Generalized Adams-Type Second Derivative Methods for Stiff Systems of ODEs

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Abstract—**In this paper, a family of second derivative generalized Adams-type methods (SDGAMs) is proposed. Here, a boundary value approach to the numerical solution of stiff initial value problems (IVPs) by means of second derivative linear multistep formulae (SDLMF) is presented. Stability analysis shows that these methods with order $p = 2k + 2$ for all values of the step-length $k \geq 1$ are all $0_{v,k-v}$-stable and $A_{v,k-v}$-stable which must be used with $(v, k - v)$-boundary conditions.**

Keywords: ***Linear Multistep Formulae, Boundary Value Methods,*** $0_{k_1,k_2}$***-stable,*** A_{k_1,k_2}***-stable.***

AMS subject classification: 65L04, 65L05

1 Introduction

Nwachukwu and Okor [25] introduced a class of second derivative generalized backward differentiation formulae (SDGBDF) applied as boundary value methods (BVMs) for stiff initial value problems (IVPs) in ordinary differential equations (ODEs)

$$y' = f(x, y),\ x \in [t_0, T],\ y(x_0) = y_0 \tag{1}$$

This class of methods which is $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with (v,k-v)-boundary conditions and order $p = k + 1$ for values of the steplength $k \geq 1$ is of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_v f_{n+v} + h^2 f'_{n+v} \tag{2}$$

where

$$v = \begin{cases} \frac{k+2}{2} & for\ even\ k \\ \frac{k+1}{2} & for\ odd\ k \end{cases}. \tag{3}$$

and is used with the following additional initial methods:

$$\sum_{j=0}^{k} \alpha_j^* y_j = h\beta_i f_i + h^2 f'_i, \qquad i = 1, 2, \cdots, v-1 \tag{4}$$

and final methods:

$$\sum_{j=0}^{k} \alpha_j^* y_j = h\beta_i f_i + h^2 f'_i, \qquad i = v+1, \cdots, N \tag{5}$$

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The SDGBDF of Nwachukwu and Okor [25] has better stability properties than the GBDF of Brugnano and Trigiante [7] which is also a BVM.

BVMs are linear multistep methods whose main feature consists in approximating a given continuous IVP by means of a discrete boundary value problem (BVP). The solution of the initial value problem is given simultaneously at all grid points. This boundary value approach circumvents the well known Dahlquist-barriers on convergence and stability. BVMs, also provide several families of methods which make them very flexible and computationally efficient. These methods have been analyzed in details in [1, 4, 5, 6, 8, 24, 25, 26]

Brugnano and Trigiante [7] derived a family of methods called the generalized Adams methods (GAMs) of order $k+1$ which are $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with $(v, k-v)$-boundary conditions for $k \geq 1$. These methods can be written as

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k} \beta_i f_{n+i} \tag{6}$$

where

$$v = \begin{cases} \frac{k}{2} & for\ \ even\ \ k \\ \frac{k+1}{2} & for\ \ odd\ \ k \end{cases} \tag{7}$$

They are conveniently used with the following set of additional initial methods,

$$y_j - y_{j-1} = h\sum_{i=0}^{k} \beta_i^{(j)} f_i, \qquad j = 1, ..., v-1, \tag{8}$$

and final ones,

$$y_j - y_{j-1} = h\sum_{i=0}^{k} \beta_{k-i}^{(j)} f_{N-i}, \quad j = N-k+v+1, ..., N. \tag{9}$$

GAMs have better stability regions than the Reverse Adams methods of Brugnano and Trigiante [2]. The Reverse Adams methods can also be used to approximate the solution of stiff problems since for $k \leq 8$ the $(1, k-1)$-Absolute stability regions are unbounded. Although they have good stability properties in comparison with the Adams-Moulton methods they do not provide very high order methods suitable for stiff problems. The trapezoidal rule which has order 2 is the only Adams-Moulton method appropriate for solving stiff problems because it has an unbounded Absolute stability region.

Still on Adams methods, Jator and Sahi [21] proposed a family of second derivative Adams-type methods (SDAMs) of order up to $p = 2k+2$ (k is the step number) for IVPs. These methods are used as initial value methods (IVMs). The class of IVMs is a subclass of BVMs

The aim of this paper is to develop a family of second derivative generalized Adams-type methods (SDGAMs) for stiff systems of ODEs (1) with higher order than the SDGBDF of Nwachukwu and Okor [25] and the GAMs of Brugnano and Trigiante [7]. The developed methods (see (10)) herein generalizes the second derivative methods of Jator and Sahi [21] and extends the GAMs of Brugnano and Trigiante to second derivative methods. The need for second derivative is to improve the stability and order of a given step length k of the GAMs, see [15]. Also the proposed methods will be implemented using BVMs as in [2, 3, 4, 8, 9, 10, 11, 12, 13, 14, 18, 21, 25, 26, 27] so that the numerical solution $(y_1, y_2, \cdots, y_N)^T$ of the initial value problem is given simultaneously at all the grid points.

This article is organized as follows: Section 2 introduces the new methods. Section 3 considers the stability analysis of the methods. Section 4 presents the implementation strategy of the methods. Finally some problems are solved to show comparison with some related works.

2 The New Methods

Let us consider the new SDGAMs of the form

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\gamma_i g_{n+i} \tag{10}$$

where

$$v = \begin{cases} \frac{k+1}{2} & for \quad odd \quad k \\ \frac{k}{2} & for \quad even \quad k \end{cases} \tag{11}$$

$y_{n+i} \approx y(x_n + ih)$, $f_{n+i} \equiv f(x_n + ih, y(x_n + ih))$ and $g_{n+i} \equiv \frac{df(x,y(x))}{dx}\Big|_{y=y_{n+i}}^{x=x_{n+i}}$

x_n is a discrete point at node point n, β_i and γ_i are parameters to be determined by imposing the formula (10) to reach its highest possible order which is $2k + 2$.

The proposed class of methods (10) can be written in the form

$$\begin{aligned} &y(x + vh) - y(x + (v-1)h) = \\ &h\textstyle\sum_{i=0}^{k}\beta_i y'(x + ih) + h^2\sum_{i=0}^{k}\gamma_i y''(x + ih) \end{aligned} \tag{12}$$

The approach here is to employ Taylor's series expansion and method of undetermined coefficients to generate the main methods and their coefficients at different values of k as shown in Tables 1.

As in [16] and [23], the local truncation error associated with (10) is the linear difference operator

$$\begin{aligned} &L[y(x); h] = y(x + vh) - y(x + (v-1)h) \\ &-h\textstyle\sum_{i=0}^{k}\beta_i y'(x + ih) - h^2\sum_{i=0}^{k}\gamma_i y''(x + ih) \end{aligned} \tag{13}$$

Assuming that $y(x)$ is sufficiently differentiable, by Taylor series expansion of $y(x+ih)$, $y'(x+ih)$ and $y''(x+ih)$, $i = 0, 1, 2, ..., k$ the $L[y(x); h]$ becomes

$$\begin{aligned} &L[y(x); h] = \\ &C_0 y(x) + C_1 h y'(x) + \cdots + C_q h^q y^{(q)}(x) + \cdots \end{aligned} \tag{14}$$

where

$$C_0 = \sum_{i=0}^{k}\alpha_i,$$

$$C_1 = 1 - \sum_{i=0}^{k}\beta_i,$$

$$C_2 = \frac{1}{2}(v^2 - (v-1)^2) - \sum_{i=1}^{k} i\beta_i - \sum_{i=0}^{k}\gamma_i,$$

$$\vdots$$

$$C_q = \frac{1}{q!}(v^q - (v-1)^q) - \frac{1}{(q-1)!}\sum_{i=1}^{k} i^{q-1}\beta_i - \frac{1}{(q-2)!}\sum_{i=1}^{k} i^{q-2}\gamma_i,$$

The method (10) is said to be of order p if

$$C_j = 0, \ j = 0(1)p \quad and \quad C_{p+1} \neq 0, \tag{15}$$

Therefore C_{p+1} is the error constant (EC) and $C_{p+1}h^{p+1}y^{p+1}(x_n)$ is the principal LTE at the point x_n. Hence the LTE is given as

$$LTE = C_{p+1}h^{p+1}y^{p+1}(x_n) + O(h^{p+2})$$

The matrix form of the order equations is given in (21)

In table 1, for $k = 1(1)10$, the order and the error constant of the SDGAMs (10) are given.

3 Stability Analysis of the methods

The methods (10) can be written compactly as

$$\rho(E) = h\sigma(E)f_n + h^2\eta(E)g_n \tag{16}$$

where $\rho(w) = w^{v-1}(w-1)$, $\sigma(w) = \sum_{i=0}^{k}\beta_i w^i$ and $\eta(w) = \sum_{i=0}^{k}\gamma_i w^i$ are the first, second and third characteristic polynomial respectively, $w \in C$ and $E^i y_n = y_{n+i}$ is the shift operator ([19]). According to [20] the stability analysis is achieved through linearization with the scalar test equations

$$y' = \lambda y \quad and \quad y'' = \lambda^2 y \tag{17}$$

which when applied to (16) gives the characteristics equation

$$\pi(w,z) = w^v - w^{v-1} - w^i \sum_{i=0}^{k} (z\beta_i + z^2\gamma_i), \qquad (18)$$

where $z = \lambda h$. Then equating (18) to zero and inserting $w = e^{t\theta}$, $t = 0(1)k$, $\theta \in [0, 2\pi]$ in (18) yields a polynomial of degree two in z. The two roots of z are functions of θ describing the stability domain of the SDGAMs (10). For even values of k the stability region of the SDGAMs (10) is given in Figure 1. For odd values of k the methods (10) whose boundary loci coincide with the imaginary axis is characterized by the following properties:

The polynomials $\rho(w)$ have skew-symmetric coefficients, $\sigma(w)$ have symmetric coefficients and $\eta(w)$ have skew-symmetric coefficients $\alpha_i = -\alpha_{k-i}$, $\beta_i = \beta_{k-i}$ and $\gamma_i = -\gamma_{k-i}$, $\quad i = 0, 1, \cdots, k$ respectively.

The new methods which are $0_{v,k-v}$-stable and $A_{v,k-v}$-stable are used with $(v, k-v)$-boundary conditions, see Figure1. For the definitions of BVM, IVM, $0_{v,k-v}$-stable and $A_{v,k-v}$-stable, see [8, 25].

4 Implementation Strategy of the Methods

In what follows, we give the implementation strategy of the new family of the Adams-type developed in the previous section for the numerical solution of systems of stiff IVPs. The SDGAMs (10) are conveniently used with the following set of additional methods which we define generally as:

initial methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k} \beta_i f_i + h^2 \sum_{i=0}^{k} \gamma_i g_i, \qquad j = 1, \ldots, v-1 \qquad (19)$$

and final methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k} \beta_{k-i} f_{N-i} + h^2 \sum_{i=0}^{k} \gamma_{k-i} g_{N-i}, \qquad (20)$$

$$j = N-k+v+1, \ldots, N$$

The coefficient of the additional methods are determined by imposing each formula to reach its highest possible order $2k+2$ as the main method. In order to solve the discrete problem, we add $v-1$ initial and $k-v$ final additional equations (conditions) since the continuous problem provides only the initial condition y_0. Hence we treat the additional conditions as unknowns. Additional methods having the same order as the main method are introduced in order to preserve the order of the main method.

The sixth order SDGAM,

$$\begin{aligned} y_{n+1} - y_n &= h\left(\frac{101}{240} f_n + \frac{8}{15} f_{n+1} + \frac{11}{240} f_{n+2}\right) \\ &+ h^2\left(\frac{13}{240} g_n - \frac{1}{6} g_{n+1} - \frac{1}{80} g_{n+2}\right), \end{aligned}$$

$$n = 1, \ldots, N-1$$

can be used with the following final additional method

$$\begin{aligned} y_N - y_{N-1} &= h\left(\frac{11}{240} f_{N-2} + \frac{8}{15} f_{N-1} + \frac{101}{240} f_N\right) \\ &+ h^2\left(\frac{1}{80} g_{N-2} + \frac{1}{6} g_{N-1} - \frac{13}{240} g_N\right) \end{aligned}$$

The eighth order SDGAM,

$$\begin{aligned} y_n - y_{n-1} &= \frac{h}{224}(3f_{n-2} + 109 f_{n-1} + 109 f_n + 3 f_{n+1}) \\ &+ \frac{h^2}{10080}(31 g_{n-2} + 1017 g_{n-1} - 1017 g_n) \\ &- \frac{h^2}{10080}(31 g_{n+1}), \end{aligned}$$

$$n = 2, \ldots, N-1$$

can be used with the following two additional (one initial and one final) methods

$$\begin{aligned} y_1 - y_0 &= \frac{h}{18144}(6893 f_0 + 8451 f_1 + 2403 f_2 + 397 f_3) \\ &+ \frac{h^2}{30240}(1283 g_0 - 7659 g_1 - 2421 g_2 - 163 g_3), \end{aligned}$$

$$\begin{aligned} y_N - y_{N-1} &= \frac{h}{18144}(397 f_{N-3} + 2403 f_{N-2}) \\ &+ \frac{h}{18144}(8451 f_{N-1} + 6893 f_N) \\ &+ \frac{h^2}{30240}(163 g_{N-3} + 2421 g_{N-2}) \\ &+ \frac{h^2}{30240}(7659 g_{N-1} - 1283 g_N). \end{aligned}$$

The tenth order SDGAM,

$$\begin{aligned} y_n - y_{n-1} &= \frac{h}{4354560}(26081f_{n-2} + 1957456f_{n-1}) \\ &+ \frac{h}{4354560}(2163456f_n + 193456f_{n+1}) \\ &+ \frac{h}{4354560}(14111f_{n+2}) + \frac{h^2}{725760}(893g_{n-2}) \\ &+ \frac{h^2}{725760}(55096g_{n-1} - 106596g_n) \\ &- \frac{h^2}{725760}(13768g_{n+1} + 515g_{n+2}), \\ &\quad n = 2, \ldots, N-2 \end{aligned}$$

can be used with the following initial method,

$$\begin{aligned} y_1 - y_0 &= \frac{h}{4354560}(1539551f_0 + 1429936f_1) \\ &+ \frac{h}{4354560}(711936f_2 + 613456f_3 + 59681f_4) \\ &+ \frac{h^2}{725760}(26051g_0 - 249656g_1) \\ &- \frac{h^2}{725760}(183708g_2 + 49720g_3 + 2237g_4) \end{aligned}$$

and the two final additional methods,

$$\begin{aligned} y_N - y_{N-1} &= \frac{h}{4354560}(14111f_{N-3} + 193456f_{N-2}) \\ &+ \frac{h}{4354560}(2163456f_{N-1} + 1957456f_N) \\ &+ \frac{h}{4354560}(26081f_{N+1}) + \frac{h^2}{725760}(515g_{N-3}) \\ &+ \frac{h^2}{725760}(13768g_{N-2} + 106596g_{N-1}) \\ &- \frac{h^2}{725760}(55096g_N + 893g_{N+1}), \end{aligned}$$

$$\begin{aligned} y_{N+1} - y_N &= \frac{h}{4354560}(59681f_{N-3} + 613456f_{N-2}) \\ &+ \frac{h}{4354560}(711936f_{N-1} + 1429936f_N) \\ &+ \frac{h}{4354560}(1539551f_{N+1}) + \frac{h^2}{725760}(2237g_{N-3}) \\ &+ \frac{h^2}{725760}(49720g_{N-2} + 183708g_{N-1}) \\ &+ \frac{h^2}{725760}(249656g_N - 26051g_{N+1}). \end{aligned}$$

5 Numerical results

All numerical computations were carried out using MATLAB. Both linear and non-linear problems were solved with boundary layers. The SDGAMs (10) of order 6, 8 and 10 are applied to numerical examples to illustrate the accuracy and the efficiency of the new methods on some systems of stiff IVPs. Note that the SDGAMs (10) of orders six, eight and ten are denoted by SDGAM6, SDGAM8 and SDGAM10 respectively.

Problem 1: Singularly Perturbed Problem ([20])

$$y_1' = -(2 + 10^4)y_1 + 10^4 y_2^2, \qquad y_2' = y_1 - y_2 - y_2^2,$$

$$y_1(0) = 1, \quad y_2(0) = 1$$

The exact solution is $y_1 = e^{-2t}, \quad y_2 = e^{-t}$

The numerical results for problem 1 are presented in Table 6. From table 6, it can be seen that the implementation using SDGAMs (10) is better than the SDGBDF of Nwachukwu and Okor [25] with respect to the steplength k.

Problem 2: Moderately stiff problem solved by Jia-Xiang and Jiao-Xun [22],

$$y' = -y - 10z, \quad y(0) = 1; \quad y(x) = e^{-x}cos10x$$

$$z' = 10y - z, \quad z(0) = 0; \quad z(x) = e^{-x}sin10x$$

This problem is solved using step sizes $h = \{0.04, 0.1, 0.4\}$ and the maximum errors $(Max||y_i - y(x_i)||)$ in the interval $0 < x < 10$ are computed. x_T are some points on the range of integration. The numerical results displayed in Table 7 show that the SDGAMs (10) is more accurate than the DBDF method of Jia-Xiang and Jiao-Xun [22], the BDF of Gear [17] and the BVM of Ehigie et al [14].

Problem 3: Linear stiff problem considered by Amodio and Mazzia [2] and Jator and Sahi [21] on the range $0 \leq x \leq 1$

$$\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3, & y_1(0) &= 1, \\ y_2' &= 19y_1 - 21y_2 + 20y_3, & y_2(0) &= 0, \\ y_3' &= 40y_1 - 40y_2 + 40y_3, & y_3(0) &= -1. \end{aligned}$$

The exact solution of the system is given by

$$\begin{aligned} y_1(x) &= \frac{1}{2}(e^{-2x} + e^{-40x}(cos(40x) + sin(40x))), \\ y_2(x) &= \frac{1}{2}(e^{-2x} - e^{-40x}(cos(40x) + sin(40x))), \\ y_3(x) &= \frac{1}{2}(2e^{-40x}(sin(40x) - cos(40x))). \end{aligned}$$

Problem 3 was solved using our methods for $k = 2$ and $k = 3$. From the details of the numerical results given in Table 8, it is obvious that our method performed excellently compared with Amodio and Mazzia [2] and Jator and Sahi [21].

Problem 4: Robertson's equation, see [20] (nonlinear problem)

$$y_1' = -0.04y_1 + 10^4 y_2 y_3, \quad y_2' = 0.04y_1 - 10^4 y_2 y_3 - 3\times 10^7 y_2^2,$$

$$y_3' = 3\times 10^7 y_2^2, \quad y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0.$$

The eighth order SDGAM and the tenth order SDGAM are implemented using Problem 4. The results are compared with the solution from the Ode15s in MATLAB. The solid lines are the solutions of the SDGAMs. Figures 2 and 3 show that the new methods coincide with the Ode15s in MATLAB.

Problem 5: Van der Pol equations (nonlinear problem), [20]

$$y_1' = y_2, \qquad y_2' = -y_1 + 10y_2(1 - y_1^2),$$

$$y_1(0) = 2, \quad y_2(0) = 0$$

The results of this problem using the SDGAMs (10) of order $p = 8$ and $p = 10$ are presented in Figures 4 and 5 respectively. It is seen from the figures that the new methods are very comparable with the Ode15s in MATLAB.

6 Conclusion

In this paper a new family of second derivative generalized Adams-type methods (SDGAMs) is considered for the numerical solution of stiff IVPs in ODEs. The new formulas are found to be $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with (v,k-v)-boundary conditions for all values of $k \geq 1$ and are of order $p = 2k + 2$. We have shown the accuracy of the proposed class of methods on some stiff (both linear and non-linear) systems. From Figures 2, 3, 4 and 5 it is observed that the proposed methods (10) compare favorably with Matlab Ode15s. From tables 6, 7 and 8 the numerical results show that the methods (10) perform better than the existing methods cited in the literature and are well suited for the integration of stiff systems in ODEs.

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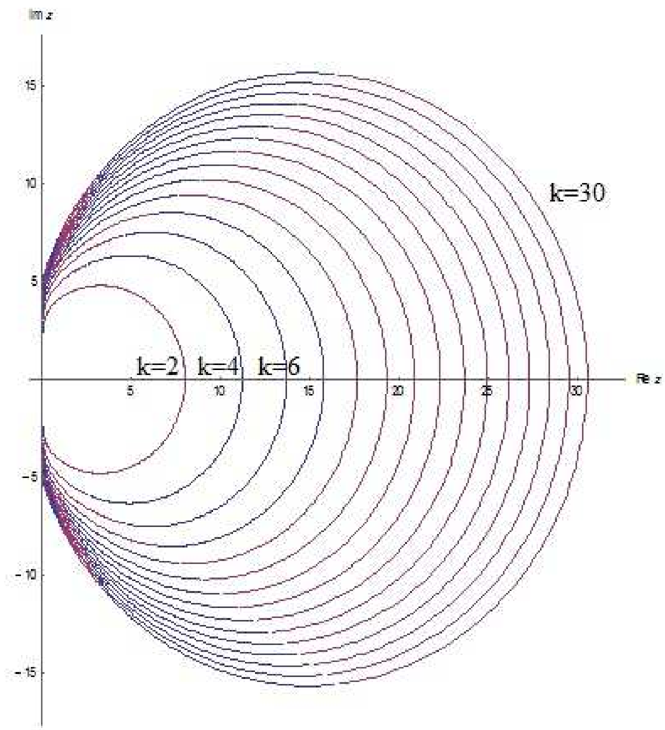

Figure 1: Stability region (exterior of closed curves) of (10), k=2 (2) 30

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ (v-1) & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ (v-1)^2 & 0 & 2 & \dots & 2*k & 2 & 2 & \dots & 2 \\ (v-1)^3 & 0 & 3 & \dots & 3*k^2 & 0 & 3*2 & \dots & 3*2*k \\ (v-1)^4 & 0 & 4 & \dots & 4*k^3 & 0 & 4*3 & \dots & 4*3*k^2 \\ (v-1)^5 & 0 & 5 & \dots & 5*k^4 & 0 & 5*4 & \dots & 5*4*k^3 \\ (v-1)^6 & 0 & 6 & \dots & 6*k^5 & 0 & 6*5 & \dots & 6*5*k^4 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ (v-1)^q & 0 & q & \dots & q*k^{(q-1)} & 0 & q*(q-1) & \dots & q*(q-1)k^{(q-2)} \end{bmatrix} \begin{bmatrix} -1 \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \\ \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_k \end{bmatrix} = \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \\ \vdots \\ v^q \end{bmatrix} \tag{21}$$

Table 1: The Coefficients, Error Constant (EC) and Order p of the classes of methods (10) for $k = 1(1)10$

k	v	β_0	β_1	β_2	β_3
1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0
2	1	$\frac{101}{240}$	$\frac{8}{15}$	$\frac{11}{240}$	0
3	2	$\frac{3}{224}$	$\frac{109}{224}$	$\frac{109}{224}$	$\frac{3}{224}$
4	2	$\frac{26081}{4354560}$	$\frac{122341}{272160}$	$\frac{313}{630}$	$\frac{12091}{272160}$
5	3	$\frac{4001}{4561920}$	$\frac{3581}{168960}$	$\frac{136267}{285120}$	$\frac{136267}{285120}$
6	3	$\frac{18227803}{56609280000}$	$\frac{156486943}{12972960000}$	$\frac{758335087}{1660538880}$	$\frac{587192}{1216215}$
7	4	$\frac{217426757}{3736212480000}$	$\frac{72234599}{29889699840}$	$\frac{384031751}{15375360000}$	$\frac{1569368687}{3321077760}$
8	4	$\frac{275618952431}{14227497123840000}$	$\frac{97292813749}{88921857024000}$	$\frac{13323334967}{814302720000}$	$\frac{1942806486353}{4234374144000}$
9	5	$\frac{27673701304843}{7224981721251840000}$	$\frac{50714562811}{200610349056000}$	$\frac{139894823711}{35416577064960}$	$\frac{3534206761517}{131650541568000}$
10	5	$\frac{331823317063312681}{275891087727230976000000}$	$\frac{970724650372992181}{9656188070453084160000}$	$\frac{218383591648975999}{105966398578360320000}$	$\frac{10765992822744097}{558807180003072000}$

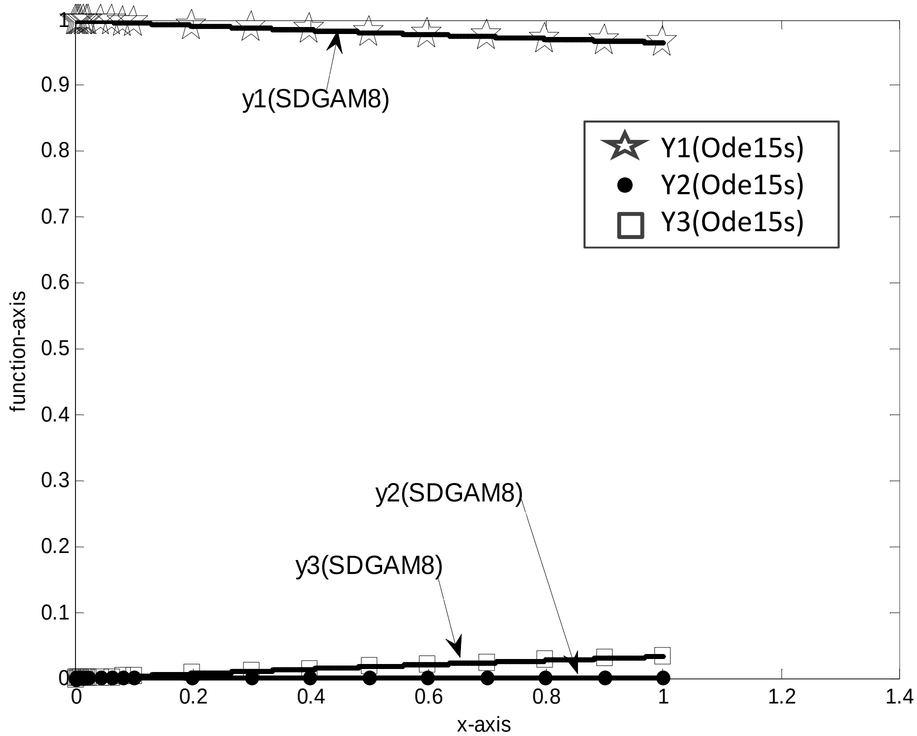

Figure 2: Numerical Results for Problem 4 using the eighth order SDGAM

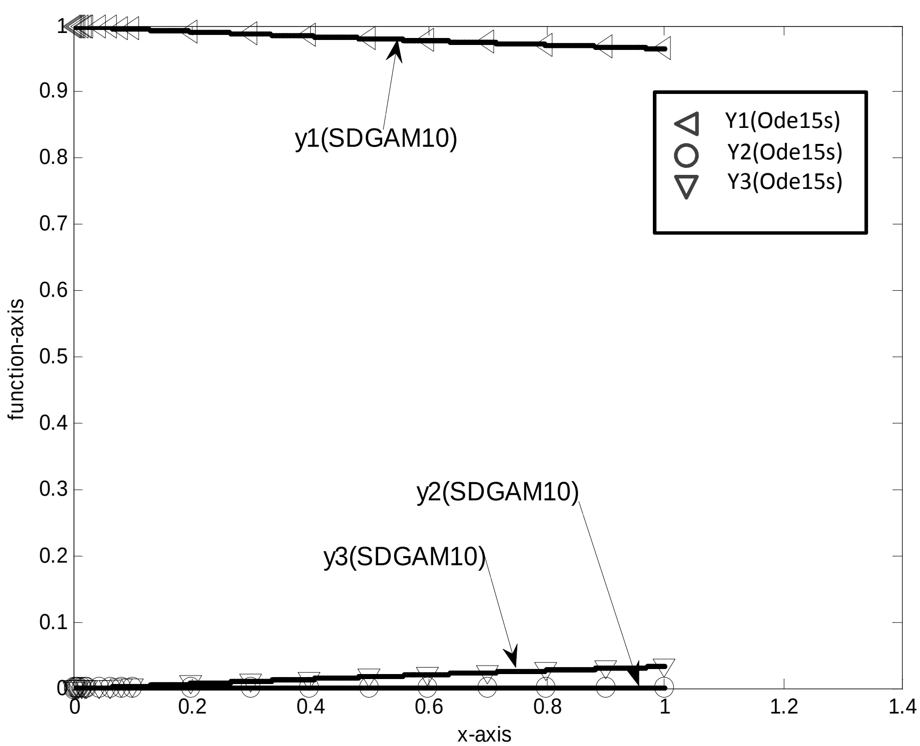

Figure 3: Numerical Results for Problem 4 using the tenth order SDGAM

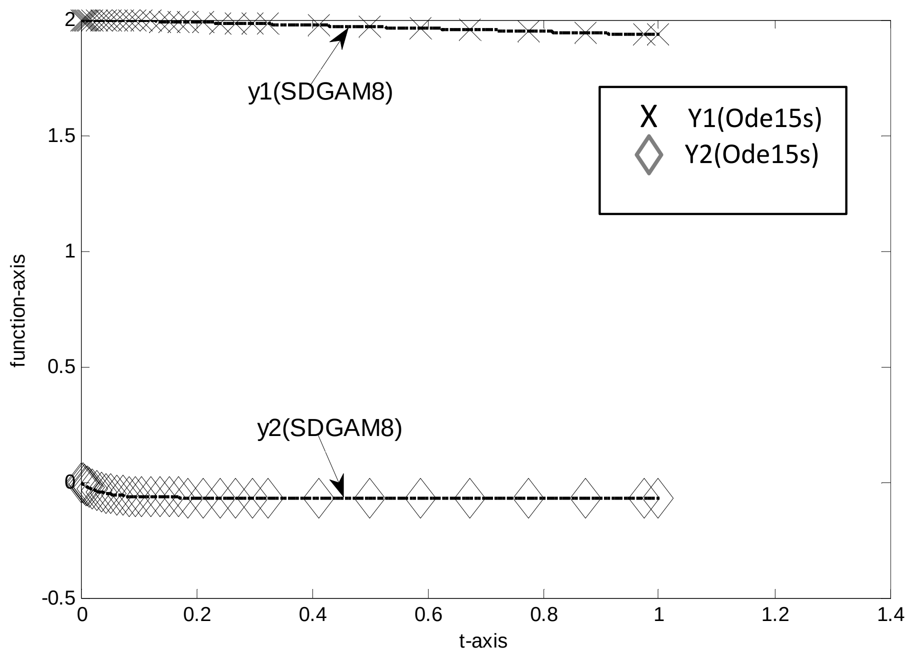

Figure 4: Numerical Results for Problem 5 using the eighth order SDGAM

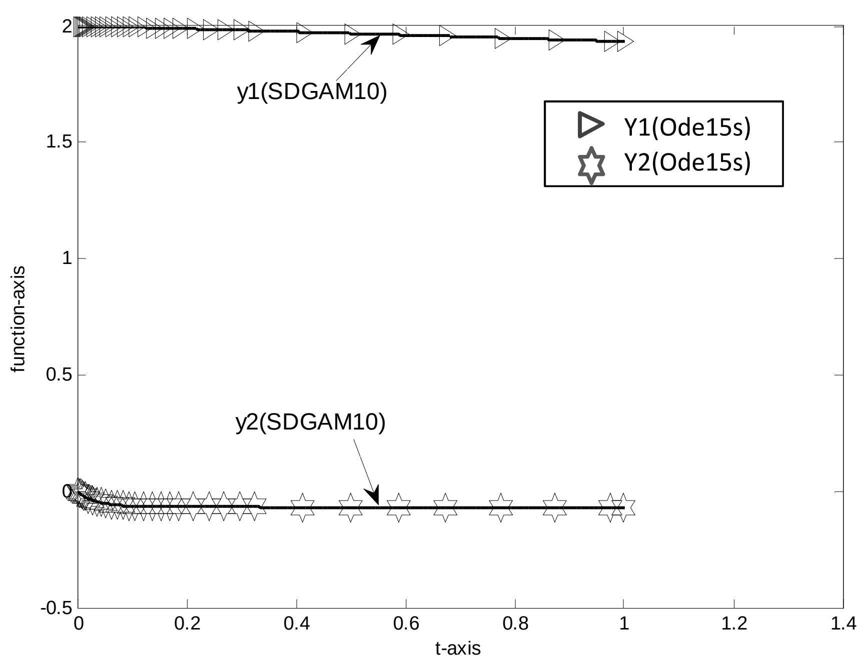

Figure 5: Numerical Results for Problem 5 using the tenth order SDGAM

Table 2: Table 1 continued

k	β_4	β_5	β_6	β_7	β_8
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	$\frac{14111}{4354560}$	0	0	0	0
5	$\frac{3581}{168960}$	$\frac{4001}{4561920}$	0	0	0
6	$\frac{68960401}{1660538880}$	$\frac{82799329}{12972960000}$	$\frac{4173367}{18869760000}$	0	0
7	$\frac{1569368687}{3321077760}$	$\frac{384031751}{15375360000}$	$\frac{72234599}{29889699840}$	$\frac{217426757}{3736212480000}$	0
8	$\frac{314527667}{661620960}$	$\frac{827205144427}{21171870720000}$	$\frac{4146181343}{488581632000}$	$\frac{331496225143}{444609285120000}$	$\frac{14004257503}{948499808256000}$
9	$\frac{34298478405773}{73139189760000}$	$\frac{34298478405773}{73139189760000}$	$\frac{3534206761517}{131650541568000}$	$\frac{139894823711}{35416577064960}$	$\frac{50714562811}{200610349056000}$
10	$\frac{47895787061277899}{104267228921856000}$	$\frac{1387127254973}{2946280837500}$	$\frac{19354866336237577}{521336144609280000}$	$\frac{3916362537630989}{399147985716480000}$	$\frac{1094574383125943}{784936285765632000}$

Table 3: Table 1 continued

k	β_9	β_{10}	γ_0	γ_1
1	0	0	$\frac{1}{12}$	$\frac{-1}{12}$
2	0	0	$\frac{13}{240}$	$\frac{-1}{6}$
3	0	0	$\frac{31}{10080}$	$\frac{113}{1120}$
4	0	0	$\frac{893}{725760}$	$\frac{6887}{90720}$
5	0	0	$\frac{313}{1774080}$	$\frac{39517}{5322240}$
6	0	0	$\frac{3777757}{62270208000}$	$\frac{59197}{16016000}$
7	0	0	$\frac{3972713}{373621248000}$	$\frac{4203911}{5748019200}$
8	0	0	$\frac{343925311}{101624979456000}$	$\frac{386887999}{1270312243200}$
9	$\frac{27673701304843}{7224981721251840000}$	0	$\frac{369733300393}{567677135241216000}$	$\frac{367112146447}{5406448907059200}$
10	$\frac{147407778627852047}{1931237614090616832000}$	$\frac{1877160528485018593}{1931237614090616832000000}$	$\frac{43303804162621}{218961180735897600000}$	$\frac{195229126044109}{7663641325756416000}$

Table 4: Table 1 continued

k	γ_2	γ_3	γ_4	γ_5	γ_6
1	0	0	0	0	0
2	$\frac{-1}{80}$	0	0	0	0
3	$\frac{-113}{1120}$	$\frac{-31}{10080}$	0	0	0
4	$\frac{-47}{320}$	$\frac{-1721}{90720}$	$\frac{-103}{145152}$	0	0
5	$\frac{143471}{1330560}$	$\frac{-143471}{1330560}$	$\frac{-39517}{5322240}$	$\frac{-313}{1774080}$	0
6	$\frac{24151013}{276756480}$	$\frac{-37969}{272160}$	$\frac{-6228251}{276756480}$	$\frac{-309979}{144144000}$	$\frac{-379397}{8895744000}$
7	$\frac{153317011}{13837824000}$	$\frac{1666591847}{14944849920}$	$\frac{-1666591847}{14944849920}$	$\frac{-153317011}{13837824000}$	$\frac{-4203911}{5748019200}$
8	$\frac{85580953}{13571712000}$	$\frac{119713951991}{1270312243200}$	$\frac{-56714731}{418037760}$	$\frac{-17444761861}{705729024000}$	$\frac{-90058579}{24429081600}$
9	$\frac{3607240110641}{2365321396838400}$	$\frac{96373826609}{6895980748800}$	$\frac{109868350967579}{965437304832000}$	$\frac{-109868350967579}{965437304832000}$	$\frac{-96373826609}{6895980748800}$
10	$\frac{548860751149819}{756902846988288000}$	$\frac{46346126414417}{5321973142886400}$	$\frac{687900349648999}{6951148594790400}$	$\frac{-383141112401}{2874009600000}$	$\frac{-011785775029757}{34755742973952000}$

Table 5: Table 1 continued

k	γ_7	γ_8	γ_9	γ_{10}	EC	p
1	0	0	0	0	$\frac{1}{720}$	4
2	0	0	0	0	$\frac{-1}{9450}$	6
3	0	0	0	0	$\frac{103}{25401600}$	8
4	0	0	0	0	$\frac{-89}{314344800}$	10
5	0	0	0	0	$\frac{379397}{28768836096000}$	12
6	0	0	0	0	$\frac{-3901}{4382752374000}$	14
7	$\frac{-3972713}{373621248000}$	0	0	0	$\frac{1964407}{435971161866240000}$	16
8	$\frac{-1364044741}{63515612160000}$	$\frac{-1964407}{752777625600}$	0	0	$\frac{724523791}{24258218831919072000 0}$	18
9	$\frac{-3607240110641}{2365321396838400}$	$\frac{-367112146447}{5406448907059200}$	$\frac{-369733300393}{567677135241216000}$	0	$\frac{22424299416863}{14159037167814523944960000 0}$	20
10	$\frac{-19412103395197}{3801409387776000}$	$\frac{-15510407670221}{302761138795315 20}$	$\frac{-30149869152983}{1532728265151283200}$	$\frac{-22424299416863}{1393389331955712000000}$	$\frac{346654620623}{333926511330781985616000000}$	22

Table 6: Absolute error in problem 1, $h = 0.01$, Error y_i=$|y_i - y(x_i)|$, $i = 1, 2$

x	y_i	Error in SDGAM8 (k=3,p=8)	Error in SDGBDF4 [25] (k=3,p=4)	Error in SDGAM10 (k=4,p=10)	Error in SDGBDF5 [25] (k=4,p=5)
1.0	y_1	1.18313×10^{-10}	3.06126×10^{-11}	2.07187×10^{-12}	3.43744×10^{-11}
	y_2	4.04676×10^{-13}	4.22623×10^{-11}	1.92618×10^{-12}	4.96455×10^{-11}
2.0	y_1	1.54753×10^{-13}	1.03235×10^{-11}	2.03823×10^{-13}	5.15573×10^{-12}
	y_2	1.47077×10^{-13}	3.96899×10^{-11}	7.08739×10^{-13}	1.91052×10^{-11}
3.0	y_1	6.23676×10^{-15}	2.39019×10^{-12}	2.61852×10^{-14}	6.49362×10^{-13}
	y_2	5.19376×10^{-14}	2.40044×10^{-11}	2.60764×10^{-13}	6.79007×10^{-12}
4.0	y_1	7.85992×10^{-16}	4.31932×10^{-13}	3.51840×10^{-15}	9.57145×10^{-14}
	y_2	2.08930×10^{-14}	1.20298×10^{-11}	9.59406×10^{-14}	2.61306×10^{-12}
5.0	y_1	9.34040×10^{-17}	8.00396×10^{-14}	4.75368×10^{-16}	1.20228×10^{-14}
	y_2	7.37951×10^{-15}	5.82196×10^{-12}	3.52695×10^{-14}	9.28587×10^{-13}
6.0	y_1	1.15866×10^{-17}	1.27167×10^{-14}	6.42813×10^{-17}	1.77133×10^{-15}
	y_2	2.60686×10^{-15}	2.56518×10^{-12}	1.29662×10^{-14}	3.57306×10^{-13}
7.0	y_1	1.87491×10^{-18}	2.18299×10^{-15}	8.69321×10^{-18}	2.22482×10^{-16}
	y_2	1.04864×10^{-15}	1.15005×10^{-12}	4.76669×10^{-15}	1.26970×10^{-13}
8.0	y_1	2.34152×10^{-19}	3.27871×10^{-16}	1.17567×10^{-18}	3.27798×10^{-17}
	y_2	3.70580×10^{-16}	4.79014×10^{-13}	1.75229×10^{-15}	4.88578×10^{-14}
9.0	y_1	2.94063×10^{-20}	4.83562×10^{-17}	1.59004×10^{-19}	4.11682×10^{-18}
	y_2	1.31676×10^{-16}	1.95919×10^{-13}	6.44206×10^{-16}	1.73602×10^{-14}
10.0	y_1	4.78731×10^{-21}	7.87909×10^{-18}	2.15042×10^{-20}	6.06524×10^{-19}
	y_2	5.32479×10^{-17}	8.33725×10^{-14}	2.36830×10^{-16}	6.67981×10^{-15}

Table 7: Maximum error, Max$||y_i - y(x_i)||$, for problem 2

Method	h	N	x_T	y_1 (Max$\|y_i - y(x_i)\|$)	y_2 (Max$\|y_i - y(x_i)\|$)
SDGAM6	0.4	25	10	1.93×10^{-6}	2.01×10^{-6}
SDGAM8	0.4	25	10	9.60×10^{-5}	7.73×10^{-5}
BVM2 [14]	0.4	25	10	3.20×10^{-5}	3.04×10^{-6}
BVM3 [14]	0.4	25	10	7.90×10^{-4}	5.84×10^{-3}
DBDF [22]	0.4	85	10	1.0×10^{-4}	
SDGAM6	0.1	50	5	9.11×10^{-6}	1.60×10^{-5}
SDGAM8	0.1	50	5	2.33×10^{-7}	8.57×10^{-7}
BVM2 [14]	0.1	50	5	6.5×10^{-5}	1.50×10^{-3}
BVM3 [14]	0.1	50	5	7.45×10^{-4}	9.5×10^{-5}
DBDF [22]	0.1	47	5	4.4×10^{-4}	
GEAR [17]	0.04	122	5	3.8×10^{-4}	
SDGAM6	0.04	125	5	1.28×10^{-7}	2.85×10^{-8}
SDGAM8	0.04	125	5	1.17×10^{-9}	2.96×10^{-10}
BVM2 [14]	0.04	125	5	7.45×10^{-6}	4.07×10^{-5}
BVM3 [14]	0.04	125	5	8.33×10^{-6}	1.32×10^{-6}

Table 8: Relative error for problem 3

	SDGAM6 $k = 2(p = 6)$	SDAM [21] $k = 2(p = 6)$	Amodio [2] $k = 5(p = 6)$	SDGAM8 $k = 3(p = 8)$	SDAM [21] $k = 3(p = 8)$	Amodio [2] $k = 7(p = 8)$
Steps	Error	Error	Error	Error	Error	Error
20	1.3×10^{-11}	2.9×10^{-3}	5.7×10^{-2}	7.3×10^{-15}	7.5×10^{-4}	2.9×10^{-2}
40	2.1×10^{-13}	7.3×10^{-5}	8.7×10^{-3}	1.3×10^{-17}	1.9×10^{-5}	6.8×10^{-3}
80	3.2×10^{-15}	1.8×10^{-6}	4.9×10^{-4}	1.3×10^{-17}	1.4×10^{-7}	7.8×10^{-5}
160	6.5×10^{-17}	3.3×10^{-8}	1.2×10^{-5}	0.0	6.4×10^{-10}	4.7×10^{-7}
320	1.3×10^{-17}	5.1×10^{-10}	2.2×10^{-7}	2.6×10^{-17}	2.5×10^{-12}	2.3×10^{-9}
640	2.6×10^{-17}	7.7×10^{-12}	3.7×10^{-9}	0.0	9.8×10^{-15}	1.3×10^{-11}

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