Exponential Stabilization for a Class of Nonlinear Uncertain Switched Systems with Time-varying Delay

Yongzhao Wang

Abstract—**In this paper, the problem of exponential stabilization for a class of switched nonlinear uncertain systems with time-varying delay is investigated based on a type of Lyapunov-Krasovskii functional, the free-weighting matrix technique and the average dwell time approach. Some exponential stability criteria are established in terms of linear matrix inequalities (LMIs) for switched nonlinear uncertain systems with delay. In addition, the feedback controller design algorithm for the system is given through the matrix deformation technique and Schur complement. Finally, a numerical example and a practical example of river pollution control are provided to show the validity and potential of the developed results.**

Index Terms—**Switched system, Time-varying delays, Average dwell time, Exponential stabilization.**

I. Introduction

It is well known that switched system is one of the most important dynamic hybrid systems, which comprises a collection of subsystems equipped with a switching law orchestrating. In recent decades, a large number of switched systems have appeared in the research field, such as industrial manufacturing, social management, aircraft control systems and artificial intelligence[1-3]. Moreover, many results on behavior analysis, property characterization and control synthesis for various types of switched systems have been obtained [4-6]. Thus, it is significance of studying switched systems, not only for their interesting theoretical properties but also for their applicability in practice.

As far as the many properties of switched systems are concerned, it is worth mentioning that stability analysis is one of the fundamental problems. Compared with the theory for linear switched system, nonlinear one in the presence of time delays is less well-developed due to its inherent complexities. In particular, it is fairly challenging to research the stability and controller design of switched nonlinear systems. In former literature[7-10], many stability results related to switched nonlinear systems have been reported. For instance, it can be obtained the sufficient conditions to guarantee the exponential stability by a common Lyapunov functional (CLF) [11-12]. However, to the best of author's knowledge, CLF might become too conservative when stability is assessed. To address this issue, scholars investigate the systems by using multiple Lyapunov functional (MLF) [13], min/max-switching [14] and average dwell time (ADT) [15-16] in recent developments. Especially, ADT plays an important role in switched system analysis and control synthesis. In this paper, our main goal is to provide a novel multiple Lyapunov-Krasovskii functional to study the exponential stabilization of a class of switched nonlinear uncertain systems with time-varying delay by average dwell time method.

On the other hand, in many practical applications, time delays and uncertainties are regularly encountered in dynamic systems, which lead to poor performance and even instability in some control systems, even make the machine unable to work properly. Therefore, the subject of stability analysis of switched systems with time-varying delays has attracted considerable attention due to strong engineering background in the past few years. For instance, process control systems[17], networked control [18] and power systems [19]. Free weighting matrix method and matrix deformation technique are adopted to reduce the conservatism of delay-dependent criteria of switched linear systems. Obviously, those methods are more realistic and of great significance to study. To mention a few, robustly exponential stability for uncertain neutral systems with time-varying delays and nonlinear perturbations are investigated by matrix deformation technique in [20]. [16] considered the mean-square exponential stability of switched stochastic neutral systems with time-varying delay under asynchronous switching by the free weight matrix method. Moreover, the issues of robustly exponential stability H_∞ control for uncertain discrete switched systems with interval time-varying delay and the new sufficient stability condition with delay dependence are presented in [21]. However, based on the above discussion, the problem of exponential stabilization for a class of switched nonlinear uncertain systems with time-varying delay has not been well reported, which motivates the present study. Specially, in order to overcome some difficulties caused by the delay and nonlinearity behavior, this paper focuses on the methods of the bounded time delays and matrix deformation technique such that the resulting system is exponentially stable.

The core of this paper is a powerful condition for establishing exponential stability of a class of switched nonlinear uncertain systems with time-varying delay. The main contributions of this paper include:

- An explicit expression of time delay is assumed to be bounded. Meanwhile, we consider the uncertainties and nonlinearity in the model, which make the systems get extensive applications in practical field compared with other papers.

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Y. Wang is with the School of Mathematics and Statistics, Anyang Normal University, Anyang, 455000, China. e-mail: wangyongzhao1987@126.com.

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• A novel multiple Lyapunov-Krasovskii functional is established to study the exponential stabilization of switched nonlinear uncertain systems by average dwell time method.

• The sufficient condition for exponential stabilization of switched nonlinear uncertain systems with delay is obtained.

• The stabilizing feedback controller of switched nonlinear systems is designed through the matrix deformation technique and Schur complement.

• A practical example of river pollution control is provided to show the validity and potential of the developed results.

The rest of the paper is organized as follows. The problem description and preliminary knowledge are presented in Section 2. In Section 3, a novel multi-Lyapunov-Krasovskii functional related to delay is constructed, and a sufficient condition for exponential stabilization of a class of switched nonlinear systems with time-varying delay is given by using average dwell time and matrix inequality. Moreover, the controllers of switched systems are designed through special matrix deformation method, which are the important conclusion of this paper. Section 4 gives a numerical example and a practical example of river pollution control to show the validity and potential of the developed results. Conclusions are shown in Section 5.

II. Problem formulation and preliminaries

In this section, we first show the switched system and recall some facts, which will be used in this paper.

The switched nonlinear uncertain system is expressed as

$$\begin{aligned}\dot{x}(t) &= \bar{A}_{1\sigma(t)}(t)x(t) + \bar{A}_{2\sigma(t)}(t)x(t-h(t)) \\ &\quad + B_{\sigma(t)}u(t) + Ef(t,x(t),x(t-h(t))), \\ x(t) &= \varphi(t), t\in[-h_M,0],\end{aligned} \tag{1}$$

where $u(t)\in R^m$ denotes control input, $x(t)\in R^n$ is state vector, $\varphi(t)\in R^n$ represents the initial condition. $B_i(i\in N)$ and E are constant matrices.

For $i\in N, \bar{A}_{1i}(t)$ and $\bar{A}_{2i}(t)$ are uncertain real-valued matrices with appropriate dimensions, which satisfy

$$\begin{aligned}&\bar{A}_{1i}(t) = A_{1i} + \Delta A_{1i}(t), \quad \bar{A}_{2i}(t) = A_{2i} + \Delta A_{2i}(t), \\ &[\Delta A_{1i}(t) \quad \Delta A_{2i}(t)] = H_iF_i(t)[M_{1i} \quad M_{2i}], \forall t\geq 0,\end{aligned} \tag{2}$$

where $\Delta A_{1i}(t)$ and $\Delta A_{2i}(t)$ are parameter uncertainty; H_i, M_{1i} and M_{2i} are known real constant matrices with appropriate dimensions; $F_i(t)$ is an unknown time-varying matrix, which satisfies the following restriction

$$F_i{}^T(t)F_i(t)\leq I. \tag{3}$$

The switching signal $\sigma(t):[0,\infty]\to N=\{1,2,\ldots,n\}$ is a piecewise continuous function, where n denotes the number of subsystems. $\{(t_0,\sigma(t_0)),(t_1,\sigma(t_1)),\cdots,(t_k,\sigma(t_k)),\cdots\}$ is the switching sequence, where t_0 is the initial switching instant and t_k denotes the kth one. $h(t)$ represents the interval time-varying delay satisfying

$$h_m\leq h(t)\leq h_M, \quad \dot{h}(t)\leq d<1. \tag{4}$$

$f(t,x(t),x(t-h(t)))$ denotes the nonlinear function and satisfies the following restriction

$$\begin{aligned}&f^T(t,x(t),x(t-h(t)))f(t,x(t),x(t-h(t))) \\ &\quad\leq x^T(t)\Gamma^T\Gamma x(t) + x^T(t-h(t)\Lambda^T\Lambda x(t-h(t),\end{aligned} \tag{5}$$

where Γ and Λ are known real constant matrices. For system (1), we consider the state feedback given by

$$u(t)=K_{\sigma(t)}x(t), \tag{6}$$

where $K_i(i\in N)$ is a feedback gain matrix.

For the convenience of calculation, define $\hat{A}_{1i}=A_{1i}+B_iK_i$. System (1) can be written as

$$\begin{aligned}\dot{x}(t) &= (\hat{A}_{1i}+H_iF_i(t)M_{1i})x(t) + Ef(t,x(t),x(t-h(t))) \\ &\quad + (A_{2i}+H_iF_i(t)M_{2i})x(t-h(t)).\end{aligned} \tag{7}$$

Definition 1. *For any $t_2>t_1\geq 0$, let $N_\sigma(t_1,t_2)$ denote the switching number of $\sigma(t)$ on an interval(t_1,t_2). If*

$$N_\sigma(t_1,t_2)\leq N_0+(t_2-t_1)/\tau_a \tag{8}$$

holds for given $N_0\geq 0,\tau_a\geq 0$, then the constant τ_a is called the average dwell time.

Without loss of generality, we set $N_0=0$ in this paper.

Definition 2.([7]) *The equilibrium $x^*=0$ of the closed-loop system (7) with switching signal $\sigma(t)$ and Feedback control $u(t)=K_{\sigma(t)}x(t)$ is said to be exponentially stable if the solution $x(t)$ of the closed-loop system (7) satisfies*

$$\|x(t)\|\leq\omega\sup_{-h\leq\theta\leq 0}\|x(t_0+\theta)\|e^{-\lambda(t-t_0)},\forall t\geq t_0, \tag{9}$$

where $\omega\geq 1$ and $\lambda>0$

Lemma 1.([21]) *Let S_1,S_2 and S_3 be symmetric matrices, $S_1=S_1^T<0$ and $S_3=S_3^T>0$, then for all $S_1+S_2S_3^{-1}S_2^T<0$ if and only if*

$$\begin{bmatrix} S_1 & S_2 \\ S_2^T & -S_3 \end{bmatrix}<0.$$

Lemma 2.([22]) *Let U,V,W and X be real matrices of appropriate dimensions with X satisfying $X^T=X$, then for all $V^TV\leq I, X+UVW+W^TV^TU^T<0$ if and only if there exists a scalar $\varepsilon>0$ such that $X+\varepsilon UU^T+\varepsilon^{-1}W^TW<0$.*

III. Main results

In this section, the criterions of exponential stabilization of the closed-loop system (7) are given.

Theorem 1. *For given positive constants α and $\mu\geq 1$, if there exist positive scalars ε_i, symmetric and positive definite matrices $P_i,Q_{1i},Q_{2i},Q_{3i},R_i$, which satisfy the following matrix inequality for all $i,j\in N,i\neq j$*

$$P_i\leq\mu P_j, \quad Q_{si}\leq\mu Q_{sj}, \quad R_i\leq\mu R_j(s=1,2,3), \tag{10}$$

$$\begin{pmatrix} \tilde{\varphi}_i^{11} & P_iA_{2i} & 0 & 0 & P_iE & P_iH_i & \varepsilon_iM_{1i}^T \\ * & \tilde{\varphi}_i^{22} & 0 & 0 & 0 & 0 & \varepsilon_iM_{2i}^T \\ * & * & \varphi_i^{33} & 0 & 0 & 0 & 0 \\ * & * & * & \varphi_i^{44} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_iI & 0 \\ * & * & * & * & * & * & -\varepsilon_iI \end{pmatrix}<0 \tag{11}$$

where

$$\tilde{\varphi}_{11i} = \hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + Q_{1i} + Q_{2i} + Q_{3i} + \delta^2 R_i + \alpha P_i + \Gamma^T \Gamma,$$
$$\tilde{\varphi}_{22i} = -(1-d)e^{-\alpha h_M} Q_{1i} + \Lambda^T \Lambda,$$
$$\varphi_{33i} = -e^{-\alpha h_m} Q_{2i}, \quad \varphi_{44i} = -e^{-\alpha h_M} Q_{3i}.$$

Then system (7) is exponentially stabilizable under the feedback control (6) for any switching signal with the average dwell time satisfying

$$\tau_a > \tau_a^* = \frac{\ln \mu}{\alpha}. \tag{12}$$

Proof: Inspired by [13], we choose the Lyapunov-Krasovskii functional as follows:

$$V(t) = V_{\sigma(t)}(t) = \sum_{j=1}^{5} V_{j\sigma(t)}(t), \tag{13}$$

where

$$V_{1\sigma(t)}(t) = x^T(t) P_{\sigma(t)} x(t),$$
$$V_{2\sigma(t)}(t) = \int_{t-h(t)}^{t} e^{\alpha(s-t)} x^T(s) Q_{1\sigma(t)} x(s) ds,$$
$$V_{3\sigma(t)}(t) = \int_{t-h_m}^{t} e^{\alpha(s-t)} x^T(s) Q_{2\sigma(t)} x(s) ds,$$
$$V_{4\sigma(t)}(t) = \int_{t-h_M}^{t} e^{\alpha(s-t)} x^T(s) Q_{3\sigma(t)} x(s) ds,$$
$$V_{5\sigma(t)}(t) = \delta \int_{-h_M}^{-h_m} \int_{t+\theta}^{t} e^{\alpha(s-t)} x^T(s) R_{\sigma(t)} x(s) ds d\theta$$

and $\delta = h_M - h_m$.

Derive the trajectory of the system, we can get

$$\begin{aligned}
\dot{V}_{1i} &= 2x^T(t) P_i \dot{x}(t) \\
\dot{V}_{2i} &= -(1-\dot{h}(t)) e^{-\alpha h(t)} x^T(t-h(t)) Q_{1i} x(t-h(t)) \\
&\quad + x^T(t) Q_{1i} x(t) - \alpha \int_{t-h(t)}^{t} e^{\alpha(s-t)} x^T(s) Q_{1i} x(s) ds \\
&\leq -(1-d) e^{-\alpha h_M} x^T(t-h(t)) Q_{1i} x(t-h(t)) \\
&\quad + x^T(t) Q_{1i} x(t) - \alpha \int_{t-h(t)}^{t} e^{\alpha(s-t)} x^T(s) Q_{1i} x(s) ds \\
\dot{V}_{3i} &= -e^{-\alpha h_m} x^T(t-h(t)) Q_{2i} x(t-h(t)) \\
&\quad + x^T(t) Q_{2i} x(t) - \alpha \int_{t-h_m}^{t} e^{\alpha(s-t)} x^T(s) Q_{2i} x(s) ds \\
\dot{V}_{4i} &= -e^{-\alpha h_M} x^T(t-h(t)) Q_{3i} x(t-h(t)) \\
&\quad + x^T(t) Q_{3i} x(t) - \alpha \int_{t-h_M}^{t} e^{\alpha(s-t)} x^T(s) Q_{3i} x(s) ds \\
\dot{V}_{5i} &= (-\alpha) \delta \int_{-h_M}^{-h_m} \int_{t+\theta}^{t} e^{\alpha(s-t)} x^T(s) R_i x(s) ds d\theta \\
&\quad + \delta^2 x^T(t) R_i x(t) - \delta \int_{t-h_M}^{t-h_m} e^{-\alpha h_M} x^T(s) R_i x(s) ds
\end{aligned} \tag{14}$$

Using inequality (5), it is easy to get

$$\begin{aligned}
&x^T(t) \Gamma^T \Gamma x(t) + x^T(t-h(t)) \Lambda^T \Lambda x(t-h(t)) \\
&- f_i^T(t, x(t), x(t-h(t))) f_i(t, x(t), x(t-h(t))) \geq 0,
\end{aligned} \tag{15}$$

Combined (14) with (15), we obtain

$$\begin{aligned}
&\dot{V}(t) + \alpha V(t) \\
&\leq x^T(t)[\hat{A}_{1i}^T P_i + P_i A_{1i} + Q_{1i} + Q_{2i} \\
&\quad + Q_{3i} + (H_i F_i(t) M_{1i})^T P_i + P_i H_i F_i(t) M_{1i} + \delta^2 R_i \\
&\quad + \alpha P_i + \Gamma^T \Gamma] x(t) - e^{-\alpha h_m} x^T(t-h_m) Q_{2i} x(t-h_m) \\
&\quad + x^T(t-h(t))[\Lambda^T \Lambda - (1-d) e^{-\alpha h_M} Q_{1i}] x(t-h(t)) \\
&\quad + x^T(t)[P_i A_{2i} + P_i H_i F_i(t) M_{2i}] x(t-h(t)) \\
&\quad + x^T(t-h(t))[A_{2i}^T P_i + (H_i F_i(t) M_{2i})^T P_i] x(t) \\
&\quad - f^T(t, x(t), x(t-h(t))) f(t, x(t), x(t-h(t))) \\
&\quad + x^T(t) P_i E f(t, x(t), x(t-h(t))) \\
&\quad + f^T(t, x(t), x(t-h(t))) E^T P_i x(t) \\
&\quad - e^{-\alpha h_M} x^T(t-h_M) Q_{3i} x(t-h_M).
\end{aligned}$$

where

$$\Upsilon_i = \begin{pmatrix} \varphi_i^{11} & \varphi_i^{12} & 0 & 0 & P_i E \\ * & \varphi_i^{22} & 0 & 0 & 0 \\ * & * & \varphi_i^{33} & 0 & 0 \\ * & * & * & \varphi_i^{44} & 0 \\ * & * & * & * & -I \end{pmatrix},$$

$$\begin{aligned}
\varphi_i^{11} &= \hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + (H_i F_i(t) M_{1i})^T P_i + \Gamma^T \Gamma \\
&\quad + P_i H_i F_i(t) M_{1i} + Q_{1i} + Q_{2i} + Q_{3i} + \delta^2 R_i + \alpha P_i, \\
\varphi_i^{12} &= P_i A_{2i} + P_i H_i F_i(t) M_{2i}, \\
\varphi_i^{22} &= -(1-d) e^{-\alpha h_M} Q_{1i} + \Lambda^T \Lambda.
\end{aligned}$$

Denote $\Upsilon_i = \Upsilon_{1i} + \Upsilon_{2i}$, here

$$\Upsilon_{1i} = \begin{pmatrix} \hat{\varphi}_i^{11} & P_i A_{2i} & 0 & 0 & P_i E \\ * & \varphi_i^{22} & 0 & 0 & 0 \\ * & * & \varphi_i^{33} & 0 & 0 \\ * & * & * & \varphi_i^{44} & 0 \\ * & * & * & * & -I \end{pmatrix}$$

$$\Upsilon_{2i} = \Omega_i F_i(t) \Sigma_{1i} + {\Sigma_{1i}}^T F_i(t)^T {\Omega_i}^T,$$
$$\hat{\varphi}_i^{11} = \hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + \Gamma^T \Gamma + Q_{1i} + Q_{2i} + Q_{3i} + \delta^2 R_i + \alpha P,$$
$$\Omega_i = \begin{pmatrix} (P_i H_i)^T & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$
$$\Sigma_{1i} = \begin{pmatrix} M_{1i} & M_{2i} & 0 & 0 & 0 \end{pmatrix}.$$

By the Lemma 1, we can get

$$\Upsilon_{1i} + {\varepsilon_i}^{-1} \Omega_i {\Omega_i}^T + \varepsilon_i {\Sigma_{1i}}^T \Sigma_{1i} < 0. \tag{16}$$

So $\Upsilon_i < 0$, which is equivalent with (10) by using Lemma 1. Then the following inequality can be derived,

$$\dot{V}_i(t) + \alpha V_i(t) \leq 0. \tag{17}$$

Hence, it follows that

$$(e^{\alpha t} V_i(t))' = \alpha e^{\alpha t} V_i(t) + e^{\alpha t} \dot{V}_i(t) \leq 0. \tag{18}$$

After calculation, we obtain

$$\dot{V}(t) \leq -\alpha V(t). \tag{19}$$

When $t \in [t_k, t_{k+1})$, both sides of (19) integrate from t_k to t_{k+1} at the same time, we have

$$V(t) = V_{\sigma(t)}(t) \leq e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(t_k), \quad t_k \leq t < t_{k+1}. \tag{20}$$

$k = N_\sigma(t, t_0) \leq (t - t_0)/\tau_a$is recalled, it is easy to obtain that

$$\begin{aligned}
V(t) &\leq e^{-\alpha(t-t_k)} \mu V_{\sigma({t_k}^-)}({t_k}^-) \leq \cdots \\
&\leq e^{-\alpha(t-t_0)} \mu^k V_{\sigma({t_0}^-)}({t_0}^-) \\
&\leq e^{-(\alpha - \ln\mu/\tau_a)(t-t_0)} V_{\sigma(t_0)}(t_0).
\end{aligned} \tag{21}$$

From equation (13), it follows that

$$V(t) \geq a\|x(t)\|^2, V(t_0) \leq b \sup_{-h_M \leq \theta \leq 0} \|x(t_0 + \theta)\|^2, \tag{22}$$

where

$$\begin{aligned}
a &= \min_{i \in N} \lambda_{\max}(P_i), \\
b &= \max_{i \in N} \lambda_{\max}(P_i) + h_M \max_{i \in N} \lambda_{\max}(Q_{1i}) \\
&\quad + h_m \max_{i \in N} \lambda_{\max}(Q_{2i}) + h_M \max_{i \in N} \lambda_{\max}(Q_{3i}) \\
&\quad + \tfrac{\delta^3}{2} \max_{i \in N} \lambda_{\max}(R_i).
\end{aligned}$$

Therefore, we have

$$\|x(t)\| \leq \sqrt{\frac{b}{a}} \sup_{-h_M \leq \theta \leq 0} \|x(t_0 + \theta)\| \, e^{-\frac{1}{2}(\alpha - \ln\mu/\tau_a)(t-t_0)}. \tag{23}$$

According to Definition 2, the closed-loop system (7) is exponentially stabilizable. This completes the proof.

Remark 1. *Refs [11] and [14] have investigated stability and stabilization of switched linear systems with time delay, and the stability criteria for switched linear systems are presented. However, stability and stabilization for switched nonlinear systems did not consider. In this paper, we consider* *switched nonlinear systems and some exponential stabilization criteria are obtained. [11] and [14] are seen as a special case of this paper.*

Theorem 2. *For given positive constants α and $\mu \geq 1$, if there exist positive scalars ε_i, symmetric and positive definite matrices $X_i, T_{1i}, T_{2i}, T_{3i}, L_i$ and Y_i, which satisfy the following matrix inequality*

$$X_j \leq \mu X_i, \quad T_{sj} \leq \mu T_{si}, \quad L_j \leq \mu L_i, (s = 1, 2, 3) \quad (24)$$

$$\begin{pmatrix} \Sigma_i & \Pi_i \\ * & \tilde{\Sigma}_i \end{pmatrix} < 0, \quad (25)$$

where

$$\Sigma_i = \begin{pmatrix} \theta_i^{11} & \theta_i^{12} & 0 & 0 & E & 0 \\ * & \theta_i^{22} & 0 & 0 & 0 & X_i\Lambda^T \\ * & * & \theta_i^{33} & 0 & 0 & 0 \\ * & * & * & \theta_i^{44} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{pmatrix},$$

$$\Pi_i = \begin{pmatrix} X_i\Gamma^T & H_i & \theta_i^{19} & \delta X_i & X_i & X_i & X_i \\ 0 & 0 & \theta_i^{29} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\tilde{\Sigma}_i = diag\{-I \ -\varepsilon_i I \ -\varepsilon_i I \ -L_i \ -T_{3i} \ -T_{2i} \ -T_{1i}\}$,
$\theta_i^{11} = A_{1i}X_i + B_iY_i + (A_{1i}X_i + B_iY_i)^T + \alpha X_i$,
$\theta_i^{12} = A_{2i}X_i, \theta_i^{22} = (1-d)e^{-\alpha h_M}(T_{1i} - 2X_i)$,
$\theta_i^{33} = e^{-\alpha h_m}(T_{2i} - 2X_i), \theta_i^{19} = \varepsilon_i X_i M_{1i}^T$,
$\theta_i^{44} = e^{-\alpha h_M}(T_{3i} - 2X_i), \theta_i^{29} = \varepsilon_i X_i M_{2i}^T$.
Then system (7) is exponentially stabilizable with the average dwell time satisfying

$$\tau_a > \tau_a^* = \frac{\ln \mu}{\alpha}. \quad (26)$$

And the controller can be designed by the following formula

$$K_i = Y_i {X_i}^{-1}, i \in N. \quad (27)$$

Proof: Because of $T_{si} > 0, L_i > 0 (s = 1, 2, 3)$, we have

$$(T_{si} - X_i)^T {T_{si}}^{-1}(T_{si} - X_i) \geq 0,$$
$$(L_i - X_i)^T {L_i}^{-1}(L_i - X_i) \geq 0.$$

Therefore, it follows that

$$\begin{aligned} T_{si} - 2X_i &\geq -X_i {T_{si}}^{-1} X_i, \\ L_i - 2X_i &\geq -X_i {L_i}^{-1} X_i. \end{aligned} \quad (28)$$

The left and right ends of inequality (25) are multiplied by the diagonal matrix $\{{X_i}^{-1}, {X_i}^{-1}, {X_i}^{-1}, {X_i}^{-1}, I, I, I, I, I, I, I, I\}$ at the same time. By using (28) we get

$$\begin{pmatrix} \hat{\Sigma}_i & \hat{\Pi}_i \\ * & \tilde{\Sigma}_i \end{pmatrix} < 0, \quad (29)$$

where

$$\Sigma_i = \begin{pmatrix} \hat{\theta}_i^{11} & \hat{\theta}_i^{12} & 0 & 0 & X_i^{-1}E & 0 \\ * & \hat{\theta}_i^{22} & 0 & 0 & 0 & \Lambda^T \\ * & * & \hat{\theta}_i^{33} & 0 & 0 & 0 \\ * & * & * & \hat{\theta}_i^{44} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{pmatrix},$$

$$\Pi_i = \begin{pmatrix} \Gamma^T & X_i^{-1}H_i & \hat{\theta}_i^{19} & \delta I & I & I & I \\ 0 & 0 & \hat{\theta}_i^{29} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\hat{\theta}_i^{11} = X_i^{-1}A_{1i} + X_i^{-1}B_iY_iX_i^{-1}$
$\quad +(X_i^{-1}A_{1i} + X_i^{-1}B_iY_iX_i^{-1})^T + \alpha X_i^{-1}$,
$\hat{\theta}_i^{12} = X_i^{-1}A_{2i}, \hat{\theta}_i^{22} = -(1-d)e^{-\alpha h_M}{T_{1i}}^{-1}$,
$\hat{\theta}_i^{33} = -e^{-\alpha h_m}{T_{2i}}^{-1}, \hat{\theta}_i^{19} = \varepsilon_i M_{1i}^T$,
$\hat{\theta}_i^{44} = -e^{-\alpha h_M}{T_{3i}}^{-1}, \hat{\theta}_i^{29} = \varepsilon_i M_{2i}^T$.
Denote

$$Y_i = K_iX_i, {X_i}^{-1} = P_i, {T_{si}}^{-1} = Q_{si}, \ {L_i}^{-1} = R_i. \quad (30)$$

Recalling Lemma 1 and (29), it is easy to obtain that system (7) is exponentially stabilizable by using Theorem 1. The proof is completed.

Remark 2. *In [13], only the exponential stabilization of the switched system is studied, and the feedback controller design algorithm for the system is also given in this paper, which greatly extend the scope of the application.*

Remark 3. *In [11], the switched system based on the common Lyapunov-Krasovskii function for all subsystems. However, we use multi-Lyapunov-Krasovskii functions. From the view of switched system and Lyapunov-Krasovskii function selection, the conservativeness of our results is lower. Compared with [21] and [22], we consider a large range of time-delay and nonlinearity, which have a greater advantage when dealing with complex systems.*

IV. Numerical examples

In this section, a numerical example and a practical example are given to illustrate the effectiveness and applicability of the proposed approach.

Example 1. Consider system (1) composed of two subsystems with the following parameters:

$$A_{11} = \begin{bmatrix} -0.8 & 0.1 \\ 0 & -0.3 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},$$
$$M_{22} = \begin{bmatrix} -0.8 & 0.1 \\ 0.1 & -0.6 \end{bmatrix}, E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$A_{12} = \begin{bmatrix} -0.7 & 0.1 \\ 0 & -0.4 \end{bmatrix}, \Lambda = \begin{bmatrix} -0.8 & 0.1 \\ 0.2 & -0.6 \end{bmatrix},$$
$$M_{11} = \begin{bmatrix} -0.5 & 0.2 \\ 0.1 & -0.4 \end{bmatrix}, H_1 = \begin{bmatrix} 0.3 & 0.6 \\ 0.3 & 0.5 \end{bmatrix},$$
$$M_{21} = \begin{bmatrix} -0.6 & 0.1 \\ 0.1 & -0.5 \end{bmatrix}, \Gamma = \begin{bmatrix} -0.6 & 0.2 \\ 0.1 & -0.4 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} -0.5 & 0 \\ 0.1 & -0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$
$$M_{12} = \begin{bmatrix} -0.4 & 0.2 \\ 0.1 & -0.3 \end{bmatrix}, H_2 = \begin{bmatrix} 0.2 & 0.6 \\ 0.25 & 0.3 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -0.6 & 0 \\ 0.1 & -0.4 \end{bmatrix},$$

$\alpha = 0.6, \mu = 1.8, h_M = 0.8, h_m = 0.2, \delta = 0.6, d = 0.4, h(t) = 0.2 + 0.4sin(t), \varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.3$.

We get that $\tau_a > 0.98$ from (26). Let

$$f(t, x(t), x(t-h(t))) = \begin{pmatrix} 0.1sin(x_1(t)) - 0.2^t \\ 0.2sin(x_2(t - h(t))) - 0.3^{t-0.1} \end{pmatrix}.$$

By solving (24) and (25) in Theorem 2, we can get

$$X_1=\begin{bmatrix}0.5390 & -0.2308\\ -0.2308 & 0.1659\end{bmatrix}, L_1=\begin{bmatrix}3.5204 & 0.2255\\ 0.2255 & 2.2966\end{bmatrix}$$
$$X_2=\begin{bmatrix}0.1717 & -0.0372\\ -0.0372 & 0.0737\end{bmatrix}, L_2=\begin{bmatrix}3.3439 & 0.7642\\ 0.7642 & 2.4166\end{bmatrix}$$
$$T_{11}=\begin{bmatrix}1.2081 & 0.0148\\ 0.0148 & 1.4106\end{bmatrix}, T_{12}=\begin{bmatrix}2.0501 & 0.4639\\ 0.4639 & 1.5501\end{bmatrix}$$
$$T_{22}=\begin{bmatrix}0.6056 & 0.0168\\ 0.0168 & 0.4150\end{bmatrix}, T_{32}=\begin{bmatrix}0.9027 & 0.1971\\ 0.1971 & 0.7718\end{bmatrix}$$
$$Y_1=\begin{bmatrix}0.3046 & 0.1897\end{bmatrix}, Y_2=\begin{bmatrix}0.1934 & 0.6822\end{bmatrix}$$
$$T_{21}=\begin{bmatrix}2.0885 & -0.5189\\ -0.5189 & 0.6469\end{bmatrix},$$
$$T_{31}=\begin{bmatrix}2.5240 & -0.5489\\ -0.5489 & 0.9485\end{bmatrix},$$

then the controller gains constructed by (27) are

$$K_1=\begin{bmatrix}2.6070 & 4.7691\end{bmatrix}, K_2=\begin{bmatrix}3.5143 & 1.0254\end{bmatrix}.$$

According to Theorem 2, we can get that the system (1) is exponentially stabilizable for any switching signal under the feedback control. State response diagrams are shown in Fig.1, where the initial state is $x(0)=(-1,1)^T$.

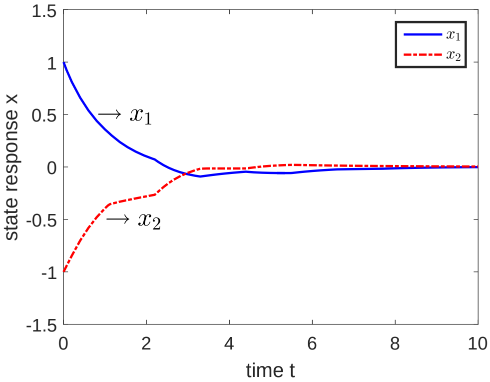

Fig. 1: State response of the closed-loop system.

Example 2. We will illustrate the effectiveness of our approach through river pollution control issues.

For ease of description, the concentrations per unit volume of biochemical oxygen demand and dissolved oxygen in a reach of a polluted river are denoted as $z(t)$ and $q(t)$, respectively. Let z^* and q^* denote the desired steady values of $z(t)$ and $q(t)$ corresponding to some measure of water quality standards, respectively. Define

$$x_1(t)=z(t)-z^*,\ x_2(t)=q(t)-q^*,\ x(t)=[x_1^T(t)\ x_2^T(t)]^T$$

Then, the dynamic equation for $x(t)$ can be written as following:

$$\begin{aligned}\dot{x}(t)=&(A_{11}+\Delta A_{11}(t))x(t)+B_1u(t)+\omega(t)\\&+(A_{21}+\Delta A_{21}(t))x(t-h(t)),\end{aligned}\tag{31}$$

where

$$A_{11}=\begin{bmatrix}-k_{10}-\eta_1-\eta_2 & 0\\ -k_{30} & -k_{20}-\eta_1-\eta_2\end{bmatrix},$$
$$A_{21}=\begin{bmatrix}\eta_2 & 0\\ 0 & \eta_2\end{bmatrix},\ B_1=\begin{bmatrix}\eta_1\\ 1\end{bmatrix},$$
$$\Delta A_{11}(t)=H_1F_1(t)M_{11},\ \Delta A_{21}(t)=H_1F_1(t)M_{21}.$$

$u(t)=[u_1^T(t)\ u_2^T(t)]^T$ is the control variable of river pollution, $k_{i0}(i=1,2,3),\eta_1$ and η_2 are known constants, $\Delta A_1(t)$ and $\Delta A_2(t)$ are uncertainty, and $\omega(t)$ is the disturbance input of the system. The physical meaning of these parameters can be found in [23-24].

In accordance with the actual situation, we assumed that the system actuators are subject to good performance or failure in this paper. Therefore, the model is divided into two subsystems for discussion. Then, the system (31) can be described as the following switched system:

$$\dot{x}(t)=\begin{cases}\bar{A}_{11}(t)x(t)+\bar{A}_{21}(t)x(t-h(t))+B_1u(t)\\ \quad +Ef(t,x(t),x(t-h(t))),\ no\ failures\ occur\\ \bar{A}_{12}(t)x(t)+\bar{A}_{22}(t)x(t-h(t))+B_2u(t)\\ \quad +Ef(t,x(t),x(t-h(t))),\ failures\ occur\end{cases}\tag{32}$$

For simulation of our purposes, we choose $k_{10}=1.6, k_{20}=1, k_{30}=1.6, \eta_1=0.3, \eta_2=0.7$, and get that

$$A_{11}=\begin{bmatrix}-2.6 & 0\\ -1.6 & -2\end{bmatrix}, A_{21}=\begin{bmatrix}0.7 & 0\\ 0 & 0.7\end{bmatrix}, B_1=\begin{bmatrix}0.3\\ 1\end{bmatrix}.$$

Let $h(t)=0.3sin(t)$,
$\omega(t)=f(t,x(t),x(t-h(t)))=\begin{pmatrix}0.1sin(x_1(t))\\ 0.2sin(x_2(t-h(t)))\end{pmatrix}$.
Then, we will use the parameters in Theorem 1 to design a set of switching sequences to stabilize the above system (32). At the same time, we choose $\alpha=0.3, \mu=1.7, H_M=0.8, h_m=0.2, \delta=0.6, d=0.2, \varepsilon_1=0.2, \varepsilon_2=0.3, h(t)=0.3sin(t)$. By solving (24) and (25), we can get

$$X_1=\begin{bmatrix}1.1804 & -0.4882\\ -0.4882 & 0.4255\end{bmatrix},$$
$$X_2=\begin{bmatrix}2.4705 & -0.1175\\ -0.1175 & 1.4638\end{bmatrix},$$
$$T_{11}=\begin{bmatrix}1.4890 & -0.2052\\ -0.2052 & 1.2590\end{bmatrix},$$
$$T_{12}=\begin{bmatrix}2.4705 & -0.1175\\ -0.1175 & 1.4638\end{bmatrix},$$
$$T_{21}=\begin{bmatrix}3.5615 & -0.6581\\ -0.6581 & 1.2357\end{bmatrix},$$
$$T_{22}=\begin{bmatrix}0.8323 & 0.0402\\ 0.0402 & 0.6673\end{bmatrix},$$
$$T_{31}=\begin{bmatrix}3.8260 & -0.6562\\ -0.6562 & 1.4274\end{bmatrix},$$
$$T_{32}=\begin{bmatrix}1.0061 & 0.1464\\ 0.1464 & 0.8410\end{bmatrix},$$
$$L_1=\begin{bmatrix}2.8345 & 0.3898\\ 0.3898 & 1.6749\end{bmatrix},$$
$$L_2=\begin{bmatrix}2.5942 & 0.6388\\ 0.6388 & 1.7737\end{bmatrix},$$
$$Y_1=\begin{bmatrix}0.8913 & 0.7621\end{bmatrix},$$
$$Y_2=\begin{bmatrix}0.4565 & 0.0185\end{bmatrix}.$$

Then the controller gains constructed by (27) are

$$K_1=\begin{bmatrix}2.8465 & 5.0571\end{bmatrix}, K_2=\begin{bmatrix}1.6449 & 0.0079\end{bmatrix}.$$

The state responses of the subsystem 1 and 2 of the system (32) with the initial condition $x(0)=(-1,1)^T$ are shown in Figs. 2 and 3, respectively. According to Theorem 2, we can get that the system (32) is exponentially stabilizable for any switched signal under the feedback control. At the same time, the state response is shown in Fig.4, where the initial state is $x(0) = (-1, 1)^T$. Therefore, the effectiveness of our approach is verified by its application in the control of river pollution process.

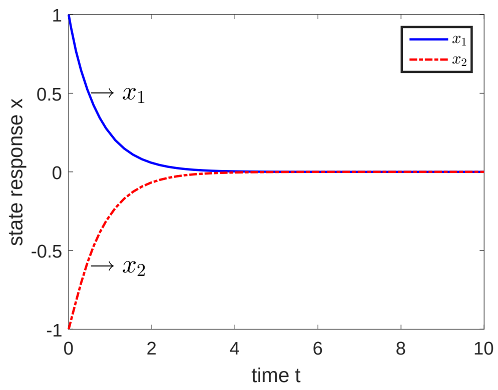

Fig. 2: State response of the subsystem 1

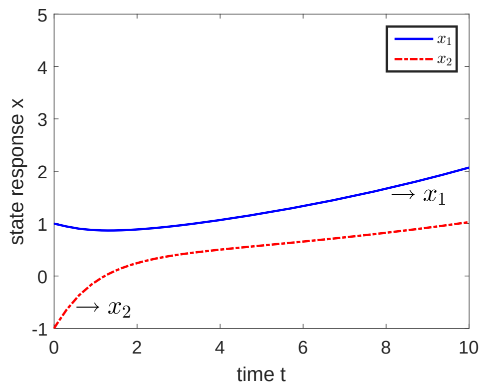

Fig. 3: State response of the subsystem 2

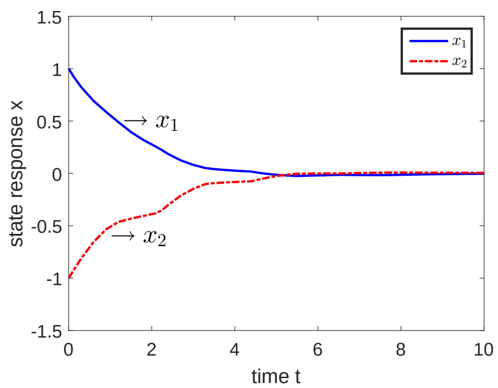

Fig. 4: State response of the system (32)

Remark 4. *[23] have investigated the pollution problem of a single reach river modelled by the dynamics of water quality subject to uncertainty in system parameters, and an adaptive controller can guarantee the closed loop system to converge, globally and exponentially. However, the performance of the actuator for river pollution model may be deviated in practical applications. This paper has solved this situation by changing the original system model to a switched system for processing.*

V. Conclusions

In this paper, the problem of exponential stabilization for a class of switched nonlinear system with time-varying delay has been studied. Based on a novel Lyapunov-Krasovskii functional, some sufficient conditions for the exponential stability of switched nonlinear system are obtained by the average dwell time approach. Moreover, the controllers of the switched system are designed through a special matrix transformation method. Finally, a numerical example and a practical example of river pollution control are provided to show the validity and potential of the developed results.

Through the research of this paper, we learned that different Lyapunov functionals and delay may lead to different conservatism. It deserves further study to choose an improved piecewise Lyapunov functional so as to reduce the conservativeness. In order to better study switched nonlinear system with time-varying delay in multiple aspects, we will further optimize Lyapunov functionals and the delay for better performance. In this paper, we did not consider stochastic term. It is well known that stochastic term is inevitable in some practical control systems, which is often the main cause for instability or undesirable system performance of a control system. Specifically, the stabilization of stochastic switched nonlinear systems will be taken as a main direction of our future research.

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