Adjacent Vertex Distinguishing Proper Edge Colorings of Bicyclic Graphs*

Xiang'en Chen† Shunyi Liu‡

Abstract—**An adjacent vertex distinguishing proper edge coloring of a graph G is a proper edge coloring of G such that no pair of adjacent vertices meets the same set of colors. Let $\chi'_a(G)$ be the minimum number of colors required to give G an adjacent vertex distinguishing proper edge coloring. In this paper, we show that $\chi'_a(G) \leq \Delta(G)+1$ for bicyclic graphs G, where $\Delta(G)$ is the maximum degree of G.**

Keywords: Adjacent vertex distinguishing proper edge coloring; Adjacent vertex distinguishing proper edge chromatic number; Bicyclic graph

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A proper edge coloring of G is a mapping $\varphi : E(G) \to \{1, 2, \dots, k\}$ such that no two adjacent edges meet the same color. Denote by $C_\varphi(v) = \{\varphi(uv)|uv \in E(G)\}$ the color set of the vertex v. We say that a proper edge coloring φ of G is *adjacent vertex distinguishing*, or an *avd*-coloring, if $C_\varphi(u) \neq C_\varphi(v)$ for any pair of adjacent vertices u and v. It is obvious that an avd-coloring exists provided that G contains no isolated edge. A k-avd-coloring of G is an avd-coloring of G using at most k colors. Let $\chi'_a(G)$ be the minimum number of colors in an avd-coloring of G. We use $d_G(u)$ to denote the degree of the vertex u of G, and $\Delta(G)$ denotes the maximum degree of G. Clearly, $\chi'_a(G) \geq \Delta(G)$, and $\chi'_a(G) \geq \Delta(G)+1$ if there exist two adjacent vertices u and v with $d_G(u) = d_G(v) = \Delta(G)$.

The adjacent vertex distinguishing proper edge coloring was first introduced by Zhang et al., and the following conjecture was proposed [17].

Conjecture 1. *(AVDPEC Conjecture) If G is a simple connected graph on at least 3 vertices and $G \neq C_5$ (a 5-cycle), then $\Delta(G) \leq \chi'_a(G) \leq \Delta(G)+2$.*

In [2], Balister et al. proved that Conjecture 1 holds for bipartite graphs and for graphs with $\Delta(G) \leq 3$. Edwards et al. [6] showed that $\chi'_a(G) \leq \Delta(G)+1$ if G is a planar bipartite graph with $\Delta(G) \geq 12$. Horňák et al. [12] showed that $\chi'_a(G) \leq \Delta(G)+2$ for all planar graphs G with $\Delta(G) \geq 12$. Akbari et al. [1] obtained $\chi'_a(G) \leq 3\Delta(G)$ for all graphs G without isolated edges. This bound was recently improved to $3\Delta(G)-1$ by Zhu et al. [19]. The best general result is due to Hatami [10] who bounded (by a probabilistic method) $\chi'_a(G)$ from above by $\Delta(G)+300$ provided that $\Delta(G) > 10^{20}$. For more on the avd-colorings of graphs, see [3–5, 7–9, 11, 13–16, 18].

A bicyclic graph is a connected graph in which the number of edges equals the number of vertices plus one. In this paper, we investigate the avd-coloring of bicyclic graphs and show that $\chi'_a(G) \leq \Delta(G)+1$ for bicyclic graphs G. This implies that Conjecture 1 holds for all bicyclic graphs.

The rest of the paper is organized as follows. In Section 2, we obtain $\chi'_a(G)$ for bicyclic graphs G without pendant vertex. This plays an important role in Section 3 where we obtain the exact value of $\chi'_a(G)$ for bicyclic graphs G with at least one pendant vertex. In Section 4, we give the conclusion of this paper.

2 Bicyclic graphs without pendant vertex

In this section, we obtain the exact value of $\chi'_a(G)$ for bicyclic graphs G without pendant vertex.

It is easy to see that if G is a bicyclic graph without pendant vertex, then G must be some H_i for $1 \leq i \leq 5$ (see Figure 1).

The following lemma is obvious.

Lemma 1. *Let P be a path of G whose internal vertices are all of degree 2 in G. If φ is a 3-avd-coloring of G, then the colors of any three consecutive edges of P are pairwise distinct.*

In what follows, we say that two vertices u and v are distinguished from each other in a given coloring if the set of colors incident to u is not equal to the set of colors

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†College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, P R China. The corresponding author. Email: chenxe@nwnu.edu.cn; xiangenchen@163.com

‡School of Science, Chang'an University, Xi'an 710064, P R China. Email: liu@chd.edu.cn

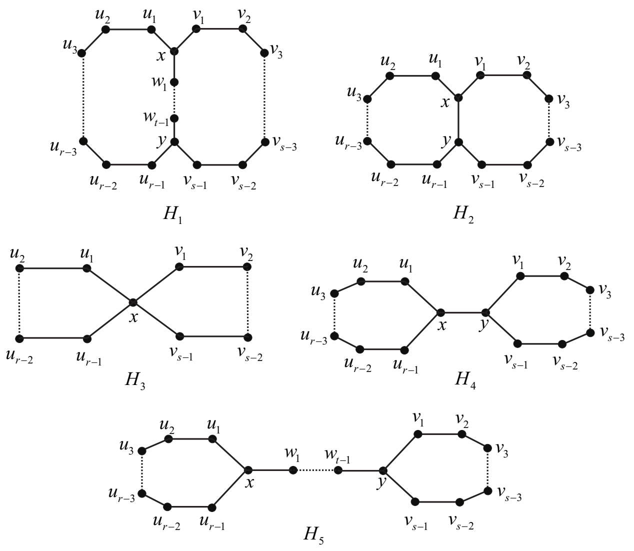

Figure 1: Bicyclic graphs without pendant vertex.

incident to v. We also say that the coloring distinguishes u and v in this case, or that u and v are *distinguishable.*

From Lemma 1 we can immediately obtain the following result.

Lemma 2. *If a graph G has a cycle C of length r so that there exists exactly one vertex of C whose degree is greater than 2 in G, where $r \equiv 1 \pmod 3$, then $\chi'_a(G) \geq 4$.*

Let $P = u_1u_2 \cdots u_r$ be a path of G. We say that "P is cyclically colored by colors 1, 2 and 3" if the colors assigned to u_1u_2, u_2u_3 and u_3u_4 are 1, 2 and 3 respectively, and u_4u_5, u_5u_6 and u_6u_7 are colored by 1, 2 and 3 respectively, and the remaining edges are colored in a similar manner until the last one $u_{r-1}u_r$ is colored. We may similarly give a definition of "s distinct edges e_1, e_2, $\cdots$, e_s are cyclically colored by colors 1, 2 and 3". We use $l(P)$ to denote the length of P.

Lemma 3. *Let C be a cycle of G of length r, where $r \equiv 1 \pmod 3$. If C has exactly two vertices of degree 3 in G such that these two 3-vertices are not adjacent in G and their respective adjacent vertices not belonging to C are also not adjacent in G, and the rest of $r-2$ vertices of C are all of degree 2 in G, then the edges incident to the vertices of C can be properly colored using 3 colors such that any two consecutive vertices of C are distinguished from each other.*

Proof. Suppose that $C = x_1x_2 \cdots x_rx_1$, where $d_G(x_1) = d_G(x_j) = 3$, $3 \leq j \leq r-1$; and $d_G(x_i) = 2$, $i \neq 1, j$. Let e_1 and e_j be the edges incident to x_1 and x_j, respectively, where e_1 and e_j are not the edges of C. Let P_1 and P_2 be the two paths connecting x_1 and x_j in C, respectively. We cyclically color e_1, x_1x_2, x_2x_3, $\cdots$, $x_{j-1}x_j$, e_j, x_jx_{j+1}, $x_{j+1}x_{j+2}$, $\cdots$, $x_{r-1}x_r$, x_rx_1 by colors 1, 2 and 3. It is easy to verify that the resulting coloring satisfies the conditions of the lemma. $\square$

We call the coloring method used in the proof of Lemma 3 the *ξ-coloring* of $C \cup \{e_1, e_j\}$. Let φ be a ξ-coloring of $C \cup \{e_1, e_j\}$. It is obvious that φ is a partial avd-coloring of G. Clearly, we can obtain a ξ-coloring such that the color of e_1 is 2 or 3 by permuting the order of colors.

Proposition 1.

$$\chi'_a(H_1) = \begin{cases} 4, & \textit{if there are exactly two numbers of } r,\ s \textit{ and } t \textit{ both congruent to 1 modulo 3;} \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. Set $P_1 = xu_1u_2 \cdots u_{r-1}y$, $P_2 = xv_1v_2 \cdots v_{s-1}y$, and $P_3 = xw_1w_2 \cdots w_{t-1}y$. Clearly $\chi'_a(H_1) \geq 3$. By the symmetry of P_1, P_2 and P_3, we only describe 10 cases in which we can find a suitable corresponding edge coloring (see Table 1).

Table 1. Avd-coloring of H_1

Conditions	P_1	P_2	P_3
$r \equiv s \equiv t \equiv 0 \pmod 3$	$(123)^{\frac{r}{3}}$	$(231)^{\frac{s}{3}}$	$(312)^{\frac{t}{3}}$
$r \equiv 1, s \equiv t \equiv 0 \pmod 3$	$(123)^{\frac{r-1}{3}}1$	$(213)^{\frac{s}{3}}$	$(312)^{\frac{t}{3}}$
$r \equiv 2, s \equiv t \equiv 0 \pmod 3$	$(123)^{\frac{r-2}{3}}12$	$(213)^{\frac{s}{3}}$	$(321)^{\frac{t}{3}}$
$r \equiv s \equiv 1, t \equiv 0 \pmod 3$	$(123)^{\frac{r-1}{3}}1$	$(231)^{\frac{s-1}{3}}2$	$(312)^{\frac{t-3}{3}}314$
$r \equiv s \equiv 2, t \equiv 0 \pmod 3$	$(123)^{\frac{r-2}{3}}12$	$(231)^{\frac{s-2}{3}}23$	$(321)^{\frac{t}{3}}$
$r \equiv 2, s \equiv 1, t \equiv 0 \pmod 3$	$(312)^{\frac{r-2}{3}}31$	$(231)^{\frac{s-1}{3}}2$	$(123)^{\frac{t}{3}}$
$r \equiv s \equiv t \equiv 2 \pmod 3$	$(231)^{\frac{r-2}{3}}23$	$(312)^{\frac{s-2}{3}}31$	$(123)^{\frac{t-2}{3}}12$
$r \equiv s \equiv 2, t \equiv 1 \pmod 3$	$(231)^{\frac{r-2}{3}}23$	$(321)^{\frac{s-2}{3}}32$	$(123)^{\frac{t-1}{3}}1$
$r \equiv 2, s \equiv t \equiv 1 \pmod 3$	$(123)^{\frac{r-2}{3}}14$	$(231)^{\frac{s-1}{3}}2$	$(312)^{\frac{t-1}{3}}3$
$r \equiv s \equiv t \equiv 1 \pmod 3$	$(231)^{\frac{r-1}{3}}2$	$(312)^{\frac{s-1}{3}}3$	$(123)^{\frac{t-1}{3}}1$

It remains to show that there exists no 3-avd-coloring when $r \equiv s \equiv 1, t \equiv 0 \pmod 3$ or $r \equiv 2, s \equiv t \equiv 1 \pmod 3$. We consider the latter case only. Suppose that φ is a 3-avd-coloring of H_1 when $r \equiv 2 \pmod 3$, $s \equiv t \equiv 1 \pmod 3$. Clearly the colors of xu_1, xv_1 and xw_1 are pairwise distinct. Without loss of generality, we assume that $\varphi(xu_1) = 1$, $\varphi(xv_1) = 2$ and $\varphi(xw_1) = 3$. It follows from Lemma 1 and $s \equiv t \equiv 1 \pmod 3$ that $\varphi(v_{s-1}y) = 2$ and $\varphi(w_{t-1}y) = 3$. Clearly, the coloring of P_1 must be $(123)^{\frac{r-2}{3}}12$ or $(132)^{\frac{r-2}{3}}13$, which results in that $\varphi(u_{r-1}y)=\varphi(v_{s-1}y)$ or $\varphi(u_{r-1}y)=\varphi(w_{t-1}y)$, a contradiction. $\square$

Proposition 2. $\chi'_a(H_i) = 4, i = 2, 3, 4.$

Proof. Since H_i has a 4-vertex or two adjacent 3-vertices, we have $\chi'_a(H_i) \geq 4$, $i = 2, 3, 4$. It remains to prove that H_i has a 4-avd-coloring, $i = 2, 3, 4$. For H_2, we assign colors 4, 2 and 3 to xy, $u_{r-1}y$ and $v_{s-1}y$, respectively. Then we cyclically color xu_1, u_1u_2, $\cdots$, $u_{r-2}u_{r-1}$ by colors 1, 3 and 4; and we cyclically color xv_1, v_1v_2, $\cdots$, $v_{s-2}v_{s-1}$ by colors 2, 4 and 1.

For H_3, we assign colors 1, 2, 3 and 4 to xu_1, xu_{r-1}, xv_1 and xv_{s-1}, respectively. The path $u_1u_2\cdots u_{r-1}$ is cyclically colored by colors 3, 4 and 1; and the path $v_1v_2\cdots v_{s-1}$ is cyclically colored by colors 1, 2 and 3.

For H_4, we assign colors 2, 4 and 4 to xy, $u_{r-1}y$ and $v_{s-1}y$, respectively. Then the path $xu_1u_2\cdots u_{r-1}$ is cyclically colored by colors 1, 2 and 3; and path $yv_1v_2\cdots v_{s-1}$ is cyclically colored by colors 3, 2 and 1.

It is easy to see that the resulting coloring is a 4-avd-coloring in each case. □

Proposition 3.

$$\chi'_a(H_5) = \begin{cases} 4, & \text{if } r \equiv 1 \pmod 3 \text{ or } s \equiv 1 \pmod 3; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. When $r \equiv 1 \pmod 3$ or $s \equiv 1 \pmod 3$, it follows from Lemma 2 that $\chi'_a(H_5) \geq 4$. So it is sufficient to give H_5 a 4-avd-coloring. We cyclically color xw_1, w_1w_2, $\cdots$, $w_{t-2}w_{t-1}$ by colors 1, 2 and 3; and we cyclically color $w_{t-1}y$, yv_1, v_1v_2, $\cdots$, $v_{s-2}v_{s-1}$ by colors 4, 3 and 2. We assign colors 1 and 4 to $v_{s-1}y$ and $u_{r-1}x$, respectively. Finally xu_1, u_1u_2, $\cdots$, $u_{r-2}u_{r-1}$ are cyclically colored by colors 3, 2 and 1. Clearly, the resulting coloring is a 4-avd-coloring of H_5.

When $r \not\equiv 1 \pmod 3$ and $s \not\equiv 1 \pmod 3$, it is sufficient to give H_5 a 3-avd-coloring. We cyclically color xw_1, w_1w_2, $\cdots$, $w_{t-1}y$ by colors 1, 2 and 3. Assume that the color of $w_{t-1}y$ is a. Set $\{1,2,3\}\backslash\{a\}=\{b,c\}$. We cyclically color yv_1, v_1v_2, $\cdots$, $v_{s-1}y$ by colors "c, a and b" or "b, c and a" with respect to $s \equiv 0$ or $2 \pmod 3$. The edges of xu_1, u_1u_2, $\cdots$, $u_{r-1}x$ can be colored in a similar manner. □

3 Bicyclic graphs with pendant vertices

In this section, we investigate the avd-coloring of bicyclic graphs with at least one pendant vertex.

Let G be a bicyclic graph, and let G_1 be the graph obtained from G by deleting all the pendant vertices of G (if G contains no pendant vertex, then $G_1 = G$). Similarly, G_2 is the graph obtained from G_1 by deleting all the pendant vertices of G_1 (if G_1 contains no pendant vertex, then $G_2 = G_1$). This process continues, and we finally obtain a graph H such that H has no pendant vertex. Denote H by $H(G)$.

Fact. *If G is a bicyclic graph, then $H(G) \in \{H_1, H_2, H_3, H_4, H_5\}$ (see Figure 1).*

We will classify all bicyclic graphs with at least one pendant vertex into three classes: *α-type*, *β-type* and *γ-type*. Let G be a bicyclic graph containing a pendant vertex. We use G_Δ to denote the subgraph of G induced by all the vertices of maximum degree of G.

We call G an *α-type* graph, if all the following conditions hold:

(1) $\Delta(G) = 3$ and G_Δ is an empty graph (i.e. a graph without edges).

(2) G has a cycle C of length r ($r \equiv 1 \pmod 3$) such that there exists exactly one vertex of C whose degree is 3 in G.

We call G a *β-type* graph, if all the following conditions hold:

(1) $\Delta(G) = 3$ and G_Δ is an empty graph.

(2) $H(G)$ is H_1, and there exist exactly two numbers of r, s and t which are both congruent to 1 modulo 3, and the other is congruent to 2 modulo 3.

(3) There exists an internal vertex z_0 of one (x, y)-path P in H_1 whose length is congruent to 2 modulo 3, and $d_P(z_0, x)$ and $d_P(z_0, y)$ are both congruent to 1 modulo 3, where $d_P(z_0, x)$ denotes the distance between z_0 and x in P.

(4) $d_G(v)$=$d_{H_1}(v)$ for $v \in V(H_1)\backslash z_0$.

If G is neither α-type nor β-type, then we call G a *γ-type* graph.

Theorem 1. *Let G be a bicyclic graph on n vertice. If G is α-type or β-type, then $\chi'_a(G) = 4$; if G is γ-type, then*

$$\chi'_a(G) = \begin{cases} \Delta(G), & \text{if } G_\Delta \text{ is an empty graph;} \\ \Delta(G)+1, & \text{otherwise.} \end{cases}$$

Proof. We divide the proof into three cases.

Case 1. G is an α-type graph.

It follows from Lemma 2 that $\chi'_a(G) \geq 4$. We prove by induction on the number of vertices of G that there is a 4-avd-coloring of G.

When $n = 10$, G must be the graph illustrated in Figure 2, and a 4-avd-coloring of G is also presented.

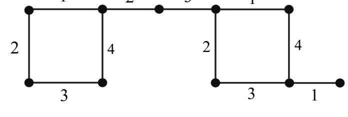

Figure 2: Basis step in Case 1.

Suppose that the theorem is true for α-type graphs with fewer than n vertices, and let G be an α-type graph with

$n \geq 11$ vertices. In fact, $H := H(G) = H_5$. Let C be the cycle of H_5 of length r such that there exists exactly one vertex of C whose degree is 3 in G and $r \equiv 1 \pmod 3$.

When G has a pendant vertex v such that the neighbor of v is not in H. Let v_0 be a pendant vertex of G such that $d(v_0, H)$ is maximum, where $d(v, H) = \min\{d_G(v, u)|u \in V(H)\}$. Clearly, $d(v_0, H) \geq 2$. Let w be the neighbor of v_0, and u the exactly one neighbor of w in G which is not a pendant vertex. Set $G' = G - v_0$. It is easy to see that G' is an α-type graph with $n-1$ vertices. By induction hypothesis, G' has a 4-avd-coloring φ. There are at least 2 colors missing from the edges incident to w (since $\Delta(G) = 3$ and wv_0 has not been colored). Hence we can assign one missing color to wv_0 such that w and u are distinguishable.

When the neighbor of each pendant vertex of G is in H. Let v be any pendant vertex of G, and w the neighbor of v. Set $G' = G - v$. It is obvious that G' is an α-type graph with $n-1$ vertices. By induction hypothesis, G' has a 4-avd-coloring. We assign a color missing from the edges incident to w to wv such that the coloring is proper. Note that G_Δ is an empty graph, it is easy to verify that the resulting coloring is a 4-avd-coloring of G.

Case 2. G is a β-type graph.

Without loss of generality, we assume that $r \equiv 2 \pmod 3$ and $s \equiv t \equiv 1 \pmod 3$, and $z_0 = u_j$ for some $2 \leq j \leq r-2$. Clearly, $\chi'_a(G) \geq 3$. We first show that $\chi'_a(G) \geq 4$ by contradiction. Suppose that φ is a 3-avd-coloring of G. Then the colors of xu_1, xv_1 and xw_1 are pairwise distinct. Without loss of generality, we assume that $\varphi(xu_1) = 3$, $\varphi(xv_1) = 2$ and $\varphi(xw_1) = 1$. From Lemma 1 it follows that $\varphi(yv_{s-1}) = 2$ and $\varphi(yw_{t-1}) = 1$. Thus $\varphi(yu_{r-1}) = 3$. Since the lengths of $xu_1u_2 \cdots u_j$ and $u_ju_{j+1} \cdots u_{r-1}y$ are both congruent to 1 modulo 3, it follows from Lemma 1 that $\varphi(u_{j-1}u_j)$=$\varphi(u_ju_{j+1})$=3, a contradiction.

It remains to show that G has a 4-avd-coloring. By induction on the number of vertices of G.

When $n = 16$, G must be the graph illustrated in Figure 3, and a 4-avd-coloring of G is also presented.

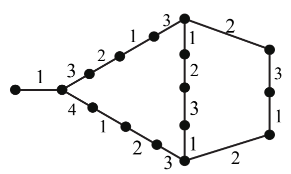

Figure 3: Basis step in Case 2.

Suppose that the theorem is true for β-type graph with fewer than n vertices, and let G be a β-type graph with $n \geq 17$ vertices. We have $H(G) = H_1$.

When G has a pendant vertex v such that the neighbor of v is not in $H(G)$. Let v_0 be a pendant vertex of G such that $d(v_0, H(G))$ is maximum. Clearly $d(v_0, H(G)) \geq 2$. Let w be the neighbor of v_0, and u the exactly one neighbor of w in G which is not a pendant vertex. Set $G' = G - v_0$. It is easy to see that G' is a β-type graph with $n-1$ vertices. By induction hypothesis, G' has a 4-avd-coloring. If $d_G(w) = 3$, then $d_G(u) = 2$ (since $\Delta(G) = 3$ and G_Δ is an empty graph). We assign a color missing from the edges incident to w to wv_0. If $d_G(w) = 2$, then there are at least 3 colors missing from the edges incident to w. Therefore we can always assign one missing color to wv_0 such that w and u are distinguishable.

When the neighbor of each pendant vertex of G is in $H(G)$. In this case G has exactly one pendant vertex, denoted by v, and the neighbor of v is u_j. The colorings of $xw_1w_2 \cdots w_{t-1}y$, $xv_1v_2 \cdots v_{s-1}y$, $xu_1u_2 \cdots u_j$ and $u_ju_{j+1} \cdots u_{r-1}y$ are $(123)^{\frac{t-1}{3}}1$, $(231)^{\frac{s-1}{3}}2$, $(321)^{\frac{j-1}{3}}3$ and $4(123)^{\frac{r-j-1}{3}}$, respectively. Finally we assign color 1 to vu_j. Clearly, the resulting coloring is a 4-avd-coloring of G.

Case 3. G is a γ-type graph.

Set

$$k(G) = \begin{cases} \Delta(G), & \text{if } G_\Delta \text{ is an empty graph;} \\ \Delta(G)+1, & \text{otherwise.} \end{cases}$$

Clearly $\chi'_a(G) \geq k(G)$. So there remains to show that G has a $k(G)$-avd-coloring. By induction on the number of vertices of G.

When $n = 5$, G must be one of the two graphs illustrated in Figure 4, and 4-avd-colorings are also presented.

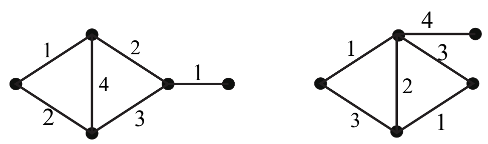

Figure 4: Basis step in Case 3.

Suppose that the theorem is true for γ-type graph with fewer than n vertices, and let G be a γ-type graph with $n \geq 6$ vertices. Let $H := H(G)$.

Now we divide the rest of the proof into four subcases.

Subcase 3.1. G has a pendant vertex v such that the neighbor of v is not in H.

Let v_0 be the pendant vertex of G such that $d(v_0, H)$ is maximum. Clearly, $d(v_0, H) \geq 2$. Let w be the neighbor of v, and u the only neighbor of w which is not a pendant vertex. Set $G' = G - v_0$. Note that G' has at least one pendant vertex.

(a) G_Δ is an empty graph.

If G' is α-type or β-type, then $\Delta(G) = 4$ and $\Delta(G') = 3$. From Case 1, Case 2 or induction hypothesis, it follows that G' has a $\Delta(G)$-avd-coloring.

If w is the vertex of maximum degree in G, then u is not the vertex of maximum degree. We assign a color missing from the edges incident to w to wv_0 such that the resulting coloring is proper.

If w is not the vertex of maximum degree in G, then w meets at most $\Delta(G) - 2$ colors, i.e., there are at least 2 colors missing from the edges incident to w. Hence there is at least one remaining color with which to color wv_0 such that w and u are distinguished from each other.

(b) G_Δ is not an empty graph.

If G' is α-type or β-type, then $\Delta(G')$=$\Delta(G)$=3. From Case 1, Case 2 or induction hypothesis, it follows that G' has a $(\Delta(G) + 1)$-avd-coloring. Clearly, w meets at most $\Delta(G) - 1$ colors (since wv_0 has not been colored), thus there are at least 2 colors missing from the edges incident to w. Therefore there is at least one remaining color with which to color wv_0 such that w and u are distinguished from each other.

Subcase 3.2. The neighbor of each pendant vertex of G is in H, and H has a vertex z of degree two in H and degree at least three in G such that $d_G(z) \neq d_G(z')$, where z' is one neighbor of z in H.

Let z'' be the other neighbor of z in H, i.e. $N_H(z) = \{z', z''\}$, where $N_H(z)$ denotes the neighborhood of z in H. Set $v \in N_G(z)\backslash\{z', z''\}$ and $G' = G - v$.

(a) G_Δ is not an empty graph.

Note that $\Delta(G')$=$\Delta(G)$. If G' has no pendant vertex, then G' has a $(\Delta(G) + 1)$-avd-coloring from Propositions 1-3. If G' has a pendant vertex, then G' has a $(\Delta(G) + 1)$-avd-coloring from Case 1, Case 2 or induction hypothesis. Since there are at least two colors missing from the edges incident to z, there is at least one remaining color with which to color vz such that the resulting coloring distinguishes z and z''. Clearly, z and z' are distinguishable (z and z' have distinct degree in G). Therefore G has a $(\Delta(G) + 1)$-avd-coloring.

(b) G_Δ is an empty graph.

Type 1: G' has no pendant vertex. It is easy to see that $G' = H_i$, where $i = 1, 3, 5$.

When $G' = H_3$, let φ be a 4-avd-coloring of H_3 obtained from the proof of Proposition 2, and we assign one color missing from the edges incident to z to zv. Clearly, the resulting coloring is a 4-avd-coloring of G.

When $G' = H_1$, let φ be a 3-avd-coloring of H_1 obtained from the proof of Proposition 1 except the cases $r \equiv s \equiv 1 \pmod 3$, $t \equiv 0 \pmod 3$ and $r \equiv 2 \pmod 3$, $s \equiv t \equiv 1 \pmod 3$. We assign one color missing from the edges incident to z to zv such that the coloring obtained is proper. Clearly the resulting coloring is a 3-avd-coloring of G (since $d_G(z')$=$d_G(z'')$=2 and $d_G(z) = 3$). So there remains to consider the cases $r \equiv s \equiv 1 \pmod 3$, $t \equiv 0 \pmod 3$ and $r \equiv 2 \pmod 3$, $s \equiv t \equiv 1 \pmod 3$.

(i) $r \equiv s \equiv 1 \pmod 3$, $t \equiv 0 \pmod 3$.

If $z = u_j$ $(2 \leq j \leq r - 2)$, then the colorings of P_2 and P_3 are $(231)^{\frac{s-1}{3}}2$ and $(123)^{\frac{t}{3}}$, respectively. The coloring of P_1 is $(312)^{\frac{j}{3}}1(321)^{\frac{r-j-1}{3}}$ (if $j \equiv 0 \pmod 3$), $(312)^{\frac{j-1}{3}}3$ $(231)^{\frac{r-j}{3}}$ (if $j \equiv 1 \pmod 3$) or $(312)^{\frac{j-2}{3}}3\ 1(312)^{\frac{r-j-2}{3}}31$ (if $j \equiv 2 \pmod 3$), where P_1, P_2 and P_3 are defined as Proposition 1. Then we properly color u_jv, and we obtain a 3-avd-coloring of G.

The case that $z = v_j$ $(2 \leq j \leq s - 2)$ can be disposed by a similar manner.

If $z = w_j$ $(2 \leq j \leq t - 2)$, then the colorings of P_1 and P_2 are $(123)^{\frac{r-1}{3}}1$ and $(321)^{\frac{s-1}{3}}3$, respectively. The coloring of P_3 is $(231)^{\frac{j}{3}}(312)^{\frac{t-j}{3}}$ (if $j \equiv 0 \pmod 3$), $(231)^{\frac{j-1}{3}}\ 2(123)^{\frac{t-j-2}{3}}12$ (if $j \equiv 1 \pmod 3$) or $(231)^{\frac{j-2}{3}}23$ $(231)^{\frac{t-j-1}{3}}\ 2$ (if $j \equiv 2 \pmod 3$). Then we properly color w_jv, and we obtain a 3-avd-coloring of G.

(ii) $r \equiv 2 \pmod 3$, $s \equiv t \equiv 1 \pmod 3$.

If $z = u_j$ $(2 \leq j \leq r - 2)$, then the colorings of P_2 and P_3 are $(231)^{\frac{s-1}{3}}2$ and $(123)^{\frac{t-1}{3}}1$, respectively. The coloring of P_1 is $(312)^{\frac{j}{3}}(132)^{\frac{r-j-2}{3}}13$ (if $j \equiv 0 \pmod 3$) or $(312)^{\frac{j-2}{3}}\ 31(213)^{\frac{r-j}{3}}$ (if $j \equiv 2 \pmod 3$). Note that G is not a β-type graph, thus $j \not\equiv 1 \pmod 3$. Then we properly color u_jv.

If $z = v_j$ $(2 \leq j \leq s - 2)$, then the colorings of P_1 and P_3 are $(231)^{\frac{r-2}{3}}23$ and $(123)^{\frac{t-1}{3}}1$, respectively. The coloring of P_2 is $(321)^{\frac{j}{3}}(231)^{\frac{s-j-1}{3}}2$ (if $j \equiv 0 \pmod 3$), $(321)^{\frac{j-1}{3}}3$ $(132)^{\frac{s-j}{3}}$ (if $j \equiv 1 \pmod 3$) or $(321)^{\frac{j-2}{3}}32(123)^{\frac{s-j-2}{3}}12$ (if $j \equiv 2 \pmod 3$). Then we properly color v_jv.

The case that $z = w_j$ $(2 \leq j \leq t - 2)$ can be disposed by a similar manner.

When $G' = H_5$, then $r \equiv 1 \pmod 3$ and $s \equiv 1 \pmod 3$ cannot both hold.

(i) $r \equiv 1 \pmod 3$ and $s \not\equiv 1 \pmod 3$.

Clearly that $z = u_j$ $(2 \leq j \leq r - 2)$. We cyclically color $xw_1, w_1w_2, \cdots, w_{t-1}y$ by colors 1, 2 and 3. Suppose that the color of $w_{t-1}y$ is a, where $a \in \{1, 2, 3\}$.

If $s \equiv 0 \pmod 3$, then the coloring of yv_1, v_1v_2, $\cdots$, $v_{s-1}y$ is $[(a + 2)a(a + 1)]^{\frac{s}{3}}$; if $s \equiv 2 \pmod 3$, then the coloring of yv_1, v_1v_2, $\cdots$, $v_{s-1}y$ is $[(a+1)(a+2)a]^{\frac{s-2}{3}}(a+$

$1)(a+2)$, where addition is taken modulo 3.

The coloring of xu_1, u_1u_2, $\cdots$, $u_{r-1}u_r$, u_rx is $(231)^{\frac{j}{3}}(321)^{\frac{r-j-1}{3}}3$, $(312)^{\frac{j-1}{3}}3(132)^{\frac{r-j}{3}}$ or $(231)^{\frac{j-2}{3}}23$ $(231)^{\frac{r-j-2}{3}}$ 23 depending on $j \equiv 0, 1$, or $2 \pmod 3$. Then we properly color vu_j, and we obtain a 3-avd-coloring of G.

(ii) $r \not\equiv 1 \pmod 3$ and $s \not\equiv 1 \pmod 3$.

Let φ be a 3-avd-coloring of H_5 obtained from the proof of Proposition 3. Then we properly color vz, and we obtain a 3-avd-coloring of G.

Type 2: G' has a pendant vertex and G' is an α-type graph.

In this case it is obvious that $3 \le \Delta(G) \le 4$ and $H(G) = H_5$. Without loss of generality, we assume that $r \equiv 1 \pmod 3$. Set $C = xu_1u_2 \cdots u_{r-1}x$.

When $\Delta(G) = 3$, we have $z = u_j$ $(2 \le j \le r-2)$. Note that $C \cup \{vu_j, xw_1\}$ satisfies the conditions of Lemma 3. It follows from Lemma 3 that $C \cup \{vu_j, xw_1\}$ has a ξ-coloring φ such that the color of xw_1 is 1. We cyclically color w_1w_2, w_2w_3, $\cdots$, $w_{t-1}y$ by colors 2, 3 and 1. Assume that the color of $w_{t-1}y$ is a, where $a \in \{1, 2, 3\}$.

If $s \equiv 0 \pmod 3$, then the coloring of $C' = yv_1v_2 \cdots v_{s-1}y$ is $[(a+2)a(a+1)]^{\frac{s}{3}}$ (starting from yv_1 in clockwise), where addition is taken modulo 3.

If $s \equiv 1 \pmod 3$, then there exists some vertex v_l of C' such that $d_G(v_l) = 3$. Let e be the pendant edge incident to v_l. By Lemma 3, $C' \cup \{e, w_{t-1}y\}$ has a ξ-coloring such that the color of $w_{t-1}y$ is a.

If $s \equiv 2 \pmod 3$, then the coloring of C' is $[(a+1)(a+2)a]^{\frac{s-2}{3}}(a+1)(a+2)$ (starting from yv_1 in clockwise), where addition is taken modulo 3.

Finally we properly color all the uncolored pendant edges and obtain a 3-avd-coloring of G.

When $\Delta(G) = 4$, then G has exactly one vertex of maximum degree. Clearly z is just the vertex of maximum degree in G, and any two 3-vertices are not adjacent in G. We cyclically color the edges of $xu_1u_2 \cdots u_{r-1}$ by colors 1, 2 and 3, and assign color 4 to $u_{r-1}x$. Starting from xw_1, we cyclically color the edges of $xw_1w_2 \cdots w_{t-1}y$ by colors 2, 3 and 1. Assume that the color of $w_{t-1}y$ is a, then we cyclically color the edges of $yv_1v_2 \cdots v_{s-1}$ by colors $a+1$, a, $a+2$, and assign color 4 to $v_{s-1}y$, where addition is taken modulo 3. Finally we properly color all the pendant edges. It is not difficult to verify, whether $z = w_j$ $(1 \le j \le t-1)$ or $z = v_j$ $(1 \le j \le s-1)$, that the resulting coloring is a 4-avd-coloring of G.

Type 3: G' has a pendant vertex and G' is a β-type graph.

Without loss of generality, we assume that $r \equiv 2 \pmod 3$ and $s \equiv t \equiv 1 \pmod 3$, and $z_0 = u_j$ for some $1 \le j \le r-1$. Clearly $3 \le \Delta(G) \le 4$. Set $P_1 = xu_1u_2 \cdots u_{r-1}y$, $P_2 = xv_1v_2 \cdots v_{s-1}y$ and $P_3 = xw_1w_2 \cdots w_{t-1}y$.

When $\Delta(G) = 4$, then $z = u_j$ is the only vertex of maximum degree. The colorings of P_2, P_3 and P_1 are $(123)^{\frac{s-1}{2}}1$, $(231)^{\frac{t-1}{2}}2$ and $(312)^{\frac{j-1}{3}}3(412)^{\frac{r-j-1}{3}}4$, respectively. Two pendant edges incident to u_j are assigned colors 1 and 2. It is obvious that the resulting coloring is a 4-avd-coloring of G.

When $\Delta(G) = 3$, let e be the pendant edge incident to u_j. There are four cases to consider.

If $z = u_i$ $(2 \le i \le j-2)$, then the colorings of P_2 and P_3 are $(123)^{\frac{s-1}{2}}1$ and $(231)^{\frac{t-1}{2}}2$, respectively. Set $P_1' = xu_1 u_2 \cdots u_i$, $P_1'' = u_iu_{i+1} \cdots u_j$ and $P_1''' = u_ju_{j+1} \cdots u_{r-1}y$. The colorings of P_1', P_1'' and P_1''' are given as follows (see Table 2):

Table 2. The colorings of P_1', P_1'' and P_1'''.

Conditions	P_1'	P_1''	P_1'''
$i \equiv 0 \pmod 3$	$(312)^{\frac{i}{3}}$	$(123)^{\frac{j-i-1}{3}}1$	$(312)^{\frac{r-j-1}{3}}3$
$i \equiv 1 \pmod 3$	$(312)^{\frac{i-1}{3}}3$	$(231)^{\frac{j-i}{3}}$	$(312)^{\frac{r-j-1}{3}}3$
$i \equiv 2 \pmod 3$	$(312)^{\frac{i-2}{3}}12$	$(321)^{\frac{j-i-2}{3}}32$	$(312)^{\frac{r-j-1}{3}}3$

Finally we properly color e and u_iv, and we obtain a 3-avd-coloring of G.

The case that $z = u_i$ $(j+2 \le i \le r-2)$ can be dealt with in a similar manner as the above case.

If $z = v_i$ $(2 \le i \le s-2)$, then the colorings of P_1 and P_3 are $(123)^{\frac{r-2}{3}}12$ and $(312)^{\frac{t-1}{3}}3$, respectively. Set $P_2' = xv_1v_2 \cdots v_i$ and $P_2'' = v_iv_{i+1} \cdots v_{s-1}y$. Then the colorings of P_2' and P_2'' are given as follows (see Table 3):

Table 3. The colorings of P_2', P_2''.

Conditions	The coloring of P_1'	The coloring of P_2''
$i \equiv 0 \pmod 3$	$(213)^{\frac{i}{3}}$	$(123)^{\frac{s-i-1}{3}}1$
$i \equiv 1 \pmod 3$	$(231)^{\frac{i-1}{3}}2$	$(321)^{\frac{s-i}{3}}$
$i \equiv 2 \pmod 3$	$(231)^{\frac{i-2}{3}}23$	$(213)^{\frac{s-i-2}{3}}21$

Finally we properly color e and v_iv, and we obtain a 3-avd-coloring of G.

The case that $z = w_j$ $(2 \le j \le t-2)$ can be dealt with in a similar manner as the above case.

Type 4: G' has a pendant vertex and G' is a γ-type graph.

By induction hypothesis, G' has a $\Delta(G)$-avd-coloring. If z is the vertex of maximum degree, then we properly color zv. If z is not the vertex of maximum degree, then there are at least two colors missing from the edges incident to z. Hence we can assign one missing color to zv such that z and z'' are distinguishable. Clearly the resulting coloring is a $\Delta(G)$-avd-coloring of G.

Subcase 3.3. The neighbor of each pendant vertex of G is in H, and H has a vertex z of degree two in H and degree at least 3 in G. For any such z, $d_G(z') = d_G(z'') = d_G(z)$, where $N_H(z) = \{z', z''\}$.

(i) $H = H_1$.

In view of the symmetry of three paths from x to y in H, G must be one of the following three graphs, where $\Delta(G) = k + 2$ and $k \geq 1$ (see Figure 5).

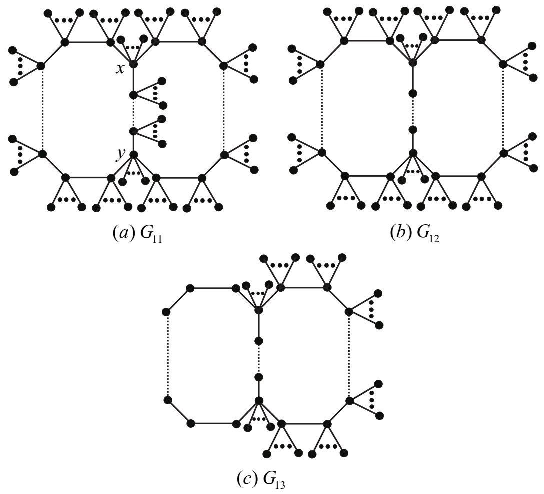

Figure 5: Illustrations in Subcase 3.3(i).

We just show that G_{11} (see Figure 5(a)) has a $(\Delta(G)+1)$-avd-coloring. The other cases can be dealt with in a similar manner. Note that in G_{11} the number of pendant edges incident to x or y is $k-1$, and the number of pendant edges incident to each of the other vertices of H_1 is k. Suppose that $k \geq 2$.

If $r+s$ is even, then we alternately color the edges of cycle $xu_1u_2\cdots u_{r-1}yv_{s-1}\cdots v_2v_1x$ starting from xu_1 by colors $k+2$ and $k+3$. The uncolored edges incident to x, u_1, u_2, $\cdots$, u_{r-1}, y, v_{s-1}, $\cdots$, v_1 are alternately colored by $\{1,2,\cdots,k\}$ and $\{2,3,\cdots,k+1\}$ such that the colors of xw_1 and yw_{t-1} are 1 and 2 respectively.

If $r+s$ is odd, the coloring of cycle $xu_1u_2\cdots u_{r-1}y\, v_{s-1}\cdots v_2v_1x$ starting from xu_1 is $[(k+2)(k+3)]^{\frac{r+s-1}{2}}2$. The uncolored edges incident to u_1, u_2, $\cdots$, u_{r-1}, y, v_{s-1}, $\cdots$, v_1 are alternately colored by $\{1,2,\cdots,k\}$ and $\{2,3,\cdots,k+1\}$ such that the color of $w_{t-1}y$ is 2. The uncolored edges incident to x or v_1 are colored by $\{1,3,4,\cdots,k+1\}$ such that the color of xw_1 is 1.

Then we cyclically color w_1w_2, w_2w_3, $\cdots$, $w_{t-2}w_{t-1}$ starting from w_1w_2 by colors 3, 4 and 1. The pendant edges incident to w_{t-1}, w_{t-2}, $\cdots$, w_2 are colored such that the missing color of these vertices are alternately $k+3$ and 2. We color the pendant edges incident to w_1 such that the missing color of w_1 is $k+2$. It is not difficult to verify that the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of G_{11}.

When $\Delta(G) = 3$ (i.e. $k = 1$), we assign colors 1, 3 and 4 to xu_1, xw_1 and xv_1, respectively. The edges of $u_1u_2\cdots u_{r-1}yv_{s-1}\cdots v_2v_1$ are cyclically colored starting from u_1u_2 by 2, 3 and 1. The pendant edge incident to each u_i $(2 \leq i \leq r-1)$ or v_j $(3 \leq j \leq s-1)$ is colored by 4, and the pendant edge incident to u_1 is colored by 3. If the colors of v_3v_2 and v_2v_1 are 1 and 2 respectively, then we assign colors 4 and 3 to the pendant edges incident to v_2 and v_1, respectively. If the colors of v_3v_2 and v_2v_1 are 2 and 3 respectively, then we assign colors 1 and 2 to the pendant edges incident to v_2 and v_1, respectively. If the colors of v_3v_2 and v_2v_1 are 3 and 1 respectively, then we assign colors 4 and 2 to the pendant edges incident to v_2 and v_1, respectively. Denote by $c(e)$ the color that has been assigned to e.

If $t = 2$, then there are three cases to consider. When $c(yu_{r-1}) = 2$ and $c(yv_{s-1}) = 3$, we exchange the colors of $u_{r-1}y$ and the pendant edge incident to u_{r-1} (i.e. we recolor $u_{r-1}y$ by color 4, and the pendant edge incident to u_{r-1} by 2). Then we assign colors 2 and 1 to w_1y and the pendant edge incident to w_1, respectively. When $c(yu_{r-1}) = 3$ and $c(yv_{s-1}) = 1$, we assign colors 2 and 4 to w_1y and the pendant edge incident to w_1, respectively. When $c(yu_{r-1}) = 1$ and $c(yv_{s-1}) = 2$, we assign colors 4 and 2 to w_1y and the pendant edge incident to w_1, respectively.

If $t \geq 3$, then there are three cases to consider. When $c(yu_{r-1}) = 2$ and $c(yv_{s-1}) = 3$, we assign color 1 to $w_{t-1}y$ and cyclically color w_1w_2, w_2w_3, $\cdots$, $w_{t-2}w_{t-1}$ by 2, 4 and 3. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 2 and 4 respectively, then the pendant edge incident to each w_i $(1 \leq i \leq t-2)$ is colored by 1, and the pendant edge incident to w_{t-1} is colored by 3. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 4 and 3 respectively, then the pendant edge incident to each w_i $(1 \leq i \leq t-3)$ is colored by 1, and the pendant edge incident to w_{t-2} or w_{t-1} is colored by 2 or 4, respectively. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 3 and 2 respectively, then the pendant edge incident to each w_i $(1 \leq i \leq t-2)$ is colored by 1, and the pendant edge incident to w_{t-1} is colored by 4.

When $c(yu_{r-1}) = 3$ and $c(yv_{s-1}) = 1$, we assign color 2 to $w_{t-1}y$ and cyclically color w_1w_2, w_2w_3, $\cdots$, $w_{t-2}w_{t-1}$ by 1, 4 and 3. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 1 and 4 respectively, then the pendant edge incident to each w_i $(1 \leq i \leq t-2)$ is colored by 2, and the pendant edge incident to w_{t-1} is colored by 3. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 4 and 3 respectively, then the pendant edge incident to each w_i $(1 \leq i \leq t-3)$ is colored by 2, and the pendant edge incident to w_{t-2} or

w_{t-1} is colored by 1 or 4, respectively. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 3 and 1 respectively, then the pendant edge incident to each w_i $(1 \leq i \leq t-2)$ is colored by 2, and the pendant edge incident to w_{t-1} is colored by 4.

When $c(yu_{r-1}) = 1$ and $c(yv_{s-1}) = 2$, we assign color 3 to $w_{t-1}y$ and cyclically color $w_1w_2, w_2w_3, \cdots, w_{t-2}w_{t-1}$ by 2, 1 and 4. Suppose that $t \geq 4$. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 2 and 1 respectively, then the pendant edge incident to each w_i $(2 \leq i \leq t-2)$ is colored by 3, and the pendant edge incident to w_1 or w_{t-1} is colored by 4. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 1 and 4 respectively, then the pendant edge incident to each w_i $(2 \leq i \leq t-2)$ is colored by 3, and the pendant edge incident to w_1 or w_{t-1} is colored by 4 or 2, respectively. If the colors of $w_{t-3}w_{t-2}$ and $w_{t-2}w_{t-1}$ are 4 and 2 respectively, then the pendant edge incident to each w_i $(2 \leq i \leq t-3)$ is colored by 3, and the pendant edge incident to w_1 or w_{t-1} is colored by 4. The pendant edge incident to w_{t-2} is colored by 1. If $t = 3$, then the pendant edge incident to w_1 or w_2 is colored by 1 or 4, respectively.

It is not difficult to see that the resulting coloring is a 4-avd-coloring of G_{11}.

(ii) $H = H_2$.

In view of the symmetry of graph, G must be one of the following two cases, where $\Delta(G) = k+2$ and $k \geq 1$ (see Figure 6).

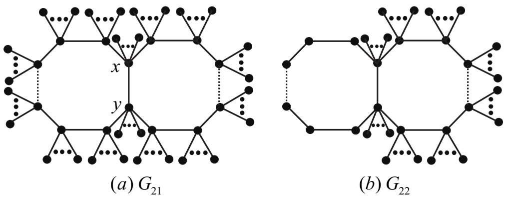

Figure 6: Illustrations in Subcase 3.3(ii).

We just show that G_{21} (see Figure 6(a)) has a $(\Delta(G)+1)$-avd-coloring, and the case G_{22} (see Figure 6(b)) can be dealt with in a similar manner. Note that in G_{21} the number of pendant edges incident to x or y is $k-1$, and the number of pendant edges incident to each of the other vertices of H is k.

When $\Delta(G) \geq 4$ (i.e. $k \geq 2$), then we alternately color $xu_1, u_1u_2, \cdots, u_{r-2}y, yv_{s-2}, v_{s-2}v_{s-1}, \cdots, v_2v_1$ by colors $k+3$ and $k+2$ starting from xu_1. Assign color 1 to xv_1. We alternately assign colors $\{1, 2, \cdots, k\}$ and $\{2, 3, \cdots, k+1\}$ to the pendant edges incident to u_1, u_2, $\cdots$, u_{r-2}, y, v_{s-2}, $\cdots$, v_2. Note that the number of pendant edges incident to y is $k-1$, so we consider here that xy is a "pendant edge" incident to y and color it by 2. The pendant edges incident to x are colored by $\{3,4,\cdots,k+1\}$. If $r+s$ is even, then the pendant edges incident to v_1 are colored by $\{2,3,\cdots,k,k+2\}$. If $r+s$ is odd, then the pendant edges incident to v_1 are colored by $\{2,3,\cdots,k+1\}$. It is not difficult to verify that the resulting coloring is a $(\Delta(G)+1)$-coloring of G_{21}.

When $\Delta(G) = 3$ (i.e. $k = 1$), we assign colors 1, 2, 3, 2 and 4 to xu_1, xv_1, xy, yu_{r-2} and yv_{s-2}, respectively. The edges of $u_1u_2 \cdots u_{r-2}$ are cyclically colored by colors 4, 3 and 1 starting from u_1u_2. We assign color 3 to the pendant edge incident to u_1. If the color of $u_{r-4}u_{r-3}$ is 3 and the color of $u_{r-3}u_{r-2}$ is 1, then the pendant edge incident to u_{r-3} is colored by 2 and the pendant edge incident to u_{r-2} is colored by 4. If the colors of $u_{r-4}u_{r-3}$ and $u_{r-3}u_{r-2}$ are 1 and 4 respectively, then the pendant edges incident to u_{r-3} and u_{r-2} are colored by 3 and 1, respectively. If the colors of $u_{r-4}u_{r-3}$ and $u_{r-3}u_{r-2}$ are 4 and 3 respectively, then the pendant edges incident to u_{r-3} and u_{r-2} are colored by 2 and 1, respectively. Then each pendant edge incident to u_2, u_3, $\cdots$, u_{r-4} is colored by 2, respectively. The edges incident to v_1, v_2, $\cdots$, v_{s-2} are colored in a similar manner. It is not difficult to see that the resulting coloring is a 4-avd-coloring of G.

(iii) $H = H_3$.

By the symmetry of graph, G must be one of the following two cases, where $\Delta(G) = k+2$ and $k \geq 2$ (see Figure 7).

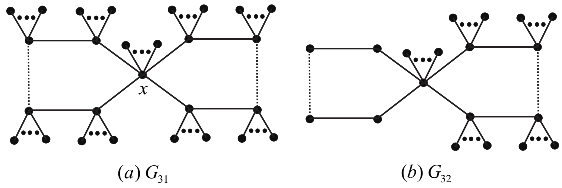

Figure 7: Illustrations in Subcase 3.3(iii).

We just show that G_{31} (see Figure 7(a)) has a $(\Delta(G)+1)$-avd-coloring. The case $G = G_{32}$ (see Figure 7(b)) can be dealt with in a similar manner. Note that in G_{31} the number of pendant edges incident to x is $k-2$, and the number of pendant edges incident to each of the other vertices of H_3 is k.

We assign colors 1, 2, 3 and 4 to the edges xu_1, xv_1, xu_{r-1} and xv_{s-1}, respectively. We alternately color the edges of $u_1u_2 \cdots u_{r-1}$ starting from u_1u_2 by colors $k+3$ and $k+2$. If the color of $u_{r-2}u_{r-1}$ is $k+3$, then the pendant edges incident to u_{r-1} are clored by $\{1,2,4,5,\cdots,k+1\}$; if the color of $u_{r-2}u_{r-1}$ is $k+2$, then the pendant edges incident to u_{r-1} are clored by $\{1,4,5,\cdots,k+1,k+3\}$. We alternately color the pendant edges incident to u_2, u_3, $\cdots$, u_{r-2} starting from u_2 by colors $\{1,2,\cdots,k\}$ and $\{2,3,\cdots,k+1\}$. The pendant edges incident to u_1 are colored by $\{2,3,\cdots,k+1\}$. Assign colors $\{5,6,\cdots,k+2\}$ to the pendant edges incident to x (if $k = 2$, then G has no pendant edge incident to x).

We alternately color the edges of $v_1v_2\cdots v_{s-1}$ starting from v_1v_2 by colors 1 and 2. The pendant edges incident to v_1 and v_{s-1} are colored by $\{4,5,\cdots,k+3\}$ and $\{3,5,6,\cdots,k+3\}$ respectively. We alternately color the pendant edges incident to v_2, v_3, $\cdots$, v_{s-2} starting from v_2 by colors $\{3,4,\cdots,k+2\}$ and $\{4,5,\cdots,k+3\}$. It is not difficult to verify that the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of G_{31}.

(iv) $H = H_4$.

By the symmetry of graph, G must be one of the following two cases, where $\Delta(G) = k+2$ and $k \geq 1$ (see Figure 8).

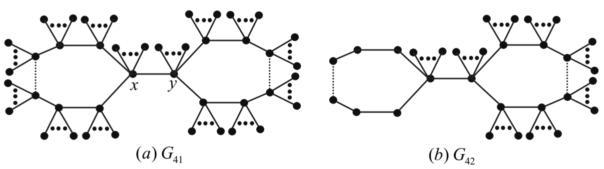

Figure 8: Illustrations in Subcase 3.3(iv).

We just show that G_{41} (see Figure 8(a)) has a $(\Delta(G)+1)$-avd-coloring. The case that $G = G_{42}$ (see Figure 8(b)) can be dealt with in a similar manner. Note that in G_{41} the number of pendant edges incident to x and y are $k-1$ and $l-1$, respectively. The number of pendant edges incident to each of u_i $(1 \leq i \leq r-1)$ is k, and the number of pendant edges incident to each v_j $(1 \leq j \leq s-1)$ is l. We assume that $k \geq l$.

If r and s are both even, then we alternately color the edges of cycle $xu_1u_2\cdots u_{r-1}x$ by colors $k+3$ and $k+2$. The pendant edges incident to u_1, u_2, $\cdots$, u_{r-1} are alternately colored by $\{1,2,\cdots,k\}$ and $\{2,3,\cdots,k+1\}$. Assign color 2 to xy and color the pendant edges incident to x by $\{3,4,\cdots,k+1\}$. Then we alternately color yv_1, v_1v_2, $\cdots$, $v_{s-2}v_{s-1}$ by colors $l+2$ and $l+3$. Assign color 1 to yv_{s-1}. The pendant edges incident to v_1, v_2, $\cdots$, v_{s-2} are alternately colored by $\{1,2,\cdots,l\}$ and $\{2,3,\cdots,l+1\}$. The pendant edges incident to v_{s-1} and y are colored by $\{2,3,\cdots,l,l+3\}$ and $\{3,4,\cdots,l+1\}$, respectively.

If r is even and s is odd, then the edges incident to x, u_1, $\cdots$, u_{r-1} are colored as the same as the above case. The edges of $yv_1v_2\cdots v_{s-1}$ are alternately colored by $l+2$ and $l+3$, and assign color 1 to yv_{s-1}. The pendant edges incident to v_1, v_2, $\cdots$, v_{s-2} are alternately colored by $\{2,3,\cdots,l+1\}$ and $\{1,2,\cdots,l\}$. The pendant edges incident to v_{s-1} and y are colored by $\{2,3,\cdots,l,l+2\}$ and $\{3,4,\cdots,l+1\}$, respectively.

If r is odd and s is even, then the edges of $xu_1u_2\cdots u_{r-1}$ are alternately colored by $k+3$ and $k+2$. Assign color 1 to xu_{r-1}. The pendant edges incident to u_1, u_2, $\cdots$, u_{r-2} are alternately colored by $\{1,2,\cdots,k\}$ and $\{2,3,\cdots,k+1\}$. The pendant edges incident to u_{r-1} and x are colored by $\{2,3,\cdots,k+1\}$ and $\{3,4,\cdots,k+1\}$, respectively. We assign color 2 to xy. The edges of cycle $yv_1v_2\cdots v_{s-1}y$ are alternately colored starting from yv_1 by colors $l+2$ and $l+3$. The pendant edges incident to v_1, v_2, $\cdots$, v_{s-1} are alternately colored by $\{1,2,\cdots,l\}$ and $\{2,3,\cdots,l+1\}$, and the pendant edges incident to y are colored by $3,4,\cdots,l+1$.

If r and s are both odd, then the edges incident to x, u_1, $\cdots$, u_{r-1} are colored as the same as the above case (i.e. the case that r is odd and s is even). The edges of $yv_1v_2\cdots v_{s-1}$ are alternately colored by $l+2$ and $l+3$. Assign color 1 to yv_{s-1}. The pendant edges incident to v_1, v_2, $\cdots$, v_{s-2} are alternately colored by $\{2,3,\cdots,l+1\}$ and $\{1,2,\cdots,l\}$. The pendant edges incident to v_{s-1} and y are colored by $\{3,4,\cdots,l+2\}$ and $\{3,4,\cdots,l+1\}$, respectively.

It is not difficult to verify that the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of G_{41}.

(v) $H = H_5$.

By the symmetry of graph, G must be one of the following six cases, where $\Delta(G) = k+2$ and $k \geq 1$ (see Figure 9).

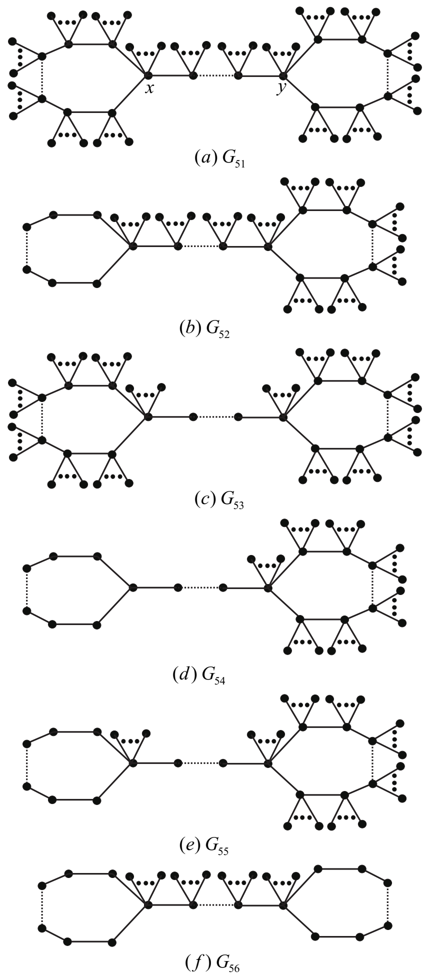

Figure 9: Illustrations in Subcase 3.3(v).

We just show that G_{51} has a $(\Delta(G)+1)$-avd-coloring, the other cases can be dealt with in a similar manner. Note that in G_{51} the number of pendant edges incident to x or y is $k-1$, and the number of pendant edges incident to each of the other vertices of H_5 is k. Suppose that $k \geq 2$.

We alternately color the edges of $u_1u_2 \cdots u_{r-1}x\ w_1w_2 \cdots\ w_{t-1}y\ v_1v_2 \cdots v_{s-1}$ starting from u_1u_2 by colors $k+2$ and $k+3$. The pendant edges incident to u_2, u_3, $\cdots$, u_{r-1}, x, w_1, $\cdots$, w_{t-1}, y, v_1, $\cdots$, v_{s-2} are alternately colored by $\{1,2,\cdots,k\}$ and $\{2,3,\cdots,k+1\}$. Note here that the number of pendant incident to x or y is $k-1$, we consider xu_1 or yv_{s-1} the "pendant edges" incident to x or y, respectively. The colors of xu_1 and yv_{s-1} are both equal to 2. The pendant edges incident to u_1 or v_{s-1} are colored by $\{1,3,4,\cdots,k+1\}$.

When $\Delta(G) = 3$ (i.e. $k = 1$), we assign colors 3, 2 and 1 to xu_1, xu_{r-1} and xw_1, respectively. The edges of $u_1u_2 \cdots u_{r-1}$ are cyclically colored by 4, 1 and 3, and the pendant edge incident to each u_i $(1 \leq i \leq r-3)$ is colored by 2. If the colors of $u_{r-3}u_{r-2}$ and $u_{r-2}u_{r-1}$ are 4 and 1 respectively, then we assign colors 3 and 4 to the pendant edges incident to u_{r-2} and u_{r-1}, respectively. If the colors of $u_{r-3}u_{r-2}$ and $u_{r-2}u_{r-1}$ are 1 and 3 respectively, then we assign colors 2 and 4 to the pendant edges incident to u_{r-2} and u_{r-1}, respectively. If the colors of $u_{r-3}u_{r-2}$ and $u_{r-2}u_{r-1}$ are 3 and 4 respectively, then we assign colors 1 and 3 to the pendant edges incident to u_{r-2} and u_{r-1}, respectively. We assign colors 2, 3 and 1 to w_1w_2, w_2w_3, $\cdots$, $w_{t-1}y$ cyclically, and color 4 to the pendant edge incident to each w_i $(1 \leq i \leq t-1)$. We assign two colors in $\{1,2,3\} \backslash \{c(w_{t-1}y)\}$ to yv_1 and yv_{s-1}. Then we color the edges incident to v_1, v_2, $\cdots$, v_{s-1} by a similar manner and obtain a 4-avd-coloring of G_{51}.

Subcase 3.4. The neighbor of each pendant vertex of G is in H, and the degree of every vertex of degree 2 in H is 2 in G. Clearly, G must be one of the following graphs illustrated in Figure 10.

Here we assume that the pendant vertices adjacent to x are x_1, x_2, $\cdots$, x_k and the pendant vertices adjacent to y are y_1, y_2, $\cdots$, y_l, where $k \geq l \geq 1$.

(i) $G = G_1$.

Clearly $\Delta(G) = k+3$. From Proposition 1, H_1 has a 4-avd-coloring φ using colors 1, 2, 3 and 4. Then we assign colors 5, 6, $\cdots$, $k+3$ to xx_2, xx_3, $\cdots$, xx_k, respectively. Similarly, we assign clors 5, 6, $\cdots$, $l+3$ to yy_2, yy_3, $\cdots$, yy_l, respectively. Then we assign colors in $\{1,2,3,4\} \backslash \{\varphi(xu_1),\ \varphi(xv_1),\ \varphi(xw_1)\}$ and $\{1,2,3,4\} \backslash \{\varphi(yu_{r-1}),\ \varphi(yv_{s-1}),\ \varphi(yw_{t-1})\}$ to xx_1 and yy_1, respectively. Clearly the resulting coloring is a $\Delta(G)$-avd-coloring of G.

(ii) $G = G_2$.

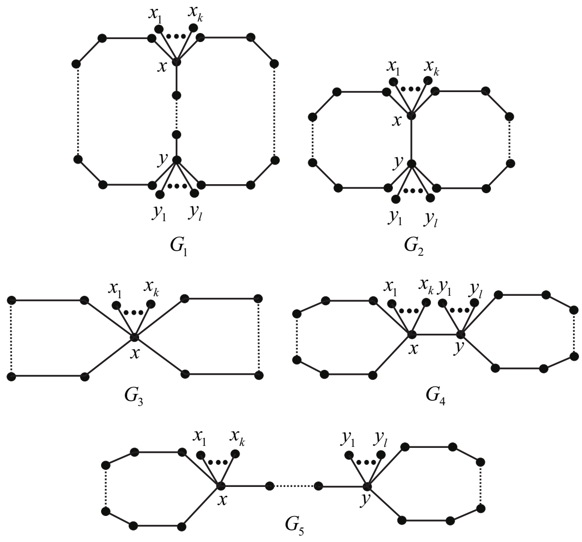

Figure 10: Illustrations in Subcase 3.4.

If $k \neq l$, then without loss of generality we assume that $k > l$. We color G in a similar manner as (i) and obtain a $\Delta(G)$-avd-coloring of G. If $k = l$, then from Proposition 2 that H_2 has a 4-avd-coloring using colors 1, 2, 3 and 4. Note here that x and y are distinguishable in H_2. We assign colors 5, 6, $\cdots$, $k+4$ to xx_1, xx_2, $\cdots$, xx_k, respectively. Similarly, we assign clors 5, 6, $\cdots$, $k+4$ to yy_1, yy_2, $\cdots$, yy_l, respectively. It is obvious that x and y are distinguished from each other, and the resulting coloring is a $(\Delta(G)+1)$-avd-coloring of G.

(iii) $G = G_3$.

By Proposition 2, H_3 has a 4-avd-coloring using colors 1, 2, 3 and 4. We assign colors 5, 6, $\cdots$, $k+4$ to the edges xx_1, xx_2, $\cdots$, xx_k, respectively. It is obvious that the resulting coloring is a $\Delta(G)$-avd-coloring of G.

(iv) $G = G_4$. This case can be dealt with in a similar manner as (ii), and we may obtain a $\Delta(G)$-avd-coloring $(k \neq l)$ or $(\Delta(G)+1)$-avd-coloring of G $(k = l)$.

(v) $G = G_5$. This case can be dealt with in a similar manner as (i), and we may obtain a $\Delta(G)$-avd-coloring of G.

Since we have dealt with all cases, the theorem is proved. □

4 Conclusion and Future Work

From Propositions 1-3 and Theorem 1, we prove that $\chi'_a(G) \leq \Delta(G)+1$ for bicyclic graphs G. This implies that Conjecture 1 holds for all bicyclic graphs. We will investigate the AVDPEC Conjecture for other graphs (such as tricyclic graphs) in the future.

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References

[1] S. Akbari, H. Bidkori, and N. Nosrati, r-strong edge colorings of graphs, Discrete Math., 306, 3005–3010, 2006.

[2] P.N. Balister, E. Györi, J. Lehel, and R.H. Schelp, Adjacent vertex distinguishing edge colorings, SIAM J. Disrete Math., 21, 237–250, 2007.

[3] J.-L. Baril, H. Kheddouci, and O. Togni, Adjacent vertex distinguishing edge colorings of meshs and hypercubes, Australasian Journal of Combinatorics, 35, 89–102, 2006.

[4] Y. Bu and W. Wang, Adjacent vertex distinguishing edge-colorings of planar graphs with girth at least six, Discuss. Math. Graph Theory, 31, 429–439, 2011.

[5] X. Chen and Z. Li, Adjacent-vertex-distinguishing proper edge colorings of planar bipartite graphs with $\Delta = 9$, 10 or 11, Inform. Process. Lett., 115, 263–268, 2015.

[6] K. Edwards, M. Horňák, and M. Woźniak, On the neighbour-distinguishing index of a graph, Graphs and Combinatorics, 22, 341–350, 2006.

[7] L. Frigerio, F. Lastaria, and N.Z. Salvi, Adjacent vertex distinguishing edge colorings of the direct product of a regular graph by a path or a cycle, Discuss. Math. Graph Theory, 31, 547–557, 2011.

[8] G.V. Ghodasra and S.G. Sonchhatra, Further resultson 3-equitable labeling, IAENG International Journal of Applied Mathematics, 45, 1–15, 2015.

[9] C. Greenhill and A. Ruciński, Neighbour-distinguishing edge colorings of random regular graphs, The Electronic Journal of Combinatorics, 13(#R77), 1–12, 2006.

[10] H. Hatami, Δ+300 is a bound on the adjacent vertex distinguishing edge chromatic number, J. Combinatorial Theory (Series B) 95, 246–256, 2005.

[11] H. Herve and M. Mickael, Adjacent vertex-distinguishing edge coloring of graphs with maximum degree Δ, J. Comb. Optim., 26, 152–160, 2013.

[12] M. Horňák, D. Huang, and W. Wang, On neighbor-distinguishing index of planar graphs, J. Graph Theory, 76, 262–278, 2014.

[13] Y. Wang, J. Chen, R. Luo, and G. Mulley, Adjacent vertex-distinguishing edge coloring of 2-degenerate graphs, J. Comb. Optim., 31, 874–880, 2016.

[14] W. Wang and Y. Wang, Adjacent vertex distinguishing edge-colorings of graphs with smaller maximum average degree, J. Comb. Optim., 19, 471–485, 2010.

[15] W. Wang and Y. Wang, Adjacent vertex-distinguishing edge colorings of K_4-minor free graphs, Appl. Math. Lett., 24, 2034–2037, 2011.

[16] C. Yan, D. Huang, D. Chen, and W. Wang, Adjacent vertex distinguishing edge colorings of planar graphs with girth at least five, J. Comb. Optim., 28, 893–909, 2014.

[17] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graphs, Appl. Math. Lett., 15, 623–626, 2002.

[18] X. Zhao and X. Chen, On the adjacent strong edge coloring of monocycle graph, Journal of Lanzhou Jiaotong University (Natural Sciences), 24, 138–140, 2005.

[19] J. Zhu, Y. Bu, and Y. Dai, Upper bounds for adjacent vertex distinguishing edge coloring, J. Comb. Optim., 35, 454–462, 2018.