

On Young Functionals Related to Certain Class of Rapidly Oscillating Sequences

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Abstract—The paper is devoted to theory of classical Young measures and associated with them Young functionals i.e. certain mappings that assign to any Carathodory function an integral with respect to related Young measure. The main result provides us with probabilistic characterization of such measures in the case where associated sequence of fast oscillating functions has uniform representation - a notion introduced in this paper. In many interesting cases this new characterization enables one to find an explicit formulae for the density functions of the Young measures and thus also to compute effectively the values of corresponding Young functionals. In other cases, where the explicit formulae for the densities cannot be received, the main result makes it possible to take advantage of Monte Carlo methods to evaluate the Young functional's values.

Index Terms—Young measures, Young functionals, fast-oscillating sequences, periodic functions, Monte Carlo.

I. INTRODUCTION

There are many physical phenomena that can be described and explained by analysis of the suitable functionals corresponding to the respective physical quantities. The natural way of analysis of such functionals can be performed with the use of variational methods, among of which the minimization methods are one of the most powerful. Their usage can be justified as a conclusion from the variational laws of nature such as, say, the least action principle. Therefore there is no surprise that variational methods have been proved useful in engineering. One may look at [1], [2], [3], [5], [9], [10], or at books [11], [13] (and the references cited there) to mention just a few examples from vast literature on this subject.

In general, we minimize the energy functional \mathcal{J} of the form

$$\mathcal{J}(u) = \int_{\Omega} H(x, u(x), \nabla u(x)) dx,$$

where $\mathbf{R}^d \supset \Omega$ is a bounded Lipschitz domain; $u: \Omega \rightarrow K \subset \mathbf{R}^l$ is a function from a suitable Sobolev space V (with K being a compact set); and $H: \Omega \times \mathbf{R}^d \times \mathbf{R}^{ld} \rightarrow \mathbf{R} \cup \{+\infty\}$ is a function satisfying certain growth and regularity conditions.

In the nonlinear elasticity theory we have $d = l = 3$, Ω is the considered elastic body, u is the displacement of Ω , H is density of the internal energy (it usually depends on some physical constants).

The functional \mathcal{J} is bounded below 'by nature' so there exists a minimizing sequence for \mathcal{J} , that is a sequence $\{u_k\} \subset V$ such that

$$\lim_{k \rightarrow \infty} \mathcal{J}(u_k) = \inf\{\mathcal{J}(v) : v \in V\}.$$

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However in general, for example when minimizing energy functionals corresponding to the multi-well potentials, the integrand does not satisfy convexity condition guaranteeing achieving by \mathcal{J} its infimum. In that case elements of the minimizing sequence are the functions that oscillate rapidly around the weak (or weak*) limit of that sequence. This situation is met in engineering for example when energy functionals of certain alloys are minimized. With the help of microscope one can observe what is by engineers called a 'microstructure' – a structure somehow between the micro- and macroscopic scale.

The following example shows the crucial role of rapid oscillations in the arising of microstructure, [13]. Consider a function sequence $\{\sin kx\}$, $k \in \mathbf{N}$, defined on an interval $(0, \frac{\pi}{2})$. On one hand we have $\int_0^{\frac{\pi}{2}} \sin(kx) dx \rightarrow \int_0^{\frac{\pi}{2}} 0 dx$, as $k \rightarrow \infty$, while on the other $\int_0^{\frac{\pi}{2}} \sin^2(kx) dx \rightarrow \int_0^{\frac{\pi}{2}} 0^2 dx$.

This in particular shows that even when the function φ is continuous and a function sequence u_k satisfies $\lim \int u_k(x) dx = \int u_0(x) dx$, the equality $\lim \int \varphi(u_k(x)) dx = \int \varphi(u_0(x)) dx$ does not have to be true in general.

More generally we can say: the existence of the weak limit of a function sequence does not guarantee even the existence of the weak limit of the sequence whose elements are compositions of the elements of the aforementioned sequence with fixed (even continuous) function.

In the context of the variational methods a generalization of the above example is presented in [11], see also [7].

One of the ways of dealing with such problems is the L. C. Young's idea: enlarging the space of admissible functions from Sobolev spaces to the space of parametrized measures $\nu := \{\nu_x\}_{x \in \Omega}$, which are now called *Young measures*. Young measures theory has a long history. It starts with the seminal work [15] where the main concept was introduced with the aim to provide extended solutions for some non-convex problems in variational calculus. After then Young developed his pioneering ideas in [16]. These "generalized limits" of oscillating sequences were called by Young himself "generalized curves" in the one dimensional case, and "generalized surfaces" in more general situation. Nowadays this concept has found applications in the solution of various non-convex optimization problems - that are at the core of many different contemporary engineering theories. They arise e.g. in optimal control, nonlinear evolution equation, variational calculus, micromagnetic phenomena in ferro-magnetic materials as well as in microstructures theory in continuum mechanics. It turns out that even the simplest form of a Young measure,

namely this where the measure does not depend on the parameter $x \in \Omega$, is important both in the theory and applications. Such Young measures are called *homogeneous ones*.

In contemporary literature the Young measures are defined under different assumptions about underlying spaces and analyzed from different standpoints. This paper focuses on the classical Young measures related to sequences of rapidly oscillating functions. Presented here results are partly based on our conference presentations [7], [8].

The structure of the paper is the following. In the next section we introduce some preliminary definitions and results from the Young measure theory. In Section III we define some classes of fast-oscillating sequences and state new proposition that provide a direct link between the Young measure theory and the probability theory. Section IV presents exemplary theoretical result that is a conclusion from the main proposition stated in Section III. This conclusion allows one to find explicit forms of the density functions of related classical Young measures in various situations. Section VI illustrates the usefulness of our new results in the problems of computing Young functionals' values Finally we make some remarks about possible further extensions.

II. PRELIMINARY DEFINITIONS AND RESULTS

Now we introduce basic notions of the Young measure theory from the point of view of nonlinear elasticity. Our presentation follows the approach taken in [13], where the reader is referred to for detailed information along with necessary notions from functional analysis and further bibliography. Another book treating Young measures thoroughly in the context the optimization theory and variational calculus is [14].

Let Ω be a nonempty, open and bounded subset of \mathbf{R}^d with smooth boundary. Denote by $L^\infty(\Omega)$ the Banach space of essentially bounded functions defined on Ω with values in a compact set $K \subset \mathbf{R}^l$. Let $\{f_n\}$ be a sequence of functions converging to some function f_0 weakly* in L^∞ and denote by φ a continuous real valued function with domain \mathbf{R}^l . By the continuity of φ the sequence $\{\varphi(f_n)\}$ is uniformly bounded in L^∞ norm and Banach-Alaoglu theorem yields the existence of the (not relabeled) subsequence such that $\varphi(f_n) \rightarrow g$ weakly* in L^∞ . However, in general g is not $\varphi(f_0)$, moreover, it is not even a function with domain in \mathbf{R}^l . To quote from [13]: 'The Young measure associated with $\{f_n\}$ furnishes the link among $\{f_n\}$, f_0 , g and φ .' We now state the basic existence theorem for Young measures in its full generality.

Theorem 2.1: (see Theorem 2.2 in [13]) Let $\Omega \subset \mathbf{R}^d$ be a measurable set and let $z_n: \Omega \rightarrow \mathbf{R}^l$ be measurable functions such that

$$\sup_n \int_\Omega h(|z_n|) dx < \infty,$$

where $h: [0, \infty) \rightarrow [0, \infty)$ is a continuous, nondecreasing function such that $\lim_{t \rightarrow \infty} h(t) = \infty$. There exist a subsequence, not relabeled, and a family of probability measures $\nu = \{\nu_x\}_{x \in \Omega}$ (the associated Young measure) depending measurably on x with the property that whenever the sequence $\{H(x, z_n(x))\}$ is weakly convergent in $L^1(\Omega)$

for any Carathéodory function $H(x, \lambda): \Omega \times \mathbf{R}^l \rightarrow \mathbf{R} \cup \{\infty\}$, the weak limit is the function

$$\overline{H}(x) = \int_{\mathbf{R}^l} H(x, \lambda) d\nu_x(\lambda).$$

The family of probability measures $\nu = \{\nu_x\}_{x \in \Omega}$ is called the Young measure associated with the sequence $\{z_n\}$.

Let us recall that the Carathéodory function is a function that is measurable with respect to the first and continuous with respect to the second variable.

It often happens that the Young measure $\nu = \{\nu_x\}_{x \in \Omega}$ does not depend on $x \in \Omega$. In this case we denote it merely by ν ; such Young measure is called homogeneous.

One may also look at the Young measure as at object associated with *any* measurable function defined on a nonempty, open, bounded subset Ω of \mathbf{R}^d with values in a compact subset K of \mathbf{R}^l . Such a conclusion can be derived from the theorem 3.6.1 in [14]. Due to this theorem it can be proved that the Young measure associated with a simple function is homogeneous and is the convex combination of Dirac measures. These Dirac measures are concentrated at the values of the simple function under consideration while coefficients of the convex combination are proportional to the Lebesgue measure of the sets on which the respective values are taken on by the function; see [12] for details and more general results concerning simple method of obtaining explicit form of Young measures associated with oscillating functions (similar, although mathematically more complicated situation, is met in elasticity when the deformed body has a laminate structure; see e.g. Section 4.6 in [11]). On the basis of this concept a more general characterization of a Young measure associated with any Borel function was introduced in [6]. The main result stated there provides direct link between the Young measure concepts and the probability theory. Namely, the following theorem holds:

Theorem 2.2: Let $f: \mathbf{R}^d \supset \Omega \rightarrow K \subset \mathbf{R}^l$ be a Borel function with Young measure ν . Then ν is the probability distribution of the random variable $Y = f(U)$, where U has a uniform distribution on Ω .

Now, let us recall the notion of classical Young measure associated with a sequence of oscillating functions $\{f_k\}$, see e.g. [14].

Definition 2.1: The classical Young measure generated by the sequence $\{f_k\}$ is a family of probability measures $\nu = \{\nu_x\}_{x \in \Omega}$ satisfying the condition:

for any Carathéodory function H

$$\int_\Omega H(x, f_k(x)) dx \xrightarrow{k \rightarrow \infty} \int_\Omega \int_K H(x, y) d\nu_x(y) dx \quad (1)$$

The application that assigns to any Carathéodory function H the integral given on the right-hand side of the above equation is called *Young functional*. Its values on H will be denoted here as $YF(H)$ while the integrals on the left-hand side of 1 will be denoted as $C(f_k, H)$

Basically, the above definition presents the original understanding of the Young measure, as introduced in his work [16]. These measures are of our main concern in this paper.

III. MAIN RESULT AND ITS THEORETICAL APPLICATIONS

A. Rapidly oscillating sequences with uniform representation

Let function $f: [a, b) \rightarrow K \subset \mathbf{R}$ be a Borel function defined on the interval $[a, b)$, $b > a$, and let $f^e: \mathbf{R} \rightarrow K$ be the periodic extension of f (with the period equal to $T = b - a$).

Let Ω be a given interval. A sequence $\{f_k\}$ of functions $f_k: \Omega \rightarrow K$, $k = 1, 2, \dots$ defined by the formula

$$f_k(x) = f^e(kx), \quad x \in \Omega \tag{2}$$

will be called a Rapidly Oscillating Sequence with Uniform representation f , and denoted as ROSU(f). In such a case we will also say that f generates rapidly oscillating sequence $\{f_k\}$. Note, that the interval Ω - the domain of elements of ROSU(f) - does not have to be the same as $[a, b)$ i.e. the domain of f .

Example 1 In this example we present illustrative plots of some elements of ROSU(f), where

$$f(x) = (x - 2)^2, \quad x \in [0, 13/4) \tag{3}$$

Its purpose is not only to illustrate the behavior of ROSU (which as a matter of fact is quite obvious) but also to illustrate the concept of classical Young measure associated with the sequence $\{f_k\}$. The plots of the function f given by (3) as well as of exemplary elements of ROSU(f) with the domain $\Omega = [0, 5)$ are presented in Fig.1. Namely it shows the plots of f_1, f_5 and f_{35} .

It is seen that the graphs of f_k are getting denser when k tends to infinity. Unfortunately, a conventional weak* cluster point of $\{f_k\}$ loses most of the information about the fast oscillations in $\{f_k\}$ because, in some sense, it takes into account only the mean values of $\{f_k\}$ - as the integrals simply do. That is why we need a new concept of the limit and here the theory of Young measures appears to be very helpful. If ν_x is the Young measure associated with $\{f_k\}$ then, roughly speaking, for any measurable set $A \subset K$ the intuitive meaning of $\nu_x(A)$ is the probability that for an *infinitesimally* small neighborhood S of $x \in \Omega$ and sufficiently large k 's we can "find" $f_k(s)$ in A , when s changes within S . In some sense it reflects the "density" of the values in K - it can be observed in the plot (d) (for f_{35}) where "more frequent values" create darker straps.

B. Characterization of the Classical Young measures generated by the ROSU(f)

It results directly from the definition of ROSU(f) that its behaviour, as k tends to infinity, is exactly the same in every neighborhood of any $x \in \Omega$. In other words, its asymptotic behavior in an arbitrarily small interval $I \subset \Omega$ does not depend on where the interval is placed within the domain. Consequently, it is obvious that the classical Young measure generated by the ROSU(f) is the homogeneous Young measure (i.e. it does not depend on $x \in \Omega$).

Now, let U_S denote a random variable uniformly distributed on S . Let us consider two functions $g_1: \Omega_1 \rightarrow K$ and $g_2: \Omega_2 \rightarrow K$. We will say that the two functions *identically transform a uniform distribution* if the distributions of the random variables $g_1(U_{\Omega_1})$ and $g_2(U_{\Omega_2})$, are the same. This fact will be denoted by $g_1 \approx g_2$. Obviously the " \approx " is the equivalence relation.

Note that if the ROSU(f) is defined on the same interval as the generating function f , then *any* of its elements transform a uniform distribution identically as the function f , i.e. $f_k \approx f$ for any $k = 1, 2, \dots$. Now let us consider the case where the ROSU(f) is defined on interval Ω that is different than the domain $[a, b)$ of the generating function f . In such a case it can be seen that $f_k(U_\Omega) \xrightarrow{D} f(U_{[a,b)})$, where \xrightarrow{D} denotes the *convergence in distribution*, see [4]. In other words the distribution of $f(U_{[a,b)})$ is a vague limit of the distributions of $f_k(U_\Omega)$. Indeed, for k 's large enough so the length of interval Ω is greater than $(b - a)/k$ we have for any measurable subset $A \subset K$:

$$|P(f_k(U_\Omega) \in A) - P(f(U_{[a,b)}) \in A)| < 1/k$$

Finally, by Theorem 2.2 we know, that the distribution of $Y = f(U_{[a,b)})$ is the Young measure associated with the function f . Thus we can formulate the following result concerning classical Young measures.

Proposition 3.1: Classical Young measure generated by ROSU(f) is the homogeneous Young measure. This measure is identical with the distribution of the random variable $Y = f(U_{[a,b)})$.

IV. THEORETICAL APPLICATIONS

On the basis of the above general characterization of the classical Young measure generated by ROSU(f) we can obtain a number of rules which allow one to find an explicit form of the classical Young measure in various specific cases. For example, let us consider the following situation.

Let $[a, b)$ be the interval-domain of the function f . Let us consider an open partition of $[a, b)$ into a number of open intervals I_1, I_2, \dots, I_n such that the intervals are pairwise disjoint and $\bigcup_{i=1}^n \bar{I}_i = [a, b)$, where \bar{A} denotes the closure of the set A .

Let function f be continuously differentiable on each interval of the partition and let $f'(x) \neq 0$ for all $x \in \bigcup_{l=1}^n I_l$.

Using the well-known probabilistic result concerning the distributions of such functions of random variables we can obtain the following corollary of the Proposition 3.1.

Corollary 4.1: The classical Young measure generated by any ROSU(f) is a homogeneous one and its density g with respect to the Lebesgue measure on K is of the following form

$$g(y) = \frac{1}{b-a} \sum_{l=1}^n |h'_l(y)| \mathbf{1}_{D_l}(y) \tag{4}$$

where h_l is the inverse of f on the interval I_l , while $D_l = f(I_l)$ is the domain of h_l . The symbol $\mathbf{1}$ stands for the characteristic function of the set indicated in its subscript.

V. APPLICATIONS TO COMPUTATION OF YOUNG FUNCTIONAL'S VALUES

To show in what way the above theoretical results work in practice, let us consider a case where both sides of the Eq. 1 can be computed precisely.

Example 1 - continuation

As previously let us consider the function (3) that is defined on the interval $[0, 13/4)$ and the ROSU(f) that is defined on the interval $[0, 5)$. Now our aim is to compute

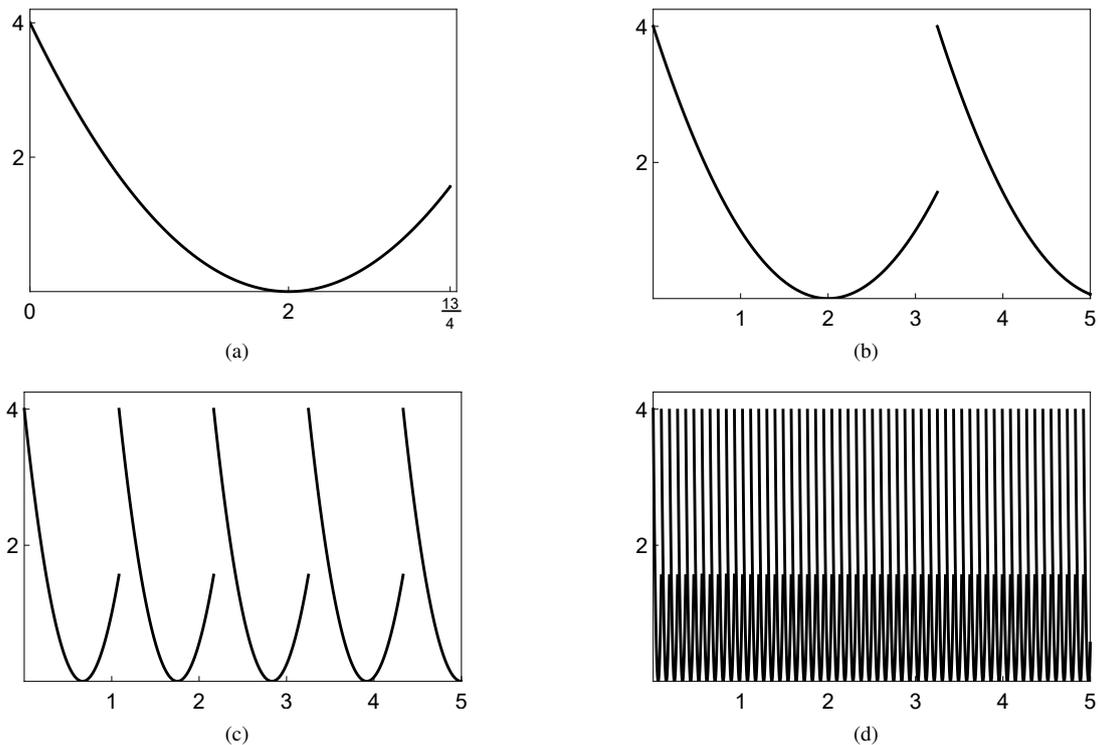


Fig. 1: A function f given by Eq. (3) and functions f_1, f_3 and f_{35} belonging to $ROSU(f)$ with the domain $\Omega = [0, 5)$. Plots of these functions are labeled as a, b, c, and d, respectively

the integrals $C(f_k, H)$ that appear on the left-hand side of the Eq. 1 for the exemplary Carathéodory function $H(x, y) = x^2y$.

Fig. 2 provides us with the graphical illustration of the $C(f_3; H)$ and $C(f_5; H)$ in this case. The values of these integrals are the areas of the surfaces presented in plot (a) and (b), respectively.

In order to compute the limit value in Eq. 1 let us note, that in the considered case

$$A_k \leq C(f_k, H) \leq B_k \tag{5}$$

where

$$\begin{aligned} A_k &= 2197(169[20k/13] - 585[20k/13]^2 + 980[20k/13]^3)/184320k^3 \\ B_k &= 2197(564 + 1939[20k/13] + 2355[20k/13]^2 + 980[20k/13]^3)/184320k^3 \end{aligned} \tag{6}$$

with $[x]$ denoting the greatest integer less than or equal to x . Because the limit of both A_k, B_k as k tends to infinity is the same and equals $L1 = 6125/144$, it is also the limit of $C(f_k, H)$ that we have been looking for.

Now let us find the limit with the help of the Eq. 1. i.e. by computation of the value of the Young functional $YF(H)$. For this purpose we need to know the classical Young measure generated by the $ROSU(f)$. With the help of Corollary 4.1 one can easily receive the following formula for its density function:

$$\begin{aligned} g(y) &= 4/(13\sqrt{y})\mathbf{1}_{[0,25/16)}(y) \\ &+ 2/(13\sqrt{y})\mathbf{1}_{[25/16,4)}(y) \end{aligned}$$

Thus the value of the Young functional in the considered case is the following (recall that the Young measure ν_x is

homogeneous in this case, so we can omit subscript "x" in the denotations) :

$$YF(H) = \int_{\Omega} \int_K H(x, y) d\nu(y) dx = \tag{7}$$

$$\int_{[0,5)} \int_{[0,4)} H(x, y) g(y) dy dx = \int_2^4 \int_0^{25/16} \frac{4x^2y}{13\sqrt{y}} dy dx + \tag{8}$$

$$\int_2^4 \int_{25/16}^4 \frac{2x^2y}{13\sqrt{y}} dy dx = \frac{6125}{144} = L1 \tag{9}$$

So, as it should be expected, the "whole information" about the rapid oscillations in this case is contained in the classical Young measure and - due to the Proposition 3.1 - can be revealed with the help of the Corollary 4.1. Because we are going to make use of this example as a kind of benchmark in our next considerations, let us also compute both the limit and the Young functional values for another Carathéodory function, namely let $H(x, y) = x \cos(y)$. In this case little calculus shows that $C(f_k; H)$ satisfy (5) with

$$\begin{aligned} A_k &= \frac{[20k/13]}{16k^2} (3\sqrt{2\pi}(F(2\sqrt{2/\pi}) + F(5/(2\sqrt{2\pi}))) + 13\sqrt{2\pi}[20k/13](F(2\sqrt{2/\pi}) + FC(5/(2\sqrt{2\pi}))) + 8(\sin(25/16) - \sin(4))) \\ B_k &= \frac{1+[20k/13]}{16k^2} (13\sqrt{2\pi}[20k/13](F(2\sqrt{2/\pi}) + F(5/(2\sqrt{2\pi}))) + 16\sqrt{2\pi}(F(2\sqrt{2/\pi}) + F(5/(2\sqrt{2\pi}))) + 8(\sin(25/16) - \sin(4))) \end{aligned} \tag{10}$$

where F denotes the special function known as the Fresnel

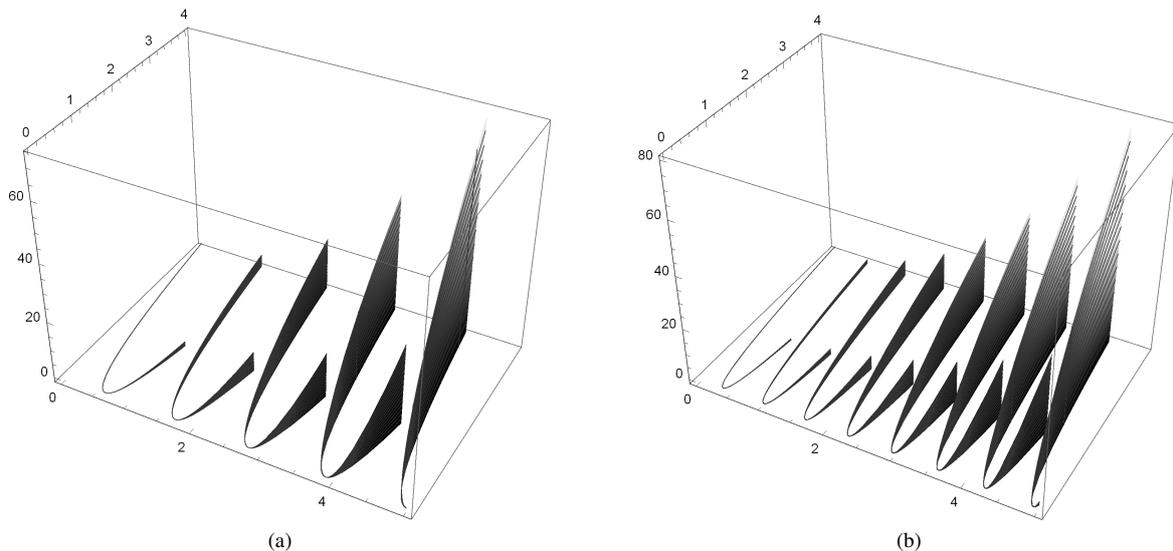


Fig. 2: Graph of two exemplary surfaces - plot (a) and plot (b) - the area of which are, respectively, the values of the integrals $C(f_3; H)$ and $C(f_5; H)$. These integrals are elements of the sequence on the left-hand side of the Eq. 1

function that is given by the following integral:

$$F(z) = \int_0^z \cos(\pi t^2 / 2) dt$$

Again, the limits of both A_k, B_k are the same, so they are the same as the limit of $C(f_k, H)$ in this case. It equals

$$L2 = (25\sqrt{2\pi}/13)(F(2\sqrt{2/\pi}) + F(5/(2\sqrt{2\pi}))) \quad (11)$$

The same value can be obtain with the help of our result. Indeed, it is easy to check, that the integral (7) in the considered case takes on the form:

$$\int_2^4 \int_0^{25/16} \frac{4x \cos(y)}{13\sqrt{y}} dy dx + \int_2^4 \int_{25/16}^4 \frac{2x \cos(y)}{13\sqrt{y}} dy dx$$

and it is equal to $L2$.

Our example is relatively simple and "smooth" - we were able to compute the limits explicitly. However even here one can notice the benefits resulting from our Proposition. It is quite clear that even in this case where the oscillations have such smooth nature, the computation of the limit of $C(f_k, H)$ for more complex Carathéodory functions could be much more difficult task than the calculation of the value of the Young functional $YF(H)$.

The possibility of derivation of explicit formulae for the density functions of classical Young measures is not the only benefit resulting from our Proposition 3.1. In many interesting cases explicit formulae for these densities cannot be found. For instance, if one wants to make use of Corollary 4.1 they have to obtain the inverses of f on particular subintervals of its domain, but it is not always possible. However in all such cases thanks to Proposition 3.1 one may use directly Monte Carlo simulations in order to evaluate values of the Young functionals.

The idea of the Monte Carlo simulation is well-known. Its main concept is to draw a sample Z_1, Z_2, \dots, Z_m of realizations of independent random variables with the same distribution as the random phenomenon under study. Based

on this sample, important information concerning stochastic characteristics of the examined distribution can be derived with the help of statistical-inference tools. Monte Carlo simulations are also frequently used to evaluate values of integrals in various situations. Because now we consider cases where the classical Young measure is not given explicitly, we adopt this approach to evaluate Young functional

$$YF(H) = \int_c^d \left(\int_a^b H(x, y) d\nu(y) \right) dx.$$

When the change of integration order is possible, then this integral is equal to

$$YF(H) = \int_a^b \left(\int_c^d H(x, y) dx \right) d\nu(y).$$

By Proposition 1, the Young measure ν is the probability distribution of the random variable $Y=f(U(a,b))$. So, we may look at the Young-functional-value as the expected value of the random variable $\int_c^d H(x, Y) dx$ and it is well-known that it can be best-estimated by its empirical mean (i.e. the arithmetic mean of its randomly generated values). Thus we propose to make use of the following procedure $YFE(H, f, a, b, c, d, N)$ to evaluate values of the Young functional:

- Step 0: Set $k = 1$
- Step 1: Set $t=Random((a,b))$
- Step 2: Set $y=f(t)$
- Step 3: Set $z[k]=INT(H(x,y),x,c,d)$
- Step 4: N times repeat Steps 1 to 3
- Step 5: Set $sample=(z[1], \dots, z[N])$
- Step 6: Set $YF=Mean(sample)$
- Step 7: Return YF

The above procedure is called with the following arguments: the formula that defines the Caratheodory function H , the formula for the function f that defines the ROSU and its domain, i.e. the interval $[a,b]$, the domain $[c,d]$ of the ROSU itself, and the size N of the sample that will be used to evaluate the Young functional value. The subroutine $Random(I)$ returns a pseudorandom number generated according to the uniform probability distribution defined on an interval I . The subroutine $INT(H(x,y), x,c,d)$ returns the value

TABLE I: Results Of Monte Carlo Evaluations of the Young Functional Values for Problems Considered in Example 1

Caratheodory integrand	Sample size	Estimated value	Relative error
$H(x, y) = x^2y$	10000	42.4312	0.0024
	20000	42.5480	0.0003
	50000	42.4768	0.0014
$H(x, y) = x \cos(y)$	10000	5.4585	0.0137
	20000	5.6104	0.0138
	50000	5.5431	0.0016

of the integral $\int_c^d H(x, y)dx$. Obviously the integration can be done numerically, because when the subroutine is called, the value of y is already fixed, as it is set up at the Step 2. The usefulness of the Monte Carlo simulations in computation of the values of the Young functionals is supported by the data presented in Table I.

As one can see, the accuracy of the Monte Carlo evaluations are really very good (let us recall that the true values are $L1$ and $L2$, given by (9) and (11), respectively). Other examples of the advantages resulting from the Monte Carlo approach to computation of Young functionals' values can be found in [8].

The pseudocode of the procedure YFE presented in this section can be implemented in various ways. In our research we use the following version coded in Wolfram Mathematica 10.5 programming environment:

```
Mean[Table[
NIntegrate[H[x, f[RandomReal[{a, b}]]],
{x, c, d}], NN]]
```

where NN is the sample size, while the remaining symbols have an analogous meaning as in the description of the YFE.

Now, let us consider the following function G :

$$G(y) = \int_c^d H(x, y)dx \tag{12}$$

If the Caratheodory integrands are such that the function G has got known open formula valid for all $y \in (a, b)$, then a faster version of YFE can be obtain by changing the original Step 3 in the following way:

Step 3: Set $z[k]=G(y)$

In our exemplary problems, it can be actually done, so we made use of the following Wolfram Mathematica code:

```
Mean[Table[G[f[RandomReal[{a, b}]]], NN]]
```

where, for the problems discussed in Example 1 $G(y) = 125y/3$ if $H(x, y) = x^2y$ and $G(y) = 25 \cos(y)/2$ if $H(x, y) = x \cos(y)$. Recall that in both cases the limits in integral (12) are $c = 0, d = 5$

We see that practical implementation of presented Monte Carlo evaluation procedure is very simple and taking into account the accuracy of the values we have obtained, it is also very competitive alternative tool for engineers.

VI. FINAL REMARK

The probability theory provides us with a number of different versions of the theorems concerning the probability distributions of functions of random variables and/or vectors.

As a consequence, various different rules for computing explicit formulae for the density functions of classical Young measures generated by $ROSU(f)$ can also be obtained on the basis of the result stated in Proposition 3.1.

We are also sure that the same approach enables development of analogous results related to rapidly oscillating sequences with uniform representation which are defined on open and bounded subsets of \mathbf{R}^d . In our opinion it is very promising direction of future research.

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