Two Analytical Methods for Fractional Partial Differential Equations with Proportional Delay

Linjun Wang, Yan Wu, Yixin Ren, Xumei Chen

***Abstract*—In this paper, two analytical methods, namely residual power series method and homotopy analysis transform method, are applied for solving the initial value problem of time fractional partial differential equations with proportional delay. By employing the methods, the explicit approximate solutions are found. The efficiency and accuracy of two methods are demonstrated by comparing the results with exact solutions as well as the solutions obtained by homotopy perturbation method and homotopy perturbation transform method. Moreover, the obtained results also show that the approximate analytical solutions are exactly same as the Taylor series expansions of the exact solutions when $\alpha = 1$. Illustrative examples through graphical representations and tables reveal that the provided methods can be used as alternatives for seeking the numerical solutions of such type of fractional partial differential equations.**

***Index Terms*—Residual power series method, Homotopy analysis transform method, Fractional partial differential equations, Proportional delay.**

I. Introduction

In recent years, the study of fractional calculus has become a hot research area. One example is fractional differential equations (FDEs), which can more accurately describe non-local models. Due to its wide applications, the FDEs have been paid close attention to. For example, they can be used to simulate many phenomena such as the mechanical properties of materials [1], [2], [3], the advection and dispersion of solutes in natural porous or fractured media [4], the description of mechanical systems subject to damping [5], the behaviour of viscoelastic and viscoelastic materials under external influences [6], the behaviour of human beings in the mathematical psychology field [7], [8].

As is known to all, it is a challenging task to obtain the exact or approximate solutions of FDEs. Therefore, various methods have been extended and developed for solving FDEs. For instance, there are Adomian decomposition method [9], [10], tanh method [11], sine-cosine method [12], differential transform method [13], variational iteration method [14], Laplace decomposition method [15], [16], homotopy perturbation method (HPM) [17], homotopy perturbation transform method (HPTM) [18], [19], fractional reduced differential transform method [20], [21] and spectral collocation method [22], [23], [24]. In addition, Khan [25] developed homotopy analysis transform method (HATM) by combining homotopy analysis method and Laplace transform method. The advantage of this technique is that it can provide us with a simple way to solve nonlinear equations by choosing auxiliary parameters properly. Recently, Arqub [26] has presented residual power series method (RPSM) to get power series solutions with rapid convergence. For more details of the mentioned methods, we refer to [27], [28], [29], [30], [31], [32] and the references therein.

In the present paper, we consider the initial value autonomous system of time fractional partial differential equations (TFPDEs) with proportional delay defined in [33]

$$\begin{cases} D_t^\alpha u(x,t) = f\big(x, u(a_0x, b_0t), \frac{\partial}{\partial x}u(a_1x,b_1t), \cdots, \\ \qquad\qquad \frac{\partial^m}{\partial x^m}u(a_mx,b_mt)\big), \\ u^k(x,0) = \psi_k(x), \end{cases} \tag{1}$$

where $a_i, b_i \in (0,1)$ for $i \in \{0,1,2,\cdots,m\}$, $\psi_k(x)$ is the initial value and f is the differential operator. For the specially selected f, system (1) represents the important models. One example is time fractional Kortewegde Vries (KdV) equation appearing in the research of shallow water waves

$$D_t^\alpha(u(x,t)) = bu\tfrac{\partial}{\partial x}u(a_0x,b_0t) + \tfrac{\partial^3}{\partial x^3}u(a_1x,b_1t), \tag{2}$$

where $0 < \alpha < 1$, b is a constant. Another example is time fractional nonlinear Klein-Gordon equation with proportional delay arising in quantum field theory to describe nonlinear wave interaction

$$\begin{aligned} D_t^\alpha(u(x,t)) &= u\frac{\partial^2}{\partial x^2}u(a_0x,b_0t) - bu(a_1x,b_1t) \\ &- F(u(a_2x,b_2t)) + h(x,t), \end{aligned} \tag{3}$$

where $1 < \alpha < 2$, b is a constant, $h(x,t)$ is a known analytical function or source term and F is a nonlinear function of $u(x,t)$. For more details of other models, please refer to [34] and the references therein.

So far, a great deal of efforts have been made on the approximate solutions for TFPDEs with proportional delay. Saker et al. [33] used HPM to obtain the numerical solutions. Singh and Kumar [35] applied extended reduced differential transform to get the approximate analytic solutions. Singh and Kumar [36] made use of HPTM to solve TFPDEs with proportional delay.

The paper suggests two analytical methods namely RPSM and HATM to find series solutions of TFPDEs with proportional delay. After some simple computations, explicit analytical solutions in the form of the power series are possible to seek.

The remnant of this paper has been organized as follows. In Section 2, Some preliminary results related to RPSM and HATM are given. In Section 3, we describe the procedures of RPSM and HATM. Numerical examples are provided to illustrate the feasibility of proposed methods in Section 4. Finally, Section 5 concludes the output of the whole paper.

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Linjun Wang, Yan Wu, Yixin Ren, Xumei Chen are with Faculty of Science, Jiangsu University, Zhenjiang, 212013. Xumei Chen is the corresponding author, e-mail: chenxumei@ujs.edu.cn.

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II. Preliminary

In this section, we collect some fundamental definitions and preliminary results of fractional calculus to RPSM and HATM which are used in the paper. The fractional derivative considered in this context is in the Caputo sense.

Definition 1. [37] The Caputo fractional derivative of $u(x,t)$ is defined by

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\ m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, \quad \alpha = m \in N, \end{cases}$$

where m is the smallest integer that exceeds α.

Here are some simple arithmetic properties for the Caputo's fractional derivative

$$D^\alpha C = 0, \text{ (C is a constant)},$$
$$D^\alpha(\gamma f(t) + \delta g(t)) = \gamma D^\alpha f(t) + \delta D^\alpha g(t),$$

where γ and δ are constants, and

$$D^\alpha t^\beta = \begin{cases} 0, & \beta \le \alpha - 1, \\ \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \beta \ge \alpha - 1. \end{cases}$$

Definition 2. [30] A power series (PS) of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^\alpha + c_2(t-t_0)^{2\alpha} + \cdots, 0 \le n-1 < \alpha \le n, \ t \le t_0,$$

is called the fractional PS about $t = t_0$.

Theorem 1. [30] Suppose that f has a fractional PS representation at $t = t_0$ of the form

$$f(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, \ t_0 \le t < t_0 + R.$$

If $f(t)$ is continuous on $[t_0, t_0+R)$ and $D^{m\alpha} f(t), \ m = 0, 1, 2, \cdots$ are continuous on (t_0, t_0+R), then $c_m = \frac{D^{m\alpha} f(t_0)}{\Gamma(1+m\alpha)}$. Here, $D^{m\alpha} = D^\alpha \cdot D^\alpha \cdot \cdots \cdot D^\alpha$ (m-times).

Remark 1. [30] The number R in Theorem 1 is called the radius of convergence of fractional PS.

Definition 3. [30] A power series of the form

$$\sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}$$

is called the multiple fractional PS about $t = t_0$.

Theorem 2. [30] Suppose that $P(x,t)$ has a multiple fractional PS representation at $t = t_0$ of the form

$$P(x,t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}, x \in I, \ t_0 \le t < t_0 + R.$$

If $D_t^{m\alpha} P(x,t), \ m = 0, 1, 2, \cdots$ are continuous on $I \times (t_0, t_0+R)$, then $f_m(x) = \frac{D_t^{m\alpha} P(x,t_0)}{\Gamma(1+m\alpha)}$.

Definition 4. [37] The Laplace transform of continuous (or an almost piecewise continuous) function $f(t)$ in $[0,\infty)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

where s is a real or complex number.

Definition 5. [37] The Laplace transform of the Caputo fractional derivative is defined as

$$L[D_t^\alpha u(x,t)] = s^\alpha L[u(x,t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} u^{(k)}(x,0), \quad n-1 < \alpha \le n.$$

III. Basic idea of RPSM and HATM

In this section, we will depict the procedures of RPSM and HATM for the initial valued autonomous system of TFPDEs with proportional delay

$$\begin{cases} D_t^\alpha u(x,t) = f(x, u(a_0x, b_0t), \frac{\partial}{\partial x}u(a_1x, b_1t), \cdots, \\ \qquad \frac{\partial^m}{\partial x^m}u(a_mx, b_mt)), \\ u(x,0) = \psi(x), \end{cases} \tag{4}$$

where $0 < \alpha \le 1$.

A. RPSM for TFPDEs

First, a detailed process of solving equation (4) by RPSM will be given.

Assume that the solution of (4) can be written in the form of series at $t = 0$ as

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \ x \in I, \ 0 \le t < R. \tag{5}$$

Following the above equation, the k-th truncated series of $u(x,t)$ can be expressed as

$$u_k(x,t) = \sum_{n=0}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{6}$$

Setting $k = 0$ together with the initial condition, the zeroth residual power series (RPS) truncated solution of $u(x,t)$ is

$$u_0(x,t) = f_0(x) = u(x,0) = \psi(x). \tag{7}$$

Consequently, equation (6) becomes

$$u_k(x,t) = \psi(x) + \sum_{n=1}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{8}$$

The residual function for (4) is defined as

$$\begin{aligned} Res_u(x,t) &= D_t^\alpha u(x,t) - f(x, u(a_0x, b_0t), \\ &\frac{\partial}{\partial x}u(a_1x, b_1t), \cdots, \frac{\partial^m}{\partial x^m}u(a_mx, b_mt)). \end{aligned} \tag{9}$$

Then, the k-th residual function takes the form

$$\begin{aligned} Res_{u,k}(x,t) &= D_t^\alpha u_k(x,t) - f(x, u_k(a_0x, b_0t), \\ &\frac{\partial}{\partial x}u_k(a_1x, b_1t), \cdots, \frac{\partial^m}{\partial x^m}u_k(a_mx, b_mt)). \end{aligned} \tag{10}$$

Computing the fractional derivative of equation (10) with respect t and taking the value $t = 0$ yields the following formula

$$D_t^{(k-1)\alpha} Res_{u,k}(x,0) = 0, \ k = 1, 2, \cdots \tag{11}$$

After solving algebraic system (11), the coefficients $f_i(x)$ $(i = 1, 2, \cdots, k)$ are got. By substituting them into formula (8), we derive the k-th RPS approximate solution.

B. HATM for TFPDEs

Next, we will introduce the simple produre of HATM. For a more detailed description of HATM, the reader is referred to [38], [25].

Taking the Laplace transform on both sides of equation (4), we have

$$\begin{aligned} &s^{\alpha} L[u(x,t)] - s^{\alpha-1}u(x,0) - L[f(x, u(a_0x, b_0t), \\ &\tfrac{\partial}{\partial x}u(a_1x, b_1t), \cdots, \tfrac{\partial^m}{\partial x^m}u(a_mx, b_mt))] = 0. \end{aligned} \tag{12}$$

On simplifying

$$\begin{aligned} &L[u(x,t)] - s^{-1}u(x,0) - s^{-\alpha}L[f(x, u(a_0x, b_0t), \\ &\tfrac{\partial}{\partial x}u(a_1x, b_1t), \cdots, \tfrac{\partial^m}{\partial x^m}u(a_mx, b_mt))] = 0. \end{aligned} \tag{13}$$

We select a linear operator as

$$\pounds[\phi(x,t;q)] = L[\phi(x,t;q)] \tag{14}$$

with the property $\pounds[c] = 0$, where c is a constant.

Meanwhile, a nonlinear operator is defined as

$$\begin{aligned} N[\phi(x,t;q)] &= L[\phi(x,t;q)] - s^{-1}\psi(x) \\ &- s^{-\alpha}L[f(x, \phi(a_0x, b_0t;q), \\ &\tfrac{\partial}{\partial x}\phi(a_1x, b_1t;q), \cdots, \\ &\tfrac{\partial^m}{\partial x^m}\phi(a_mx, b_mt;q))]. \end{aligned} \tag{15}$$

where $q \in [0,1]$, $\phi(x,t;q)$ is a real function about x, t and q.

We construct the zeroth-order deformation equation:

$$\begin{aligned} &(1-q)\pounds[\phi(x,t;q) - u_0(x,t)] \\ &= q\hbar H(x,t)N[\phi(x,t;q)], \end{aligned} \tag{16}$$

where $\hbar$ is a nonzero auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, $u_0(x,t)$ is an initial guess value of $u(x,t)$ and $\phi(x,t;q)$ is an unknown function. Let $H(x,t) = 1$. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \phi(x,t;1) = u(x,t), \tag{17}$$

respectively. As a result, when p increases from 0 to 1, the function $\phi(x,t;q)$ varies from the initial guess value $u_0(x,t)$ to the solution $u(x,t)$. It can also be expanded in Taylor series about q

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} q^m u_m(x,t), \tag{18}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}.$$

By choosing proper auxiliary linear operator, the initial guess value, the auxiliary parameter $\hbar$ and the auxiliary function, the series (18) converges at $q = 1$. Then, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{19}$$

which must be one of the solutions of the original equation.

For simplicity, we define the vectors

$$\vec{u}_n = (u_0(x,t), u_1(x,t), u_2(x,t), \cdots, u_n(x,t)). \tag{20}$$

We deduce the m-th order deformation equation by differentiating equation (16) m times with the parameter q and then setting $q = 0$. The m-th order deformation equation could be written as

$$\begin{aligned} &L[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\ &= \hbar H(x,t)R_m(\vec{u}_{m-1}, x, t). \end{aligned} \tag{21}$$

Using the inverse Laplace transform on both sides of the above equation, we have

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) \\ &+ \hbar L^{-1}[H(x,t)R_m(\vec{u}_{m-1}, x, t)], \end{aligned} \tag{22}$$

where

$$R_m(\vec{u}_{m-1}, x, t) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x,t;q)}{\partial q^{m-1}}|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Let $H(x,t) = 1$, and we can compute $u_m(x,t)$ for $m \geq 1$. Hence, the M-th order approximate solution can be represented as

$$u(x,t) = \sum_{m=0}^{M} u_m(x,t). \tag{23}$$

When $M \to \infty$, an accurate approximation of (4) is given:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{24}$$

IV. Applications of RPSM and HATM to TFPDEs with Proportional Delay

In this section, two examples are given to illustrate the validity, reliability and accuracy of RPSM as well as HATM.

Application 1. Consider time fractional generalized Burgers equation with proportional delay as given in [33], [39]:

$$\begin{cases} D_t^{\alpha}u(x,t) - u(\frac{x}{2}, \frac{t}{2})\frac{\partial}{\partial x}u(x, \frac{t}{2}) = \frac{\partial^2}{\partial x^2}u(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}u(x,t) \\ u(x,0) = x \end{cases} \tag{25}$$

For $\alpha = 1$, the exact solution of equation (25) is

$$u(x,t) = xe^t. \tag{26}$$

We will first use RPSM to solve (25). Suppose that the solution to equation (25) can be written in the form of series:

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{27}$$

Therefore, the k-th truncated series of $u(x,t)$ could be represented as:

$$u_k(x,t) = \sum_{n=0}^{k} f_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{28}$$

According to the process described in Section III, we get the zeroth RPS truncated solution of $u(x,t)$ as

$$u_0(x,t) = f_0(x) = x. \tag{29}$$

The first RPS approximate solution could be written as

$$u_1(x,t) = x + x\frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{30}$$

The second RPS approximate solution is

$$u_2(x,t) = x + x\frac{t^\alpha}{\Gamma(1+\alpha)} + x2^{-1}(1+2^{1-\alpha})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (31)$$

We proceed as above and deduce the following results

$$\begin{aligned} u_3(x,t) &= x + x\frac{t^\alpha}{\Gamma(1+\alpha)} + x2^{-1}(1+2^{1-\alpha})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ x4^{-1}(1+2^{1-\alpha}+2^{1-2\alpha}+2^{2-3\alpha} \\ &+ \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}2^{1-2\alpha})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ u_4(x,t) &= x + x\frac{t^\alpha}{\Gamma(1+\alpha)} + x2^{-1}(1+2^{1-\alpha})\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ x4^{-1}(1+2^{1-\alpha}+2^{1-2\alpha}+2^{2-3\alpha} \\ &+ \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}2^{1-2\alpha})\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + x(2^{-1-3\alpha} \\ &+ 2^{-1-4\alpha} + 2^{-1-5\alpha} + 2^{-2-\alpha} + 2^{-2-2\alpha} \\ &+ 2^{-2-3\alpha} + 2^{-3} + 2^{-6\alpha} + (2^{-2-2\alpha} \\ &+ 2^{-1-5\alpha})\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} + (2^{-1-3\alpha} \\ &+ 2^{-4\alpha})\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)})\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}. \end{aligned}$$

If we repeat the process of RPSM, the higher degree of approximate solution will be obtained.

In the special case of $\alpha = 1$, the solution is given by

$$u(x,t) = (1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\cdots)x,$$

which is exactly same as the Taylor series expansion of the exact solution.

Next, we will use HATM to solve equation (25). Taking the Laplace transform on both sides in equation (25), we get

$$\begin{aligned} &L[u(x,t)] - s^{-1}x - s^{-\alpha}L[\frac{\partial^2}{\partial x^2}u(x,t) \\ &+\frac{\partial}{\partial x}u(x,\frac{t}{2})u(\frac{x}{2},\frac{t}{2}) + \frac{1}{2}u(x,t)] = 0 \end{aligned} \quad (32)$$

On the basis of HATM, we choose the linear operator as

$$\pounds[\phi(x,t;q)] = L[\phi(x,t;q)], \quad (33)$$

and the nonlinear operator as

$$\begin{aligned} N[\phi(x,t;q)] &= L[\phi(x,t;q)] - s^{-1}x \\ &- s^{-\alpha}L[\frac{\partial^2}{\partial x^2}\phi(x,t;q) \\ &+ \frac{\partial}{\partial x}\phi(x,\frac{t}{2};q)\phi(\frac{x}{2},\frac{t}{2};q) \\ &+ \frac{1}{2}\phi(x,t;q)]. \end{aligned} \quad (34)$$

Using the above definition, with assumption $H(x,t) = 1$, we obtain the m-th order deformation equation

$$\pounds[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1},x,t), \quad (35)$$

where

$$\begin{aligned} R_m(\vec{u}_{m-1},x,t) &= L[u_{m-1}(x,t)] - (1-\chi_m)s^{-1}x \\ &- s^{-\alpha}L[\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} \\ &+ \sum_{k=0}^{m-1}\frac{\partial u_k(x,\frac{t}{2})}{\partial x}u_{m-1-k}(\frac{x}{2},\frac{t}{2}) \\ &+ \frac{1}{2}u_{m-1}(x,t)]. \end{aligned}$$

Operating the inverse Laplace transform on both sides of (35), we get

$$\begin{aligned} u_m(x,t) &= (\hbar+\chi_m)u_{m-1}(x,t) - (1-\chi_m)\hbar x \\ &- \hbar L^{-1}\Big(s^{-\alpha}L[\frac{\partial^2}{\partial x^2}u_{m-1}(x,t) \\ &+ \sum_{k=0}^{m-1}\frac{\partial u_k(x,\frac{t}{2})}{\partial x}u_{m-1-k}(\frac{x}{2},\frac{t}{2}) \\ &+ \frac{1}{2}u_{m-1}(x,t)]\Big). \end{aligned} \quad (36)$$

Noting $u_0(x,t) = x$, we have

$$\begin{aligned} u_1(x,t) &= -\frac{\hbar x t^\alpha}{\Gamma(1+\alpha)}, \\ u_2(x,t) &= -\frac{(1+\hbar)\hbar x t^\alpha}{\Gamma(1+\alpha)} + (2^{-\alpha}+2^{-1})\frac{\hbar^2 x t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ u_3(x,t) &= \frac{-(1+\hbar)^2\hbar x t^\alpha}{\Gamma(1+\alpha)} + (1+2^{1-\alpha})\frac{(1+\hbar)\hbar^2 x t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &- \Big(2^{-3\alpha}+2^{-1-2\alpha}+2^{-1-\alpha}+4^{-1} \\ &+ \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}2^{-1-2\alpha}\Big)\frac{\hbar^3 x t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ u_4(x,t) &= -\frac{(1+\hbar)^3\hbar x t^\alpha}{\Gamma(1+\alpha)} + (\frac{3}{2}+2^{-\alpha} \\ &+ 2^{1-\alpha})\frac{(1+\hbar)^2\hbar^2 x t^{2\alpha}}{\Gamma(1+2\alpha)} - \Big(2^{-\alpha}+2^{-2\alpha} \\ &+ 2^{-3\alpha}+2^{-1}+2^{-1-\alpha}+2^{-1-2\alpha}+2^{1-3\alpha} \\ &+ 2^{-2}+(2^{-1-2\alpha} \\ &+ 2^{-2\alpha})\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2}\Big)\frac{(1+\hbar)\hbar^3 x t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &+ \Big(2^{-1-3\alpha}+2^{-1-4\alpha}+2^{-1-5\alpha}+2^{-2-\alpha} \\ &+ 2^{-2-2\alpha}+2^{-2-3\alpha}+2^{-3}+2^{-6\alpha} \\ &+ (2^{-2-2\alpha}+2^{-1-5\alpha})\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}+(2^{-1-3\alpha} \\ &+ 2^{-4\alpha})\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\Big)\frac{\hbar^4 x t^{4\alpha}}{\Gamma(1+4\alpha)}. \end{aligned}$$

Therefore, the fourth approximate solution of (25) is given as

$$u(x,t) = \sum_{m=0}^{4}u_m(x,t). \quad (37)$$

The higher degree of approximate solution can be obtained in the same manner.

Remark 2. There exists the equivalence of the solutions by RPSM and HATM when selecting $h = -1$ in HATM.

TABLE I
THE ABSOLUTE ERRORS OF $u(x,t)$ FOR APPLICATION 1

(x,t)	E_{HPTM}	E_{RPS}	E_{HATM}
(0.25,0.25)	2.122401E-06	2.122401E-06	2.122401E-06
(0.25,0.50)	7.094268E-05	7.094268E-05	7.094268E-05
(0.25,0.75)	5.634807E-04	5.634807E-04	5.634807E-04
(0.25,1.00)	2.487124E-03	2.487124E-03	2.487124E-03
(0.50,0.25)	4.244802E-06	4.244802E-06	4.244802E-06
(0.50,0.50)	1.418854E-04	1.418854E-04	1.418854E-04
(0.50,0.75)	1.126961E-03	1.126961E-03	1.126961E-03
(0.50,1.00)	4.974248E-03	4.974248E-03	4.974248E-03
(0.75,0.25)	6.369688E-06	6.367203E-06	6.367203E-06
(0.75,0.50)	2.128250E-04	2.128280E-04	2.128280E-04
(0.75,0.75)	1.690020E-03	1.690442E-03	1.690442E-03
(0.75,1.00)	7.461370E-03	7.461371E-03	7.461371E-03

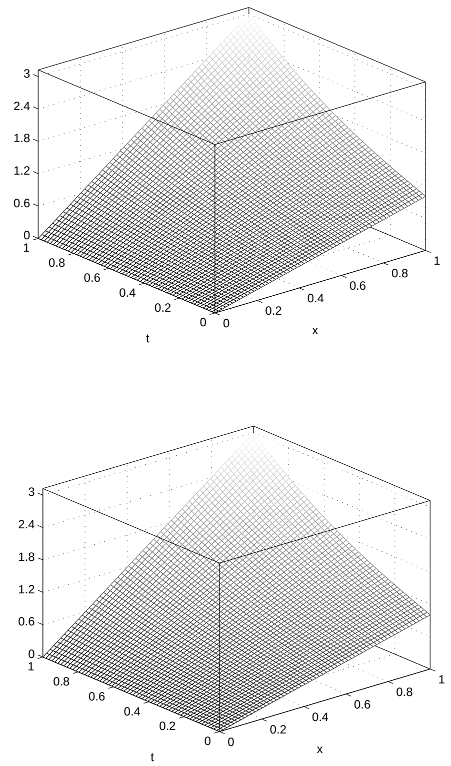
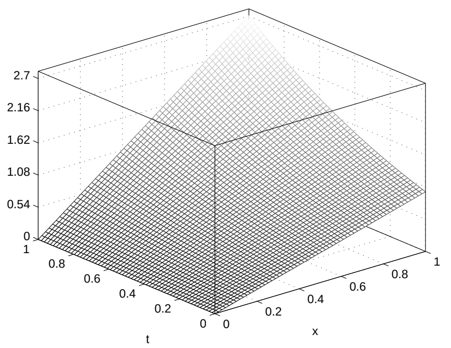


Fig. 1. The fourth approximate solutions of RPSM and HATM for Application 1 when $\alpha = 0.9$, $\hbar = -1$.



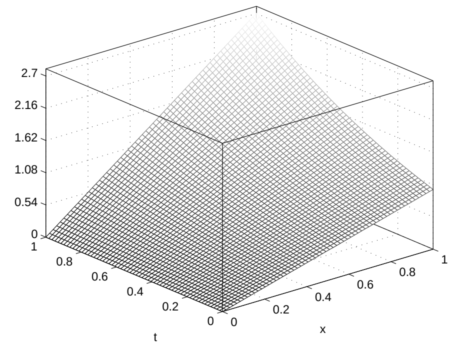


Fig. 2. The fourth approximate solutions of RPSM and HATM for Application 1 when $\alpha = 1$, $\hbar = -1$.

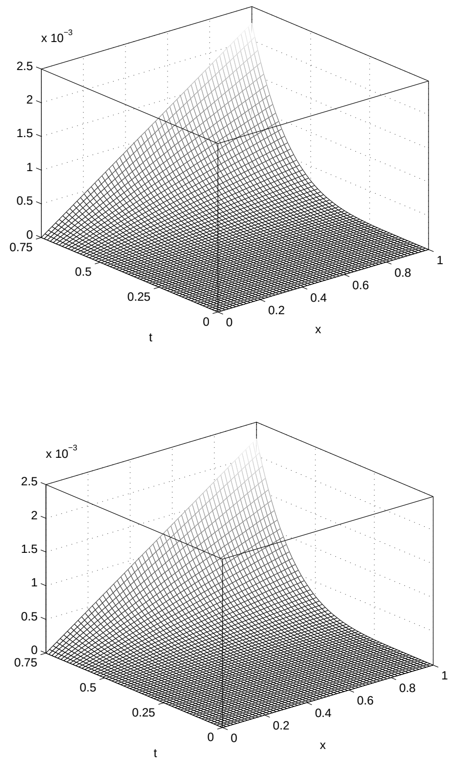


Fig. 3. The errors of RPSM and HATM for Application 1 when $\alpha = 1$, $\hbar = -1$.

We plot the numerical solutions for $\alpha = 0.9, 1$ and absolute errors in Figures 1-3. When $\alpha = 1$, the comparison results of the absolute errors by different methods are demonstrated in Table I.

Application 2. We consider the following TFPDEs with proportional delay [33], [39]:

$$\begin{cases} D_t^{\alpha} u(x,t) = u(x, \frac{t}{2}) \frac{\partial^2}{\partial x^2} u(x, \frac{t}{2}) - u(x,t), \\ u(x,0) = x^2, \end{cases} \tag{38}$$

with the exact solution $u(x,t) = x^2 e^t$, when $\alpha = 1$.

Similarly, RPSM will be employed at first. Then, the zeroth RPS approximate solution is given

$$u_0(x,t) = f_0(x) = x^2. \tag{39}$$

The first RPS approximate solution can be expressed as

$$u_1(x,t) = x^2 + x^2 \frac{t^{\alpha}}{\Gamma(1+\alpha)}. \tag{40}$$

Other higher degree of approximate solutions can be represented as

$$\begin{aligned}
u_2(x,t) &= x^2 + x^2\frac{t^\alpha}{\Gamma(1+\alpha)} + x^2(2^{2-\alpha}-1)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\
u_3(x,t) &= x^2 + x^2\frac{t^\alpha}{\Gamma(1+\alpha)} + x^2(2^{2-\alpha}-1)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&+ x^2\Big(1-2^{2-\alpha}-4^{1-\alpha}+2^{4-3\alpha}\\
&+ 2^{1-2\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\\
u_4(x,t) &= x^2 + x^2\frac{t^\alpha}{\Gamma(1+\alpha)} + x^2(2^{2-\alpha}-1)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&+ x^2\Big(1-2^{2-\alpha}-4^{1-\alpha}+2^{4-3\alpha}\\
&+ 2^{1-2\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\\
&- 64^{-\alpha}x^2\Big((2^\alpha-4)\frac{4^{1+\alpha}\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\\
&+ 2^{1+\alpha}(8^\alpha-4)\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha))^2} - 64 + 2^{4+\alpha}\\
&+ 2^{4+3\alpha} - 2^{1+5\alpha} + 4^{2+\alpha} - 2^{2+5\alpha} - 2^{2+3\alpha}\\
&+ 64^\alpha\Big)\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}.
\end{aligned}$$

When $\alpha = 1$, the solution is reduced to

$$u(x,t) = (1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\cdots)x^2,$$

which is exactly same as the Taylor series expansion of the exact solution.

For HATM, by assuming $H(x,t) = 1$, we construct the zeroth-order deformation equation

$$(1-q)\pounds[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)], \quad (41)$$

and the m-th order deformation equation

$$\pounds[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}, x, t), \quad (42)$$

where

$$\begin{aligned}
R_m(\vec{u}_{m-1},x,t) &= L[u_{m-1}(x,t)] - (1-\chi_m)x^2 s^{-1}\\
&- s^{-\alpha}L[\sum_{k=0}^{m-1}\frac{\partial^2 u_k(x,\frac{t}{2})}{\partial x^2}u_{m-1-k}(x,\frac{t}{2})\\
&- u_{m-1}(x,t)].
\end{aligned}$$

Using the inverse Laplace transform on both sides in (42), we get

$$\begin{aligned}
u_m(x,t) &= (\hbar+\chi_m)u_{m-1}(x,t) - (1-\chi_m)\hbar x^2\\
&- \hbar L^{-1}(s^{-\alpha}L[\sum_{k=0}^{m-1}\frac{\partial^2 u_k(x,\frac{t}{2})}{\partial x^2}u_{m-1-k}(x,\frac{t}{2})\\
&- u_{m-1}(x,t)]) \quad (43)
\end{aligned}$$

In view of the initial approximation value $u_0(x,t) = x^2$ and the iterative scheme (43), we deduce the following results:

$$\begin{aligned}
u_1(x,t) &= -\frac{\hbar x^2 t^\alpha}{\Gamma(1+\alpha)},\\
u_2(x,t) &= -\frac{(1+\hbar)\hbar x^2 t^\alpha}{\Gamma(1+\alpha)} - (1-2^{2-\alpha})\frac{h^2x^2t^{2\alpha}}{\Gamma(1+2\alpha)},\\
u_3(x,t) &= -\frac{(1+\hbar)^2\hbar x^2 t^\alpha}{\Gamma(1+\alpha)} + (2^{3-\alpha}\\
&- 2)\frac{(1+\hbar)\hbar^2x^2t^{2\alpha}}{\Gamma(1+2\alpha)} + (2^{2-2\alpha}\\
&+ 2^{2-\alpha} - 2^{4-3\alpha} - 1\\
&- 2^{1-2\alpha}\frac{\Gamma(1+2\alpha)}{(\Gamma(1+\alpha)^2)})\frac{\hbar^3x^2t^{3\alpha}}{\Gamma(1+3\alpha)},\\
u_4(x,t) &= -\frac{(1+\hbar)^3\hbar x^2 t^\alpha}{\Gamma(1+\alpha)} + (2^{3-\alpha} + 2^{2-\alpha}\\
&- 3)\frac{(1+\hbar)^2\hbar^2x^2t^{2\alpha}}{\Gamma(1+2\alpha)} + (2^{2-\alpha} + 2^{2-2\alpha}\\
&+ 2^{3-\alpha} + 2^{3-2\alpha} - 2^{4-3\alpha} - 3 - (2^{1-2\alpha}\\
&+ 2^{2-2\alpha})\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2})\frac{(1+\hbar)\hbar^3x^2t^{3\alpha}}{\Gamma(1+3\alpha)}\\
&+ (2^{2-\alpha} + 2^{2-2\alpha} + 2^{2-3\alpha} - 2^{4-3\alpha}\\
&- 2^{4-4\alpha} - 2^{4-5\alpha} - 1 + 2^{6-6\alpha}\\
&+ (2^{3-5\alpha} - 2^{1-2\alpha})\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - (2^{2-3\alpha}\\
&- 2^{4-4\alpha})\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)})\frac{\hbar^4x^2t^{4\alpha}}{\Gamma(1+4\alpha)}.
\end{aligned}$$

Hence, the fourth approximate solution of equation (38) is given as

$$u(x,t) = \sum_{m=0}^{4} u_m(x,t). \quad (44)$$

We can acquire the higher degree of approximate solution in the similar way.

Remark 3. There exists the equivalence of the solutions by RPSM and HATM when selecting $h = -1$ in HATM.

As in Application 1, the graphical results and absolute errors are presented in Figures 4-6 and Table II.

From Figures 1-6, it is clear that we get very good approximate solutions through RPSM and HATM. There are nearly no differences between the graphs of numerical solutions obtained by two methods. It should also be noted from Table I-II that all the methods including RPSM, HATM, HPM and HPTM can reach almost the same errors.

TABLE II
THE ABSOLUTE ERRORS OF $u(x,t)$ FOR APPLICATION 2

(x,t)	E_{HPM}	E_{RPS}	E_{HATM}
(0.25,0.25)	5.300000E-07	5.306003E-07	5.306003E-07
(0.25,0.50)	1.773500E-05	1.773567E-05	1.773567E-05
(0.25,0.75)	1.408700E-04	1.408702E-04	1.408702E-04
(0.25,1.00)	6.217800E-04	6.217809E-04	6.217809E-04
(0.50,0.25)	2.123000E-06	2.122401E-06	2.122401E-06
(0.50,0.50)	7.094300E-05	7.094268E-05	7.094268E-05
(0.50,0.75)	5.634830E-04	5.634807E-04	5.634807E-04
(0.50,1.00)	2.487123E-03	2.487124E-03	2.487124E-03
(0.75,0.25)	4.776000E-06	4.775402E-06	4.775402E-06
(0.75,0.50)	1.596200E-04	1.596210E-04	1.596210E-04
(0.75,0.75)	1.267830E-03	1.267832E-03	1.267832E-03
(0.75,1.00)	5.596030E-03	5.596028E-03	5.596028E-03

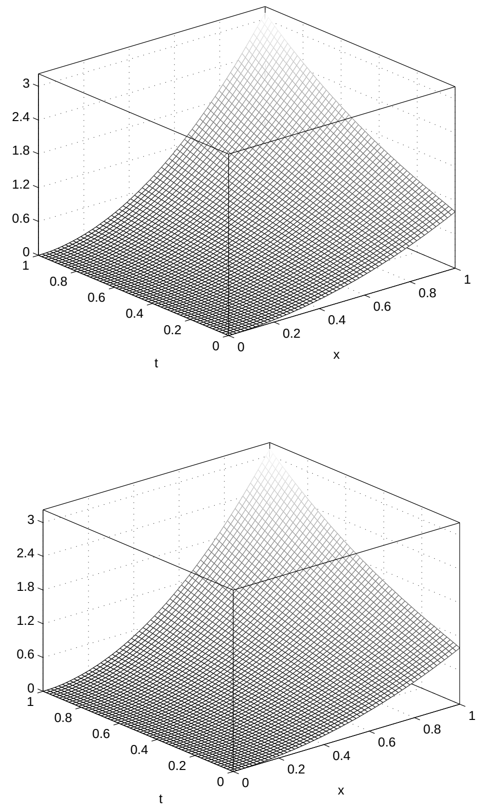


Fig. 4. The fourth approximate solutions of RPSM and HATM for Application 2 when $\alpha = 0.9, \hbar = -1$.

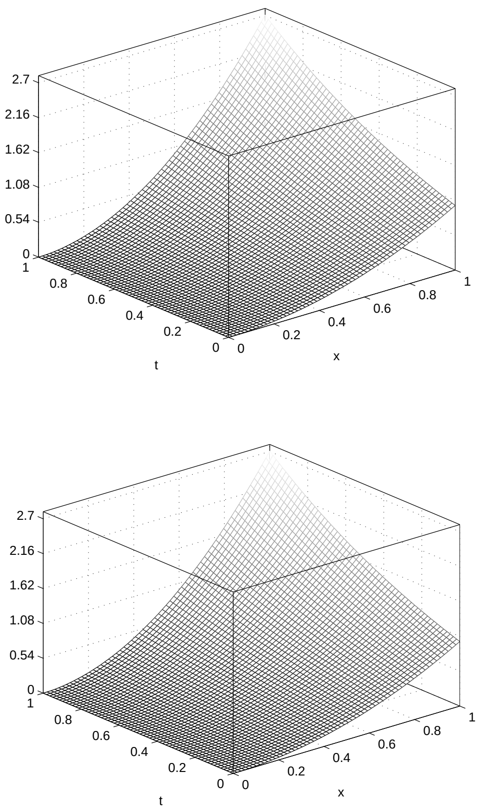


Fig. 5. The fourth approximate solutions of RPSM and HATM for Application 2 when $\alpha = 1, \hbar = -1$.

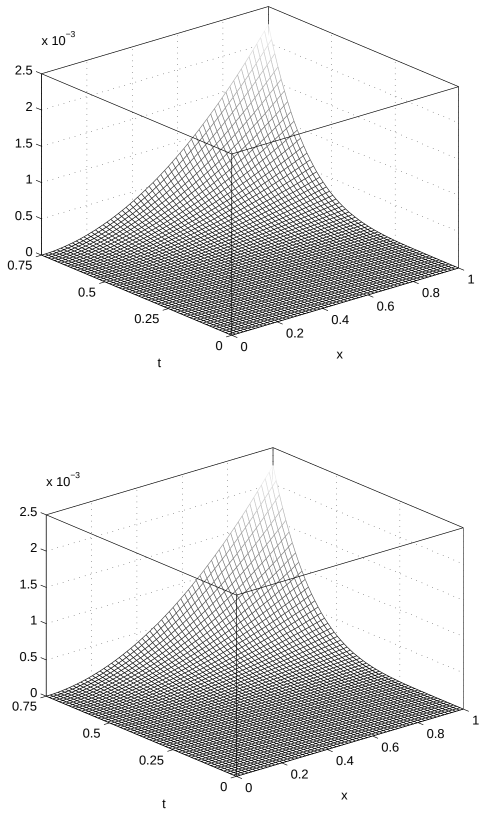


Fig. 6. The figures of the errors of RPSM and HATM for Application 2 when $\alpha = 1, \hbar = -1$.

V. Conclusion

In this paper, RPSM and HATM have been successfully employed to solve the initial value autonomous system of TFPDEs with proportional delay. The results of graphs and tables show that both of the proposed methods yield very efficient and accurate approaches to solve the initial value autonomous system of TFPDEs with proportional delay. Comparing the results with the other methods, we may safely conclude that both of the methods can be used as alternatives for solving this type of system. The given examples also reveal that there exists the equivalence of the solutions by two methods when selecting $h = -1$ in HATM.

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