A Novel Iterative Method for Nonlinear Equations

Guanghui Liu, Cunyun Nie and Jie Lei

Abstract—**Based on the idea that the iterative error function can be applied to extrapolation to zero for SAM, a novel acceleration iterative scheme is obtained by using the quadratic interpolation error function. Furthermore, the new scheme is extended to the vector case and nonlinear equations, and the convergence order of the method proposed is always greater than or equal to two. Numerical results confirm the validity of the theoretical results.**

Index Terms—**the Steffensen iteration, quadratic extrapolation, acceleration method, nonlinear equations.**

I. Introduction

Nonlinear problems arise in many fields, such as mathematical models, (partial) differential equations, algebraic equations, optimization algorithms and so on. It is very important to construct efficient methods to evaluate the roots or approximate solution of the nonlinear problems. In the last few decades, many methods have been constructed. Some of them are based on the contracting mapping principle, such as the fixed-point method, Newton iterative method, Secant method and some acceleration methods ([1], [4]); some of them are based on the geometrical ideas, such as Halley method ([2]), Cauchy method ([3]), King-werner method ([5], [6]) and multi-step Ostrowski method ([7]); and others are based on Taylor expansion and M$\ddot{u}$ller method. In addition, some researchers have developed many new iterative methods. For example, the authors presented some modifications for Newton method and Secant method ([9], [11]). Jiang and Chun put the Golden-section idea into iterative method and then produced geometrical iterative schemes ([13]). Nie and Li studied the nonlinear equation being approximated as a quadratic equation, and they proposed a new iterative method to find the roots ([14]). Fdil in [17] considered an iterative non-overlapping domain decomposition method for solving optimal boundary control problems governed by parabolic equations. Bumbariu ([16]) and Liu and Ma ([18]) presented two improved acceleration iterative methods for solving nonlinear equations. One of them uses Aithen type, and the other uses the first step of the θ-algorithm type.

The acceleration method is an important branch of the methods of solving nonlinear equations. The Aitken method and Steffensen method are two well-known acceleration methods([4]), and the latter can also be seen as the approximate method to extrapolate the linear interpolation function to zero for which some researchers provided some modifications ([19], [20]). It is well known that when the convergence speed of the iterative methods is slow, the effective use of them will be quite limited. Therefore, the convergence speed is an important factor that must be considered.

Motivated by the above discussions, the main objective of this paper is to design a novel iterative scheme to accelerate convergence speed. Firstly, the convergence order two for the Steffensen method is proved and a new scheme is derived from the idea that the quadratic interpolation function of iterative error is extrapolated to zero for a nonlinear equation. Secondly, the scalar scheme is extended to the vector case and equations. Finally, numerical tests verify the validity of the new scheme.

The remainder of this paper is organized as follows: In the next section, some preliminaries are given. In Section III, a novel scheme for both scalar and vector cases is presented, respectively. In Section IV, some numerical tests are given to illustrate our main results.

II. Some Preliminaries

Consider the following nonlinear equation

$$f(x) = 0, \tag{1}$$

which can be rewritten as a fixed-point problem

$$x = \phi(x),$$

where f is a given nonlinear smooth function, and $\phi : [a, b] \to [a, b]$ is the corresponding iteration function. The fixed-point iteration of the function ϕ is given by

$$x_{k+1} = \phi(x_k) \text{ for } k = 0, 1, 2, \cdots, \tag{2}$$

where the initial guess x_0 is given.

Definition 2.1. (Rate of Convergence) Let $\{x_k\}_{k=0}^{\infty}$ be a sequence in R^n that converges to $x^* \in R^n$ and assume that $x_k \neq x^*$ for each k. We say that the rate of convergence of $\{x_k\}$ to x^* is of order r, with asymptotic error constant C_r, if

$$\lim_{k\to\infty} \frac{\| x_{k+1} - x^* \|}{\| (x_k - x^*) \|^r} = C_r, \tag{3}$$

where x^* is the only exact solution of (1), $r \geq 1$ and $C_r > 0$. If $r = 1$, we say that convergence is linear. If $r = 2$, then the method converges quadratically, and if $r = 3$, we say it converges cubically, and so on.

Lemma 2.1: Assume that ϕ is the fixed-point function of the function f, then the following are true:

a. If $\phi(x) \in C[a, b]$, and $\phi(x) \in [a, b]$ for all $x \in [a, b]$, then ϕ has a fixed-point in $[a, b]$;

b. If $\phi'(x)$ exists for all $x \in (a, b)$ and there exists a positive

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constant $k < 1$ with $| \phi^{'}(x) | \leq k$ for all $x \in (a, b)$, then the fixed-point iteration (2) is linear convergent, and

$$\lim_{k\to\infty} \frac{x_{k+1} - x^*}{x_k - x^*} = C_1 = | \phi^{'}(x^*) |, \quad (4)$$

where $C_1 \neq 0$.

From (4), it is easy to obtain

$$x_{k+1} - x^* = C_1(x_k - x^*) + o(x_k - x^*). \quad (5)$$

It is well known that the Steffensen acceleration method is in the following form

$$y_k = \phi(x_k),$$

$$z_k = \phi(y_k)$$

and

$$\tilde{x}_{k+1} = x_k - \frac{(y_k - x_k)^2}{z_k - 2y_k + x_k} \text{ for } k = 0, 1, 2, \cdots. \quad (6)$$

Thus, we have the following Lemma.

Lemma 2.2: Assume that ϕ is the fixed-point function of the function f, that $\phi^{'}(x)$ exists, and that the fixed-point iteration (2) is linear convergent, then

$$\tilde{x}_{k+1} - x_k = -(x_k - x^*) + o(x_k - x^*). \quad (7)$$

Proof: It follows from (5) and (6) that

$$\begin{aligned} \tilde{x}_{k+1} - x_k &= -\frac{(y_k - x^* + x^* - x_k)^2}{z_k - x^* - 2(y_k - x^*) + (x_k - x^*)} \\ &= -\frac{(C_1 - 1)^2(x_k - x^*)^2 + o(x_k - x^*)^2}{(C_1 - 1)^2(x_k - x^*) + o(x_k - x^*)} \\ &= -(x_k - x^*) + o(x_k - x^*). \end{aligned}$$

The proof is complete. ■

By Lemma 2.2, we can get the following theorem.

Theorem 2.1: Assume that ϕ is the fixed-point function of the function f, that $\phi^{'}(x)$ exists, and that the fixed-point iteration (2) is linear convergent, then

$$\lim_{k\to\infty} \frac{\tilde{x}_{k+1} - x^*}{(x_k - x^*)^2} = C_2, \quad (8)$$

where $\tilde{x}_{k+1}$ is defined by (6) and C_2 is an arbitrary constant.

Proof: Let

$$\varepsilon(x^*) = \phi(x^*) - x^*,$$

$$\varepsilon(x_k) = y_k - x_k$$

and

$$\varepsilon(y_k) = z_k - y_k,$$

where x^* is the root of Equation (1) and is also the fixed-point of the function ϕ, and x_k, y_k and z_k are defined as in (6). Consider the following linear interpolation

$$\varepsilon(x) = L_1(x) + R_1(x), \quad (9)$$

where

$$L_1(x) = \frac{x - y_k}{x_k - y_k}\varepsilon(x_k) + \frac{x - x_k}{y_k - x_k}\varepsilon(y_k) \quad (10)$$

and

$$R_1(x) = \frac{\varepsilon^{''}(\xi_1)}{2}(x - x_k)(x - y_k). \quad (11)$$

The core idea of the Steffensen acceleration method is that the error function ε is extrapolated as "zero." That is to say

$$L_1(x) = 0. \quad (12)$$

Hence, the iterative error of this method is $R_1(x)$. It follows from (12) that

$$x = x_k - \frac{(y_k - x_k)^2}{z_k - 2y_k + x_k}.$$

Letting $\tilde{x}_{k+1} = x$, from the error term $R_1(x)$ in (11) we find

$$R_1(\tilde{x}_{k+1}) = \frac{\varepsilon^{''}(\xi_1)}{2}(\tilde{x}_{k+1} - x_k)(\tilde{x}_{k+1} - y_k). \quad (13)$$

By Lemma 2.2, we conclude

$$\tilde{x}_{k+1} - x_k = -(x_k - x^*) + o(x_k - x^*)$$

and

$$\tilde{x}_{k+1} - y_k = -C_2(x_k - x^*) + o(x_k - x^*).$$

Hence, we obtain

$$R_1(\tilde{x}_{k+1}) = C_2(x_k - x^*)^2 + o(x_k - x^*)^2.$$

Therefore, the relation (8) holds. The proof is complete. ■

Remark 2.1: The proof of Theorem 2.1 can be also written as

$$\begin{aligned} &\tilde{x}_{k+1} - x^* \\ &= x_k - x^* - \frac{((y_k - x^*) - (x_k - x^*))^2}{(z_k - x^*) - 2(y_k - x^*) + (x_k - x^*)} \\ &= x_k - x^* - \frac{(C_1 - 1)^2(x_k - x^*)^2 + o(x_k - x^*)^2}{(C_1 - 1)^2(x_k - x^*) + o(x_k - x^*)} \\ &= C_2(x_k - x^*)^2 + o(x_k - x^*)^2. \end{aligned}$$

Remark 2.2: The convergence order for the fixed-point iteration is generally linear, so the fixed-point method converges slowly and its effective use is quite limited. The convergence order of the Steffensen acceleration method is at least two from the aspect of interpolation theory. In order to speed up the convergence, a novel acceleration method based on the extrapolated idea will be constructed in this paper.

III. A NEW ACCELERATION SCHEME

Consider the quadratic interpolation for the error function ε. Let

$$\varepsilon(x) = L_2(x) + R_2(x), \quad (14)$$

where

$$L_2(x) = l_{xk}\varepsilon(x_k) + l_{yk}\varepsilon(y_k) + l_{xk}\varepsilon(z_k), \quad (15)$$

$$\begin{aligned} l_{xk} &= \frac{(x - y_k)(x - z_k)}{(x_k - y_k)(x_k - z_k)}, \\ l_{yk} &= \frac{(x - x_k)(x - z_k)}{(y_k - x_k)(y_k - z_k)}, \\ l_{zk} &= \frac{(x - x_k)(x - y_k)}{(z_k - x_k)(z_k - y_k)}, \end{aligned}$$

$$R_2(x) = \frac{\varepsilon'''(\xi_2)}{2}(x - x_k)(x - y_k)(x - z_k) \quad (16)$$

and

$$\varepsilon(z_k) = \phi(z_k) - z_k = r_k - z_k,$$

here $y_k = \phi(x_k), z_k = \phi(y_k)$ and $r_k = \phi(z_k)$.

Similar to the function ε in (9), the function $\varepsilon(x)$ in (14) can be also extrapolated as "zero."

Letting $L_2(x) = 0$, we obtain the following new scheme

$$\hat{x}_{k+1} = x_k - \frac{\varepsilon(x_k)^2}{\varepsilon(y_k) - \varepsilon(x_k)} + \frac{\varepsilon(y_k)\varepsilon(x_k)(\varepsilon(y_k)^2 - \varepsilon(x_k)\varepsilon(z_k))}{(\varepsilon(z_k) - \varepsilon(y_k))(\varepsilon(z_k) - \varepsilon(x_k))(\varepsilon(y_k) - \varepsilon(x_k))}. \quad (17)$$

One can easily see that the iterative error of this method is $R_2(\hat{x}_{k+1})$.

For the new proposed acceleration technique we have the following new convergence results.

Theorem 3.1: Assume that ϕ is the fixed-point function of the function f, that $\phi'(x)$ exists, and that the fixed-point iteration (2) is linear convergent, then

$$\lim_{k\to\infty} \frac{\hat{x}_{k+1} - x^*}{(x_k - x^*)^3} = C_3, \quad (18)$$

where $\hat{x}_{k+1}$ is defined by (17) and C_3 is an arbitrary constant.

The proof of Theorem 3.1 is similar to that of Theorem 2.1 and hence is omitted.

Remark 3.1: The result in Theorem 3.1 can be extended to nth-order interpolation function for the iterative error, and the error is at least convergent of order n.

Remark 3.2: The Stiffensen acceleration method (6) and the new scheme (17) can also be extended to nonlinear equations.

Let

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}, \quad (19)$$

where

$$\mathbf{F} = (f_1(\mathbf{X}), f_2(\mathbf{X}), \cdots, f_n(\mathbf{X}))^T$$

and

$$\mathbf{X} = (x_1, x_2, \cdots, x_n).$$

Similarly, let

$$\mathbf{X} = \Phi(\mathbf{X}), \quad (20)$$

where

$$\Phi(\mathbf{X}) = (\phi_1(\mathbf{X}), \phi_2(\mathbf{X}), \cdots, \phi_n(\mathbf{X}))^T.$$

If $\mathbf{X}^* = \Phi(\mathbf{X}^*)$, $\mathbf{X}^*$ is also called as a fixed-point of the function Φ.

The fixed-point iteration is similar to (2), and one has

$$\mathbf{X}^{(k+1)} = \Phi(\mathbf{X}^{(k)}), \quad (21)$$

where

$$\Phi(\mathbf{X}^{(k)}) = (\phi_1(\mathbf{X}^{(k)}), \phi_2(\mathbf{X}^{(k)}), \cdots, \phi_n(\mathbf{X}^{(k)}))^T.$$

Lemma 3.1: Assume that $\phi_i \in C^1(D)$, that $D \subset R^n$ and that there exists a constant $K < 1$ satisfying

$$\left|\frac{\partial \phi_i(\mathbf{X})}{\partial x_j}\right| \le \frac{K}{n} \ \ for\ \mathbf{X} \in D\ and\ \ j = 1, 2, \cdots, n,$$

then for any $\mathbf{X}^{(0)} \in D$, the serial $\{\mathbf{X}^{(k)}\}$ generated by (21) is convergent to $X^* \in D$, and

$$\|\mathbf{X}^{(k)} - \mathbf{X}\|_\infty \le \frac{K^k}{1-K}\|\mathbf{X}^{(1)} - \mathbf{X}^{(0)}\|_\infty,$$

where $\mathbf{X}^{(k)} \to \mathbf{X}^{(*)}$ means that $x_i^{(k)} \to x_i^{(*)}$, as $k \to \infty$ for any $i = 1, 2, \cdots, n$.

So, from Lemma 3.1 the Steffensen acceleration method for nonlinear equations is as follows:

$$\mathbf{Y}^{(k)} = \Phi(\mathbf{X}^{(k)}),$$

$$\mathbf{Z}^{(k)} = \Phi(\mathbf{Y}^{(k)}) = \Phi(\Phi(\mathbf{X}^{(k)}))$$

and

$$\tilde{x_i}^{(k+1)} = x_i^{(k)} - \frac{(y_i^{(k)} - x_i^{(k)})^2}{z_i^{(k)} - 2y_i^{(k)} + x_i^{(k)}} \text{ for } k = 0, 1, 2, \cdots. \quad (22)$$

Similarly, the new scheme for nonlinear equations is as follows:

$$\hat{x_i}^{(k+1)} = x_i^{(k)} - \frac{\varepsilon(x_i^{(k)})^2}{\varepsilon(y_i^{(k)}) - \varepsilon(x_i^{(k)})} + \frac{\varepsilon(y_i^{(k)})\varepsilon(x_i^{(k)})(\varepsilon(y_i^{(k)})^2 - \varepsilon(x_i^{(k)})\varepsilon(z_i^{(k)}))}{(\varepsilon(z_i^{(k)}) - \varepsilon(y_i^{(k)}))(\varepsilon(z_i^{(k)}) - \varepsilon(x_i^{(k)}))(\varepsilon(y_i^{(k)}) - \varepsilon(x_i^{(k)}))}$$

for $i = 1, 2, \cdots$. (23)

IV. Numerical tests

In this section, two examples are displayed for nonlinear equation and equations, respectively.

Example 4.1: The test function $f(x) = x^3 + 4x^2 - 10$ has a unique root

$$\bar{x} = 1.36523001341409684576080682898166660783\cdots.$$

The iteration equation can be rewritten as fifth-order iterative methods as follows:

$$(a)\quad x = \phi(x) = x - (x^3 + 4x^2 - 10);$$

$$(b)\quad x = \phi(x) = (\frac{10}{x} - 4x)^{\frac{1}{2}};$$

$$(c)\quad x = \phi(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}};$$

$$(d)\quad x = \phi(x) = (\frac{10}{4+x})^{\frac{1}{2}};$$

$$(e)\quad x = \phi(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}.$$

The Steffensen acceleration method and the new scheme will be applied to finding the approximate roots to x^*, respectively. The numerical results for Example 4.1 with the initial guess $x_0 = 1.5$ are shown in Tables I–III.

In these tables, $*$ means that the iteration is divergent, and the iteration will stop as

$$|x_{k+1} - x_k| < 10^{-6}.$$

TABLE I
RESULTS FOR THE FIXED-POINT ITERATION

N	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.81649658	1.28695376	1.34839972	1.3733333
2	6.73242187	2.99690871	1.40254078	1.36737636	1.36526201
3	-469.72001	$(-8.65)^{\frac{1}{2}}$	1.34545837	1.36495702	1.36523001
4	102754548	*	1.3751724	1.36526475	1.36523001
5	*		1.36009420	1.36522559	
6			1.36784697	1.36523058	
7			1.36388701	1.36522994	
8			1.36591676		
$\vdots$					
20			1.36523022		

TABLE II
RESULTS FOR THE STEFFENSEN ACCELERATION METHOD

N	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	0.93494423	1.33687476	1.36188649	1.36526519	1.36471271
2	1.00503287	1.36313034	1.36522825	1.36523000	1.36522996
3	1.07546268	1.36522011	1.36523004		
$\vdots$					
12	1.36523001				

TABLE III
RESULTS FOR THE NEW ACCELERATION SCHEME

N	(a)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5
1	1.365304947	1.421581864	1.365230083	1.365229845
2	1.474315405	1.382641673		
3	1.365267873	1.366221428		
4	1.365230203	1.365230203		
5	1365230083	1.365231395		

From these tables, one can see that the new acceleration scheme based on quadratic finite element approximation is the most quickly convergent. The iterative number is the least, only once. Since the new method is improved from quadratic interpolation, Equation (b) cannot be iterated four times and can't be used in the new method. This is a defect.

Example 4.2: Find the approximate roots $\mathbf{X}^* \in [1,2]$ of the following equations

$$\begin{cases} 3x_1 - cos(x_2x_3) - 0.5 = 0, \\ x_1^2 - 81(x_2+0.1)^2 + \sin x_3 + 1.06 = 0, \\ e^{-x_1x_2} + 20x_3 + \frac{10\pi-3}{3} = 0. \end{cases}$$

Choose the fixed-point functions as follows:

$$\begin{cases} x_1 = \frac{1}{3}(cos(x_2x_3) - 0.5), \\ x_2 = -0.1 + \frac{1}{9}\sqrt{x_1^2 + \sin x_3 + 1.06}, \\ x_3 = -\frac{1}{20}(e^{-x_1x_2} + \frac{10\pi-3}{60}). \end{cases}$$

The vector case of Steffensen acceleration method and the new scheme are also applied to these equations. Numerical results are displayed in Tables IV–VI.

Similarly, in Tables IV-VI, the iteration will stop as

$$\|\mathbf{X}^{(k+1)} - \mathbf{X}^{(k)}\|_\infty < 10^{-6}.$$

Obviously, the numerical results verify the validity of our theoretical results.

TABLE IV
RESULTS FOR THE THE FIXED-POINT ITERATION

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\| X^{(k)} - X^{(k-1)} \|_\infty$
0	0.1	0.1	-0.1	
1	0.49998334	0.00944115	-0.52310127	0.42310125
2	0.49999595	0.00002557	-0.52336329	0.00941558
3	0.5	0.00001234	-0.52359813	0.00023482
4	0.5	0.00000003	-0.52359843	0.00001230
5	0.5	0.00000002	-0.52359873	0.00000032

TABLE V
RESULTS FOR THE STEFFENSEN ACCELERATION METHOD

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\| X^{(k)} - X^{(k-1)} \|_\infty$
0	0.1	0.1	-0.1	
1	0.49999595	0.00002557	-0.52336329	0.42336330
2	0.5	0	-0.52359843	0.00023514
3	0.5	0	-0.52359873	0.00000030

TABLE VI
RESULTS FOR THE NEW ACCELERATION SCHEME

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\| X^{(k)} - X^{(k-1)} \|_\infty$
0	0.1	0.1	-0.1	
1	0.50000191	0.00002557	-0.52575636	0.42575637
2	0.5	-0.00000015	-0.52359873	0.000215763
3	0.5	0	-0.52359873	0.00000016

V. CONCLUSION

Fixed-point iterative method is one of the most powerful numerical methods for solving nonlinear equations. However, it requires multiple iterations to find the exact or approximate solution of nonlinear equations which is a little bit expensive in actual applications. Based on the idea that the iterative error function is extrapolated to zero for the Steffensen acceleration method, a novel acceleration scheme is constructed. The new scheme converges to exact solution more quickly than the Steffensen acceleration method and others. Furthermore, both the Steffensen acceleration method and the new scheme are extended to nonlinear equations. Numerical results verify the validity of the new scheme. Further studies on how to design a better method to accelerate the convergence rate and reduce the computation cost are still essential.

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REFERENCES

[1] Akbar Azam, and B. E. Rhoades, "Some Fixed Point Theorems for a Pair of Selfmaps of a Cone Metric Space," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 3, pp193-197, 2012
[2] E. Halley, "Methodus Nova Accurata et Facilis Inveniendi Radices AEquationum Quarumcumque Generaliter, Sine Praevia Reductione. Per Edm. Halley," *Philosophical Transactions*, vol. 18, pp136-148, 1964
[3] Sergio Amat, Sonia Busquier, and J. M. Gutierrez, " Geometric Constructions of Iterative Functions to Solve Nonlinear Equations," *Journal of Computational and Applied Mathematics*, vol. 157, no. 1, pp197-205, 2003
[4] R. L. Burden, and J. D. Fairs, "Numerical Analysis," *Beijing : Higher Education Press*, Ed. 7, pp43-80, 2001
[5] R. F. King, "Tangent Methods for Nonlinear Equations," *Numerische Mathematik*, vol. 18, no. 4, pp298-304, 1971
[6] Wilhelm Werner, " *über* ein Verfahren der Ordnung $1+\sqrt{2}$ zur Nullstellenbestimmung," *Numerische Mathematik*, vol. 32, no. 3, pp333-342, 1979
[7] A. M. Ostrowski, " Solution of Equations and Systems of Equations," *New York: Academic Press*, 1973
[8] P. Veninin, and R. Nascimbene, "A new Fixed-Point Algorithm for Hardening Plasticity Based on Non-linear Mixed Variational Inequalities," *International Journal for Numerical Methods in Engineering*, vol. 57, no, 1, pp83-102, 2003

[9] Z. J. Bai, and W. K. Ching, "A Smoothing Newtons Method for the Construction of a Damped Vibrating Systern From Noisy Test Eigendata," *Numerical Linear Algebra with Applications* , vol. 16, no. 2, pp109-128, 2009
[10] L. Zh. Lu, "Newton Iterations for a Non-symmetric Algebraic Riecati Equation," *Numerical Linear Algebra with Applications* , vol. 12, no. 2, pp191-200, 2005
[11] F. H. Ren, and YSAAD, "Two Classes of Multisecant Methods for Nonlinear Acceleration," *Numerical Linear Algebra with Applications*, vol. 16, no. 3, pp197-221, 2009
[12] K. J. Sheng, W. X. Hua, and Y. TLi, "Some Eighth-Order Root-Finding Three-step Methods," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 3, pp536-544, 2010
[13] L. X. Jiang, and Q. Chun, "Some Geometrical Iteration Methods for Nonlinear Equations," *Applied Mathematics: A Journal of Chinese Universities*, vol. 23, no. 1, pp25-30, 2008
[14] C. Y. Nie, and Z. H. Li, "A new Method for Finding the Root of Nonlinear Equation," *Journal of Yangzhou University (Natural Science Edition)*, vol. 13, no. 2, pp17-19, 2010
[15] H. Fukhar-ud-din, A. R. Khan, and M. A. A. Khan, "A New Implicit Algorithm of Asymptotically Quasi-nonexpansive Maps in Uniformly Convex Banach Spaces," *IAENG International Journal of Applied Mathematics*, vol. 42, no. 3, pp171-175, 2012
[16] O. Bumbariu, "A new Aitken Type Method for Accelerting Iterative Sequences," *Applied Mathematics and Computation*, vol. 219, no. 1, pp78-82, 2012
[17] A. Fdil, "A new Method for Solving a Nonliear Equation with Error Estimations," *Applied Numerical Mathematics*, vol. 21, no. 4, pp417-429, 1996
[18] W. Y. Liu, and K. Y. Ma, "An Iterative Non-overlapping Domain Decomposition Method for Optimal Boundary Control Problems Governed by Parabolic Equations," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 3, pp291-297, 2016
[19] J. Z. Deng, " Improvement on Steffensen Method for Acceleration Convergnece of the Iteration," *Journal of Xi'An Jiaotong University*, vol. 37, no. 3, pp99-104, 1993
[20] S. Amata, S. Busquier, J. A. Ezquerro, and M. A. Hernandez-Veronb, "A Steffensen Type Method of two Steps in Banach Spaces with Applications," *Journal of Computational and Applied Mathematics*, vol. 291, no. 1, pp317-331, 2016