: Give the ranking order of all alternatives based on $S(\alpha_i, \alpha^*)(i = 1, 2, 3, 4)$, that is $A_4 \succ A_2 \succ A_3 \succ A_1$.

Step4: Get the best alternative and the worst alternative. The best alternative is A_4 , and the worst alternative is A_1.

V. Conclusion

The indeterminate information and inconsistent information existing commonly in many cases cannot be deal with utilizing FSs and IFSs, and the single-valued neutrosophic set (SVNS) proposed can be better to process the information.

In this paper, we have extended the NWBM operator to accommodate the single-valued neutrosophic environment. A novel SVNNWBM aggregation operator based on Einstein operational rules is firstly proposed to avoid the weaknesses of some impractical operations, and some desirable properties of the new operator are also analyzed. In order to demonstrate the effectiveness and application of the proposed method, an illustrative example based on the SVNNWBM operator is presented. The extended SVNNWBM aggregation operator can not only capture the interrelation between different single-valued neutrosophic numbers, but also avoid some impractical operations in some cases.

The proposed method in this paper compared with the earlier method developed by Ye [8], which has the following advantages. Firstly, it can capture the interrelationship between input arguments under single-valued neutrosophic environment, which considerate the indeterminacy information besides truth and falsity information. Secondly, it has the ability to handle irrational operations in some cases than the other methods. Thirdly, if p=1, q=0, then the SVNNWBM aggregation operators can be reduced to the operator developed by Ye [8], so the operators proposed by Ye[8] is only an special case of the aggregation operator proposed in this paper .Therefore, the main advantage of the method developed in this paper is that it can provide a more feasible and general aggregation operator, and which makes the final result more practical than the other traditional methods in real decision-making problems to multi-criteria decision analysis.

In the future, we will extend this proposed method to other neutrosohpic environments, and apply it to the other domains, such as approximate reasoning, pattern recognition.

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