Applying Generalized Rough Set Concepts to Approximation Spaces of Semigroups

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***Abstract*—This paper presents generalized rough sets in approximation spaces based on portions of successor classes induced by arbitrary binary relations between two universes. Based on this definition, some interesting properties are investigated. In semigroup structures, the notions of rough semigroups, rough ideals and rough completely prime ideals in approximation spaces induced by preorder and compatible relations are proposed. Some related results and examples are discussed and provided. In the end, the relationships between rough semigroups (resp. rough ideals and rough completely prime ideals) and their homomorphic images are verified. These associations are presented in understanding of necessary and sufficient conditions.**

***Index Terms*—generalized rough set, approximation space, semigroup, rough semigroup, rough ideal, rough completely prime ideal, binary relation, preorder relation, compatible relation.**

I. INTRODUCTION

THE Pawlak's rough set theory is a influential tool for several assessment and decision problems of uncertain data in information technology. This classical theory was introduced by Pawlak [1]. The Pawlak's rough set has been regarded as an approximation processing model in an approximation space induced by an equivalence relation on the single universe. For a given equivalence relation on a universal set and given a non-empty subset of the universal set, the Pawlak's rough set of the given set is refered to as a pair of the Pawlak's upper and Pawlak's lower approximations where the difference between the Pawlak's upper and Pawlak's lower approximations (The Pawlak's boundary region) is a non-empty set. The Pawlak's upper approximation is the union of all the equivalence classes which have a non-empty intersection with the given set. The Pawlak's lower approximation is the union of all the equivalence classes which are subset of the given set. As mentioned above, the Pawlak's rough set model is referred to as a mathematical tool for an assessment and decision space in several information systems, including algebraic systems [2]–[15], expert systems with applications [16], knowledge-based information systems [17], computers and engineerings [18], measurements [19], approximate reasonings [20] etc. under intelligent information fields of artificial intelligence.

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Based on the notion of the Pawlak's roughness model induced by an equivalence relation, generalizations of such the model have been being constructed by many researchers. Particularly, a generalization of Pawlak's rough set based on an arbitrary binary relation (briefly, binary relation) on the single universe. In 1998, Yao [22] introduced a Yao's rough set based on successor neighborhoods induced by a binary relation $[SN_\theta(u) := \{u' \in U : (u, u') \in \theta\}$ denotes a successor neighborhood of u induced by a binary relation θ on a universal set U where u is an element in $U]$. In 2016, Mareay [23] introduced a new rough set by using cores of successor neighborhoods induced by a binary relation $[CSN_\theta(u) := \{u' \in U : SN_\theta(u) = SN_\theta(u')\}$ denotes a core of a successor neighborhood of u induced by a binary relation θ on a universal set U where u is an element in $U]$. Observer that the Yao's rough set and the Mareay's rough set are generalizations of the Pawlak's rough set whenever the binary relation is an equivalence relation.

The semigroup structure (see [24]) is an algebraic system with respect to wide applications. For employments of Pawlak's rough set theory in semigroup, Kuroki [4] introduced the notions of upper and lower approximation semigroups (resp. ideals) in semigroups induced by congruence relations, and provided sufficient conditions of upper and lower approximation semigroups (resp. ideals) in 1997. In 2006, Xiao and Zhang [7] introduced the notions of upper and lower approximation completely prime ideals in semigroups induced by congruence relations, and provided sufficient conditions of upper and lower approximation completely prime ideals. They examined the relationship between upper and lower approximation completely prime ideals (resp. ideals) and their homomorphic images under homomorphism problems. In 2016, Wang and Zhan [13] proposed the notions of upper and lower approximation semigroups (resp. ideals and completely prime ideals) induced by specific congruence relations, and also provided sufficient conditions of upper and lower approximation semigroups (resp. ideals and completely prime ideals).

Based on the generalized rough set in sense of Mareay, the main point of this work is a extended concept, i.e., a generalization of the Mareay's rough set induced by a binary relation between two universes will be established. Then we apply the novel rough set in semigroups for approximation processings. After providing some fundamentals of binary relations and semigroups in Section II, we propose a generalization of the Mareay's rough set in an approximation space of a universal set based on portions of successor classes induced by a binary relation between two universes and investigate some interesting properties in Section III. In Section IV, we introduce concepts of rough semigroups, rough ideals and rough completely prime ideals in approximation spaces of semigroups under preorder and compatible relations. Next, we provide sufficient conditions and examples of them. In Section V, we verify relationships between rough semigroups (resp. rough ideals and rough completely prime ideals) and their homomorphic images. Lastly, this paper is concluded with some remarks and discussions in Section VI.

II. Preliminaries

In this section we reconsider some important definitions which will be employed in the following sections.

Throughout this paper, U and V denote two non-empty universe.

Definition 1. [21] Let $\Xi(U \times V)$ be a family of all subsets of $U \times V$. An element in $\Xi(U \times V)$ is referred to as a *binary relation from* U *to* V. An element in $\Xi(U \times V)$ is called a *binary relation on* U if $U = V$.

Definition 2. [21] Let Λ be a binary relation from U to V. Λ is called *serial* if for all $u \in U$, there exists $v \in V$ such that $(u, v) \in \Lambda$.

Definition 3. [21] Let Λ be a binary relation on U.

(1) Λ is called *reflexive* if for all $u \in U$, $(u, u) \in \Lambda$.
(2) Λ is called *transitive* if for all $u_1, u_2, u_3 \in U$, $(u_1, u_2) \in \Lambda$ and $(u_2, u_3) \in \Lambda$ implies $(u_1, u_3) \in \Lambda$.
(3) Λ is called *symmetric* if for all $u_1, u_2 \in U$, $(u_1, u_2) \in \Lambda$ implies $(u_2, u_1) \in \Lambda$.
(4) If Λ is reflexive and transitive, then Λ is called a *preorder*.
(5) If Λ is reflexive, transitive and symmetric, then Λ is called an *equivalence relation*.

A *semigroup* [24] $(S, \odot)$ is defined as an algebraic system where S is a non-empty set and $\odot$ is an associative binary operation on S. Throughout this paper, S denotes a semigroup. A non-empty subset X of S is called a *subsemigroup* [25] of S if $XX \subseteq X$. A non-empty subset X of S is called a *left (right) ideal* [25] of S if $SX \subseteq X$ $(XS \subseteq X)$, and if it is both a left ideal and a right ideal of S, then it is called an *ideal* [25]. An ideal X of S is called a *completely prime ideal* [25] of S if for all $s_1, s_2 \in S$, $s_1 s_2 \in X$ implies $s_1 \in X$ or $s_2 \in X$. Let Λ be a binary relation on S. Then, Λ is called *compatible* if for all $s_1, s_2, s_3 \in S$, $(s_1, s_2) \in \Lambda$ implies $(s_1 s_3, s_2 s_3) \in \Lambda$ and $(s_3 s_1, s_3 s_2) \in \Lambda$.

III. Generalized rough sets

In this section we establish a generalization of the Mareay's rough set induced by a binary relation between two universes and provide a real-world example and verify some appealing properties.

Definition 4. Let Λ be a binary relation from U to V. For an element $u \in U$,

$$S_\Lambda(u) := \{v \in V : (u, v) \in \Lambda\}$$

is called a *successor class* of u induced by Λ.

Remark 1. If Λ is a serial relation from U to V, then $S_\Lambda(u) \neq \emptyset$ for all $u \in U$.

Definition 5. Let Λ be a binary relation from U to V. For an element $u_1 \in U$,

$$PS_\Lambda(u_1) := \{u_2 \in U : S_\Lambda(u_2) \subseteq S_\Lambda(u_1)\}$$

is called a *portion of the successor class* of u_1 induced by Λ.

We denote by $\mathcal{PS}_\Lambda(U)$ a collection of $PS_\Lambda(u)$ for all $u \in U$.

Directly from Definition 5, we can obtain to Proposition 1 as the following.

Proposition 1. *Let Λ be a binary relation from U to V. Then the following statements hold.*

(1) For all $u \in U$, $u \in PS_\Lambda(u)$.
(2) For all $u_1, u_2 \in U$, $u_1 \in PS_\Lambda(u_2)$ if and only if $PS_\Lambda(u_1) \subseteq PS_\Lambda(u_2)$.

Proposition 2. *Let Λ be a binary relation on U. Then we have the following statements.*

(1) If Λ is reflexive, then $PS_\Lambda(u) \subseteq S_\Lambda(u)$ for all $u \in U$.
(2) If Λ is transitive, then $S_\Lambda(u) \subseteq PS_\Lambda(u)$ for all $u \in U$.
(3) If Λ is a preorder, then $S_\Lambda(u)$ and $PS_\Lambda(u)$ are identical classes for all $u \in U$.

Proof: (1) It is clear that $PS_\Lambda(u) \subseteq S_\Lambda(u)$ for all $u \in U$ whenever Λ is reflexive.

(2) Let $u_1 \in U$ and let $u_2 \in S_\Lambda(u_1)$. Then, $(u_1, u_2) \in \Lambda$. We only need to show that $S_\Lambda(u_2) \subseteq S_\Lambda(u_1)$. Suppose that $u_3 \in S_\Lambda(u_2)$. Then, $(u_2, u_3) \in \Lambda$. Since Λ is transitive, we have $(u_1, u_3) \in \Lambda$. Thus $u_3 \in S_\Lambda(u_1)$. Hence we get $S_\Lambda(u_2) \subseteq S_\Lambda(u_1)$. Therefore, $u_2 \in PS_\Lambda(u_1)$. Consequently, $S_\Lambda(u_1) \subseteq PS_\Lambda(u_1)$.

(3) From items (1) and (2), it follows that this statement holds. ∎

In the following, we give a generalization of the Mareay's roughness model induced by a binary relation.

Definition 6. Let Λ be a binary relation from U to V. The triple $(U, V, \mathcal{PS}_\Lambda(U))$ is referred to as an *approximation space* based on $\mathcal{PS}_\Lambda(U)$ (briefly, $\mathcal{PS}_\Lambda(U)$-*approximation space*). If $U = V$, then $(U, V, \mathcal{PS}_\Theta(U))$ is replaced by a pair $(U, \mathcal{PS}_\Theta(U))$.

Definition 7. Let $(U, V, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space. For a non-empty subset X of U, we define three sets as follows:

$\overline{\Lambda}(X) := \{u \in U : PS_\Lambda(u) \cap X \text{ is a non-empty set}\}$,
$\underline{\Lambda}(X) := \{u \in U : PS_\Lambda(u) \subseteq X\}$ and
$\Lambda_{bnd}(X) := \overline{\Lambda}(X) - \underline{\Lambda}(X)$.

Then

(1) $\overline{\Lambda}(X)$ denotes an *upper approximation of* X in $(U, V, \mathcal{PS}_\Lambda(U))$ (briefly, $\mathcal{PS}_\Lambda(U)$-upper approximation of X).
(2) $\underline{\Lambda}(X)$ denotes a *lower approximation of* X in $(U, V, \mathcal{PS}_\Lambda(U))$ (briefly, $\mathcal{PS}_\Lambda(U)$-lower approximation of X).
(3) $\Lambda_{bnd}(X)$ denotes a *boundary region of* X in $(U, V, \mathcal{PS}_\Lambda(U))$ (briefly, $\mathcal{PS}_\Lambda(U)$-boundary region of X).
(4) If $\Lambda_{bnd}(X) \neq \emptyset$, then $\Lambda(X) := (\overline{\Lambda}(X), \underline{\Lambda}(X))$ is called a *rough set of* X *in* $(U, V, \mathcal{PS}_\Lambda(U))$ (briefly, $\mathcal{PS}_\Lambda(U)$-rough set of X).
(5) If $\Lambda_{bnd}(X) = \emptyset$, then X is called a *definable set* in $(U, V, \mathcal{PS}_\Lambda(U))$ (briefly, $\mathcal{PS}_\Lambda(U)$-definable set).

We give an example as the following.

Example 1. *Let $U := \{u_1, u_2, u_3, u_4, u_5\}$ be a set of electrical discharge machines (EDM) in an automobile industry of leading company and let $V := \{v_1, v_2, v_3, v_4\}$ be a set of components of each elements in U.*

Define TABLE I by an information of the damage values (Bad and Medium) of electrical discharge machines in U with respect to components in V as the following detail.

TABLE I
THE INFORMATION TABLE OF DAMAGE VALUES

	v_1	v_2	v_3	v_4
u_1	Medium	Bad	Bad	Medium
u_2	Medium	Bad	Bad	Medium
u_3	Medium	Bad	Bad	Bad
u_4	Medium	Medium	Medium	Bad
u_5	Bad	Bad	Medium	Medium

For a binary relation $\Lambda \in \Xi(U \times V)$ and elements $u \in U$, $v \in V$, a pair $(u, v) \in \Lambda$ is defined as a bad damage value of the electrical discharge machine u with respect to the component v under Λ. Then the binary relation Λ is a set $\{(u_1, v_2), (u_1, v_3), (u_2, v_2), (u_2, v_3), (u_3, v_2), (u_3, v_3), (u_3, v_4), (u_4, v_4), (u_5, v_1), (u_5, v_2)\}$. Suppose that a measurement expert committee assign $X := \{u_1, u_3, u_5\}$ which is a non-empty set of electrical discharge machines for the discharge under a global evaluation. Then the assessment of X in approximation space $(U, V, \mathcal{PS}_\Lambda(U))$ is derived by a process as the following. According to Definition 4, it follows that

$S_\Lambda(u_1) := \{v_2, v_3\}$,
$S_\Lambda(u_2) := \{v_2, v_3\}$,
$S_\Lambda(u_3) := \{v_2, v_3, v_4\}$,
$S_\Lambda(u_4) := \{v_4\}$ *and*
$S_\Lambda(u_5) := \{v_1, v_2\}$.

According to Definition 5, it follows that

$PS_\Lambda(u_1) := \{u_1, u_2\}$,
$PS_\Lambda(u_2) := \{u_1, u_2\}$,
$PS_\Lambda(u_3) := \{u_1, u_2, u_3, u_4\}$,
$PS_\Lambda(u_4) := \{u_4\}$ *and*
$PS_\Lambda(u_5) := \{u_5\}$.

According to Definition 7, it follows that

$\overline{\Lambda}(X) := \{u_1, u_2, u_3, u_5\}$,
$\underline{\Lambda}(X) := \{u_5\}$ *and*
$\Lambda_{bnd}(X) := \{u_1, u_2, u_3\}$.

Therefore, $\Lambda(X) := (\{u_1, u_2, u_3, u_5\}, \{u_5\})$ is a $\mathcal{PS}_\Lambda(U)$-rough set of X. As a consequence,

(1) u_1, u_2, u_3 and u_5 are possibly electrical discharge machines for the discharge,
(2) u_5 is a certainly electrical discharge machine for the discharge and
(3) u_1, u_2 and u_3 cannot be determined whether three machines are electrical discharge machines for the discharge or not.

The following remark is immediate consequences of Definition 7 and the existence of Example 1 with respect to Definition 7.

Remark 2. Every Mareay's rough set in [23] is a rough set in Definition 7, but the converse is not true in general. Therefore, the rough set in Definition 7 is considered as a generalization of the Mareay's rough set whenever $U = V$ and the relation " $\subseteq$ " is substituted by the equality " $=$ " in Definition 5.

The existence of Example 1 leads to the following definition.

Definition 8. Let $(U, V, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space and let X be a non-empty subset of U. $\overline{\Lambda}(X)$ is called a *non-empty $\mathcal{PS}_\Lambda(U)$-upper approximation* of X in $(U, V, \mathcal{PS}_\Lambda(U))$ if $\overline{\Lambda}(X)$ is a non-empty subset of U. Analogously, we can define non-empty $\mathcal{PS}_\Lambda(U)$-lower approximations. The $\mathcal{PS}_\Lambda(U)$-rough set $\Lambda(X)$ of X in $(U, V, \mathcal{PS}_\Lambda(U))$ is referred to as a *non-empty $\mathcal{PS}_\Lambda(U)$-rough set* if $\overline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-upper approximation and $\underline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-lower approximation.

Proposition 3. *Let $(U, V, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space. If X and Y are non-empty subsets of U, then we have the following statements.*

(1) $\overline{\Lambda}(U) = U$ and $\underline{\Lambda}(U) = U$.
(2) $\overline{\Lambda}(\emptyset) = \emptyset$ and $\underline{\Lambda}(\emptyset) = \emptyset$.
(3) $X \subseteq \overline{\Lambda}(X)$ and $\underline{\Lambda}(X) \subseteq X$.
(4) $\overline{\Lambda}(X \cup Y) = \overline{\Lambda}(X) \cup \overline{\Lambda}(Y)$ and $\underline{\Lambda}(X \cap Y) = \underline{\Lambda}(X) \cap \underline{\Lambda}(Y)$.
(5) $\overline{\Lambda}(X \cap Y) \subseteq \overline{\Lambda}(X) \cap \overline{\Lambda}(Y)$ and $\underline{\Lambda}(X \cup Y) \supseteq \underline{\Lambda}(X) \cup \underline{\Lambda}(Y)$.
(6) If $X \subseteq Y$, then $\overline{\Lambda}(X) \subseteq \overline{\Lambda}(Y)$ and $\underline{\Lambda}(X) \subseteq \underline{\Lambda}(Y)$.

Proof: (1)-(3) follow from Proposition 1 (1). The proofs (4), (5) and (6) are straightforward. ■

Definition 9. Let $(U, V, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space and let X be a non-empty subset of U. If $\underline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-lower approximation of X in $(U, V, \mathcal{PS}_\Lambda(U))$ and $\underline{\Lambda}(X)$ is a proper subset of X, then X is called a *set over non-empty interior set*.

Proposition 4. *Let $(U, V, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space and let X be a non-empty subset of U. If X is a set over non-empty interior set, then $\Lambda(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-rough set of X in $(U, V, \mathcal{PS}_\Lambda(U))$.*

Proof: Suppose that X is a set over non-empty interior set. Then we have that $\underline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-lower approximation and $\underline{\Lambda}(X) \subset X$. By Proposition 3 (3), we obtain that $X \subseteq \overline{\Lambda}(X)$. Thus we get $\overline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-upper approximation. We shall verify that $\Lambda_{bnd}(X)$ is a non-empty set. Suppose that $\Lambda_{bnd}(X) = \emptyset$. Then we have $\overline{\Lambda}(X) = \underline{\Lambda}(X)$. From Proposition 3 (3), once again, it follows that $\underline{\Lambda}(X) = X$, which is a contradiction. Thus $\Lambda_{bnd}(X)$ is a non-empty set. As a consequence, $\Lambda(X)$ is a non-empty $\mathcal{PS}_\Lambda(U)$-rough set of X. ■

Proposition 5. *Let $(U, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space and let $(U, \mathcal{PS}_\Upsilon(U))$ be an $\mathcal{PS}_\Upsilon(U)$-approximation space. If $\Lambda \subseteq \Upsilon$ where Λ is reflexive and Υ is transitive, then $\overline{\Lambda}(X) \subseteq \overline{\Upsilon}(X)$ for every non-empty subset X of U.*

Proof: Let X be a non-empty subset of U. Then we prove that $\overline{\Lambda}(X) \subseteq \overline{\Upsilon}(X)$. Infact, let $u_1 \in \overline{\Lambda}(X)$. Then we have $PS_\Lambda(u_1) \cap X$ is a non-empty set. Thus there exists $u_2 \in U$ such that $u_2 \in PS_\Lambda(u_1) \cap X$. Hence $S_\Lambda(u_2) \subseteq S_\Lambda(u_1)$. Since Λ is reflexive, we have $(u_2, u_2) \in \Lambda$. Whence we get that $u_2 \in S_\Lambda(u_2) \subseteq S_\Lambda(u_1)$. Thus $(u_1, u_2) \in \Lambda$.

Since $\Lambda \subseteq \Upsilon$, $(u_1, u_2) \in \Upsilon$. We shall verify that $S_\Upsilon(u_2) \subseteq S_\Upsilon(u_1)$. Let now $u_3 \in S_\Upsilon(u_2)$. Then, $(u_2, u_3) \in \Upsilon$. Since Υ is a transitive relation, we have $(u_1, u_3) \in \Upsilon$. Thus we get $u_3 \in S_\Upsilon(u_1)$, which yields $S_\Upsilon(u_2) \subseteq S_\Upsilon(u_1)$. Hence $u_2 \in PS_\Upsilon(u_1)$. Thus we have $u_2 \in PS_\Upsilon(u_1) \cap X$. Hence $PS_\Upsilon(u_1) \cap X$ is a non-empty set, which yields $u_1 \in \overline{\Upsilon}(X)$. This implies that $\overline{\Lambda}(X) \subseteq \overline{\Upsilon}(X)$. ∎

Proposition 6. *Let $(U, \mathcal{PS}_\Lambda(U))$ be an $\mathcal{PS}_\Lambda(U)$-approximation space and let $(U, \mathcal{PS}_\Upsilon(U))$ be an $\mathcal{PS}_\Upsilon(U)$-approximation space. If $\Lambda \subseteq \Upsilon$ where Λ is reflexive and Υ is transitive, then $\underline{\Upsilon}(X) \subseteq \underline{\Lambda}(X)$ for every non-empty subset X of U.*

Proof: Let X be a non-empty subset of U. Then we prove that $\underline{\Upsilon}(X) \subseteq \underline{\Lambda}(X)$. Indeed, suppose that $u_1 \in \underline{\Upsilon}(X)$. Then we get that $PS_\Upsilon(u_1) \subseteq X$. We shall show that $PS_\Lambda(u_1) \subseteq PS_\Upsilon(u_1)$. Let $u_2 \in PS_\Lambda(u_1)$. Then we have $S_\Lambda(u_2) \subseteq S_\Lambda(u_1)$. Since Λ is a reflexive relation, we have $(u_2, u_2) \in \Lambda$. Hence we get $u_2 \in S_\Lambda(u_2)$, and so $u_2 \in S_\Lambda(u_1)$. Thus we get $(u_1, u_2) \in \Lambda$. By the assumption, we obtain that $(u_1, u_2) \in \Upsilon$. Now, we shall prove that $S_\Upsilon(u_2) \subseteq S_\Upsilon(u_1)$. Let $u_3 \in S_\Upsilon(u_2)$. Then, $(u_2, u_3) \in \Upsilon$. Since Υ is a transitive relation, we have $(u_1, u_3) \in \Upsilon$. Whence $u_3 \in S_\Upsilon(u_1)$. Hence $S_\Upsilon(u_2) \subseteq S_\Upsilon(u_1)$. Thus $u_2 \in PS_\Upsilon(u_1)$. Whence we get $PS_\Lambda(u_1) \subseteq PS_\Upsilon(u_1) \subseteq X$. Therefore, $u_1 \in \underline{\Lambda}(X)$. As a consequence, $\underline{\Upsilon}(X) \subseteq \underline{\Lambda}(X)$. ∎

IV. Roughness in semigroups

In this section we introduce rough semigroups, rough ideals and rough completely prime ideals in semigroups induced by preorder and compatible relations. Then we provide sufficient conditions of them and give some interesting properties and examples.

Definition 10. Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space. $(S, \mathcal{PS}_\Lambda(S))$ is called an $\mathcal{PS}_\Lambda(S)$-*approximation space type PCR* if Λ is a preorder and compatible relation.

Proposition 7. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR. Then,*

$$(PS_\Lambda(s_1))(PS_\Lambda(s_2)) \subseteq PS_\Lambda(s_1 s_2)$$

for all $s_1, s_2 \in S$.

Proof: Let s_1 and s_2 be two elements in S. Suppose that $s_3 \in (PS_\Lambda(s_1))(PS_\Lambda(s_2))$. Then there exist $s_4 \in PS_\Lambda(s_1)$, $s_5 \in PS_\Lambda(s_2)$ such that $s_3 = s_4 s_5$. Thus $S_\Lambda(s_4) \subseteq S_\Lambda(s_1)$ and $S_\Lambda(s_5) \subseteq S_\Lambda(s_2)$. Hence we get that $S_\Lambda(s_4 s_5) \subseteq S_\Lambda(s_1 s_2)$. Indeed, we suppose that $s_6 \in S_\Lambda(s_4 s_5)$. Then, $(s_4 s_5, s_6) \in \Lambda$. Since Λ is reflexive, we have (s_4, s_4) and (s_5, s_5) are in Λ, and so $s_4 \in S_\Lambda(s_4)$ and $s_5 \in S_\Lambda(s_5)$. Whence $s_4 \in S_\Lambda(s_1)$ and $s_5 \in S_\Lambda(s_2)$. Thus $(s_1, s_4) \in \Lambda$ and $(s_2, s_5) \in \Lambda$. Since Λ is compatible, we have $(s_1 s_2, s_4 s_5) \in \Lambda$. Since Λ is transitive, we have $(s_1 s_2, s_6) \in \Lambda$. Whence we get that $s_6 \in S_\Lambda(s_1 s_2)$. Hence we obtain that $S_\Lambda(s_4 s_5) \subseteq S_\Lambda(s_1 s_2)$, which yields $s_3 = s_4 s_5 \in PS_\Lambda(s_1 s_2)$. As a consequence, $(PS_\Lambda(s_1))(PS_\Lambda(s_2)) \subseteq PS_\Lambda(s_1 s_2)$. ∎

We give to Example 2 as the following.

Example 2. *Let $S := \{s_1, s_2, s_3, s_4, s_5\}$ be the semigroup with multiplication rules defined by the TABLE II.*

TABLE II
The multiplication table on S

$\cdot$	s_1	s_2	s_3	s_4	s_5
s_1	s_1	s_2	s_3	s_2	s_5
s_2	s_2	s_2	s_3	s_2	s_5
s_3	s_3	s_3	s_3	s_3	s_3
s_4	s_2	s_2	s_3	s_2	s_5
s_5	s_5	s_5	s_3	s_5	s_5

Define $\Lambda := \{(s_1, s_1), (s_2, s_2), (s_2, s_5), (s_3, s_2), (s_3, s_3), (s_3, s_5), (s_4, s_4), (s_5, s_2), (s_5, s_5)\}$. Then it is easy to check that Λ is a preorder and compatible relation. Thus successor classes of each elements in S induced by Λ are as follows:
$S_\Lambda(s_1) := \{s_1\}$,
$S_\Lambda(s_2) := \{s_2, s_5\}$,
$S_\Lambda(s_3) := \{s_2, s_3, s_5\}$,
$S_\Lambda(s_4) := \{s_4\}$ *and*
$S_\Lambda(s_5) := \{s_2, s_5\}$.
Hence by Proposition 2 (3), we obtain that
$PS_\Lambda(s_1) = S_\Lambda(s_1)$,
$PS_\Lambda(s_2) = S_\Lambda(s_2)$,
$PS_\Lambda(s_3) = S_\Lambda(s_3)$,
$PS_\Lambda(s_4) = S_\Lambda(s_4)$ *and*
$PS_\Lambda(s_5) = S_\Lambda(s_5)$.
Here it is straightforward to verify that for all $s, s' \in S$

$$(PS_\Lambda(s))(PS_\Lambda(s')) \subseteq PS_\Lambda(ss').$$

Observe that, in Example 2, it does not holds in general for an equality case. We consider the following example.

Example 3. *Let $S := \{s_1, s_2, s_3, s_4, s_5\}$ be the semigroup with multiplication rules defined by the TABLE III.*

TABLE III
The multiplication table on S

$\cdot$	s_1	s_2	s_3	s_4	s_5
s_1	s_1	s_1	s_3	s_1	s_5
s_2	s_1	s_2	s_3	s_1	s_5
s_3	s_3	s_3	s_3	s_3	s_3
s_4	s_1	s_1	s_3	s_4	s_5
s_5	s_5	s_5	s_3	s_5	s_5

Define $\Lambda := \{(s_1, s_1), (s_1, s_2), (s_1, s_4), (s_2, s_1), (s_2, s_2), (s_2, s_4), (s_3, s_3), (s_3, s_5), (s_4, s_1), (s_4, s_2), (s_4, s_4), (s_5, s_5)\}$. Then it is easy to check that Λ is a preorder and compatible relation. Thus successor classes of each elements in S induced by Λ are as follows:
$S_\Lambda(s_1) := \{s_1, s_2, s_4\}$,
$S_\Lambda(s_2) := \{s_1, s_2, s_4\}$,
$S_\Lambda(s_3) := \{s_3, s_5\}$,
$S_\Lambda(s_4) := \{s_1, s_2, s_4\}$ *and*
$S_\Lambda(s_5) := \{s_5\}$.
Thus by Proposition 2 (3), we obtain that
$PS_\Lambda(s_1) = S_\Lambda(s_1)$,
$PS_\Lambda(s_2) = S_\Lambda(s_2)$,
$PS_\Lambda(s_3) = S_\Lambda(s_3)$,
$PS_\Lambda(s_4) = S_\Lambda(s_4)$ *and*
$PS_\Lambda(s_5) = S_\Lambda(s_5)$.
Here it is straightforward to check that for all $s, s' \in S$

$$(PS_\Lambda(s))(PS_\Lambda(s')) = PS_\Lambda(ss').$$

Considering this point, the property can be considered as a special case of Proposition 7. This important example leads to Definition 11 as the following.

Definition 11. Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR. The collection $\mathcal{PS}_\Lambda(S)$ is called a *complete collection induced by* Λ (briefly, Λ*-complete*) if for all $s_1, s_2 \in S$,

$$(PS_\Lambda(s_1))(PS_\Lambda(s_2)) = PS_\Lambda(s_1 s_2).$$

Definition 12. Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR. If $\mathcal{PS}_\Lambda(S)$ is a complete collection induced by Λ, then Λ is called a *complete relation*. $(S, \mathcal{PS}_\Lambda(S))$ is called an $\mathcal{PS}_\Lambda(S)$*-approximation space type CR* if Λ is complete.

Proposition 8. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR. Then,*

$$(\overline{\Lambda}(X))(\overline{\Lambda}(Y)) \subseteq \overline{\Lambda}(XY)$$

for every non-empty subsets X, Y of S.

Proof: Let X, Y be two non-empty subsets of S and let $s_1 \in (\overline{\Lambda}(X))(\overline{\Lambda}(Y))$. Then there exists $s_2 \in \overline{\Lambda}(X)$ and exists $s_3 \in \overline{\Lambda}(Y)$ such that $s_1 = s_2 s_3$. Thus we get $PS_\Lambda(s_2) \cap X$ and $PS_\Lambda(s_3) \cap Y$ are non-empty sets. Then there exist $s_4, s_5 \in S$ such that $s_4 \in PS_\Lambda(s_2) \cap X$ and $s_5 \in PS_\Lambda(s_3) \cap Y$. From Proposition 7, it follows that

$$s_4 s_5 \in (PS_\Lambda(s_2))(PS_\Lambda(s_3)) \subseteq PS_\Lambda(s_2 s_3).$$

Note that $s_4 s_5 \in XY$. Thus $PS_\Lambda(s_2 s_3) \cap XY$ is a non-empty set, which yields $s_1 = s_2 s_3 \in \overline{\Lambda}(XY)$. Therefore we get $(\overline{\Lambda}(X))(\overline{\Lambda}(Y)) \subseteq \overline{\Lambda}(XY)$. ■

Proposition 9. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. Then,*

$$(\underline{\Lambda}(X))(\underline{\Lambda}(Y)) \subseteq \underline{\Lambda}(XY)$$

for every non-empty subsets X, Y of S.

Proof: Let X and Y be two non-empty subsets of S and let $s_1 \in (\underline{\Lambda}(X))(\underline{\Lambda}(Y))$. Then there exist $s_2 \in \underline{\Lambda}(X)$ and $s_3 \in \underline{\Lambda}(Y)$ such that $s_1 = s_2 s_3$. Hence we get that $PS_\Lambda(s_2) \subseteq X$ and $PS_\Lambda(s_3) \subseteq Y$. Since Λ is complete,

$$PS_\Lambda(s_2 s_3) = PS_\Lambda(s_2) PS_\Lambda(s_3) \subseteq XY.$$

Whence we obtain $PS_\Lambda(s_2 s_3) \subseteq XY$. Hence we get that $s_1 = s_2 s_3 \in \underline{\Lambda}(XY)$. Therefore, $(\underline{\Lambda}(X))(\underline{\Lambda}(Y)) \subseteq \underline{\Lambda}(XY)$. ■

In what follows, a rough set in semigroups will be proposed. We consider to Example 4 as the following.

Example 4. *According to Example 3, we let $X := \{s_2, s_3, s_5\}$ be a subset of S. Then we have $\overline{\Lambda}(X) = S$ and $\underline{\Lambda}(X) := \{s_3, s_5\}$. Here it is easy to verify that $\overline{\Lambda}(X)$ and $\underline{\Lambda}(X)$ are subsemigroups, ideals and completely prime ideals of S. Moreover, we also have $\Lambda_{bnd}(X)$ is a non-empty set. Existences of subsemigroups, ideals and completely prime ideals of S induced by preorder and compatible relations in this example lead to Definition 13 as the following.*

Definition 13. Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR and let X be a non-empty subset of S. The non-empty $\mathcal{PS}_\Lambda(S)$-upper approximation $\overline{\Lambda}(X)$ of X in $(S, \mathcal{PS}_\Lambda(S))$ is called an $\mathcal{PS}_\Lambda(S)$*-upper approximation semigroup* if it is a subsemigroup of S. The non-empty $\mathcal{PS}_\Lambda(S)$-lower approximation $\underline{\Lambda}(X)$ of X in $(S, \mathcal{PS}_\Lambda(S))$ is called a $\mathcal{PS}_\Lambda(S)$*-lower approximation semigroup* if it is a subsemigroup of S. The non-empty $\mathcal{PS}_\Lambda(S)$-rough set $\Lambda(X)$ of X in $(S, \mathcal{PS}_\Lambda(S))$ is called a $\mathcal{PS}_\Lambda(S)$*-rough semigroup* if $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup and $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation semigroup. Similarly, we can define $\mathcal{PS}_\Lambda(S)$-rough (completely prime) ideals.

Theorem 1. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR. If X is a subsemigroup of S, then $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup.*

Proof: Suppose that X is a subsemigroup of S. Then, $XX \subseteq X$. By Proposition 3 (3), we obtain that

$$\emptyset \neq X \subseteq \overline{\Lambda}(X).$$

Hence we get $\overline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(S)$-upper approximation. From Proposition 3 (6), it follows that $\overline{\Lambda}(XX) \subseteq \overline{\Lambda}(X)$. By Proposition 8, we get

$$(\overline{\Lambda}(X))(\overline{\Lambda}(X)) \subseteq \overline{\Lambda}(XX) \subseteq \overline{\Lambda}(X).$$

Hence $\overline{\Lambda}(X)$ is a subsemigroup of S. Thus $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup. ■

Theorem 2. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is a subsemigroup of S with $\underline{\Lambda}(X)$ is a non-empty set, then $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation semigroup.*

Proof: Suppose that X is a subsemigroup of S. Then, $XX \subseteq X$. Obviously, $\underline{\Lambda}(X)$ is a non-empty $\mathcal{PS}_\Lambda(S)$-lower approximation. From Proposition 3 (6), it follows that $\underline{\Lambda}(XX) \subseteq \underline{\Lambda}(X)$. By Proposition 9, we obtain that

$$(\underline{\Lambda}(X))(\underline{\Lambda}(X)) \subseteq \underline{\Lambda}(XX) \subseteq \underline{\Lambda}(X).$$

Thus $\underline{\Lambda}(X)$ is a subsemigroup of S. Therefore, $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation semigroup. ■

The following corollary is immediate consequences of Proposition 4, Theorem 1 and Theorem 2.

Corollary 1. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is a subsemigroup of S over non-empty interior set, then $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough semigroup.*

Observe that, in Corollary 1, the converse is not true in general. We present an example as the following.

Example 5. *According to Example 3, suppose that $X := \{s_2, s_4, s_5\}$ is a subset of S, then we have that $\overline{\Lambda}(X) := \{s_1, s_2, s_4, s_5\}$ and $\underline{\Lambda}(X) := \{s_5\}$. Thus $\Lambda_{bnd}(X)$ is a non-empty set. Hence it is straightforward to check that $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup and $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation semigroup. However, X is not a subsemigroup of S. Consequently, $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough semigroup, but X is not a subsemigroup of S.*

Theorem 3. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type PCR. If X is an ideal of S, then $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal.*

Proof: Suppose that X is an ideal of S. Then we have $SX \subseteq X$. From Proposition 3 (6), it follows that $\overline{\Lambda}(SX) \subseteq$

$\overline{\Lambda}(X)$. By Proposition 3 (1), we obtain that $\overline{\Lambda}(S) = S$. From Proposition 8, it follows that

$$S(\overline{\Lambda}(X)) = (\overline{\Lambda}(S))(\overline{\Lambda}(X)) \subseteq \overline{\Lambda}(SX) \subseteq \overline{\Lambda}(X).$$

Hence $\overline{\Lambda}(X)$ is a left ideal of S.

Similarly, we can prove that $\overline{\Lambda}(X)$ is a right ideal of S. Therefore, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal. ■

Theorem 4. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is an ideal of S with $\underline{\Lambda}(X)$ is a non-empty set, then $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation ideal.*

Proof: Suppose that X is an ideal of S. Then, $SX \subseteq X$. From Proposition 3 (6), we get $\underline{\Lambda}(SX) \subseteq \underline{\Lambda}(X)$. By Proposition 3 (1), we obtain that $\underline{\Lambda}(S) = S$. From Proposition 9, it follows that

$$S(\underline{\Lambda}(X)) = (\underline{\Lambda}(S))(\underline{\Lambda}(X)) \subseteq \underline{\Lambda}(SX) \subseteq \underline{\Lambda}(X).$$

Thus $\underline{\Lambda}(X)$ is a left ideal of S.

Similarly, we can prove that $\underline{\Lambda}(X)$ is a right ideal of S. Thus $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation ideal. ■

The following corollary is immediate consequences of Proposition 4, Theorem 3 and Theorem 4.

Corollary 2. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is an ideal of S over non-empty interior set, then $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough ideal.*

Observe that, in Corollary 2, the converse is not true in general. We present an example as the following.

Example 6. *According to Example 3, if $X := \{s_3, s_4, s_5\}$ is a subset of S, then we have $\overline{\Lambda}(X) = S$ and $\underline{\Lambda}(X) := \{s_3, s_5\}$. Thus we get $\Lambda_{bnd}(X)$ is a non-empty set. Obviously, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal, and it is straightforward to check that $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation ideal. However, X is not an ideal of S. Consequently, $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough ideal, but X is not an ideal of S.*

Theorem 5. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is a completely prime ideal of S, then $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal.*

Proof: Suppose that X is a completely prime ideal of S. Then we prove that $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal. In fact, since X is an ideal of S, by Theorem 3, we have that $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal. Let $s_1, s_2 \in S$ be such that $s_1 s_2 \in \overline{\Lambda}(X)$. Then, by the Λ-complete property of $\mathcal{PS}_\Lambda(S)$, we get that

$$(PS_\Lambda(s_1))(PS_\Lambda(s_2)) \cap X = PS_\Lambda(s_1 s_2) \cap X$$

is a non-empty set. Thus there exist $s_3 \in PS_\Lambda(s_1)$, $s_4 \in PS_\Lambda(s_2)$ such that $s_3 s_4 \in X$. Since X is a completely prime ideal, $s_3 \in X$ or $s_4 \in X$. Thus $PS_\Lambda(s_1) \cap X$ is a non-empty set or $PS_\Lambda(s_2) \cap X$ is a non-empty set. Hence $s_1 \in \overline{\Lambda}(X)$ or $s_2 \in \overline{\Lambda}(X)$. Therefore, $\overline{\Lambda}(X)$ is a completely prime ideal of S. As a consequence, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal. ■

Theorem 6. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is a completely prime ideal of S with $\underline{\Lambda}(X)$ is a non-empty set, then $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation completely prime ideal.*

Proof: Suppose that X is a completely prime ideal of S with $\underline{\Lambda}(X) \neq \emptyset$. Then, X is an ideal of S. Thus by Theorem 4, we have $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation ideal. Let $s_1, s_2 \in S$ be such that $s_1 s_2 \in \underline{\Lambda}(X)$. Since Λ is complete, we have

$$(PS_\Lambda(s_1))(PS_\Lambda(s_2)) = PS_\Lambda(s_1 s_2) \subseteq X.$$

Now, we suppose that $s_1 \notin \underline{\Lambda}(X)$. Then we have $PS_\Lambda(s_1)$ is not a subset of X. Thus there exists $s_3 \in S$ such that $s_3 \in PS_\Lambda(s_1)$ but $s_3 \notin X$. For each $s_4 \in PS_\Lambda(s_2)$,

$$s_3 s_4 \in (PS_\Lambda(s_1))(PS_\Lambda(s_2)) \subseteq X.$$

Whence $s_3 s_4 \in X$. Since X is a completely prime ideal and $s_3 \notin X$, we have $s_4 \in X$. Thus we get $PS_\Lambda(s_2) \subseteq X$, which yields $s_2 \in \underline{\Lambda}(X)$. Hence we get $\underline{\Lambda}(X)$ is a completely prime ideal of S. Therefore, $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation completely prime ideal. ■

The following corollary is immediate consequences of Proposition 4, Theorem 5 and Theorem 6.

Corollary 3. *Let $(S, \mathcal{PS}_\Lambda(S))$ be an $\mathcal{PS}_\Lambda(S)$-approximation space type CR. If X is a completely prime ideal of S over non-empty interior set, then $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough completely prime.*

Observe that, in Corollary 3, the converse is not true in general. We present an example as the following.

Example 7. *According to Example 3, if $X := \{s_1, s_3, s_5\}$ is a subset of S, then we have $\overline{\Lambda}(X) = S$ and $\underline{\Lambda}(X) := \{s_3, s_5\}$. Hence we get $\Lambda_{bnd}(X) \neq \emptyset$. Obviously, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal, and it is straightforward to check that $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation completely prime ideal. Here we can verify that X is an ideal of S, but it is not a completely prime ideal of S since $s_2 s_4 = s_1 \in X$ but $s_2 \notin X$ and $s_4 \notin X$. As a consequence, $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough completely prime ideal, but X is not a completely prime ideal of S.*

V. Homomorphic Images of Roughness in Semigroups

In this section we verify relationships between rough semigroups (resp. rough ideals, rough completely prime ideals) and their homomorphic images. Throughout this section, we suppose that T denotes a semigroup.

Proposition 10. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. Then the following statements hold.*

(1) For all $s_1, s_2 \in S$, $s_1 \in PS_\Lambda(s_2)$ if and only if $f(s_1) \in PS_\Upsilon(f(s_2))$.

(2) $f(\overline{\Lambda}(X)) = \overline{\Upsilon}(f(X))$ for every non-empty subset X of S.

(3) $f(\underline{\Lambda}(X)) \subseteq \underline{\Upsilon}(f(X))$ for every non-empty subset X of S.

(4) If f is injective, then $f(\underline{\Lambda}(X)) = \underline{\Upsilon}(f(X))$ for every non-empty subset X of S.

(5) *If Υ is a preorder and compatible relation, then Λ is a preorder and compatible relation.*

Proof: (1) Let $s_1, s_2 \in S$ be such that $s_1 \in PS_\Lambda(s_2)$. Then, $f(s_1), f(s_2) \in T$ and $S_\Lambda(s_1) \subseteq S_\Lambda(s_2)$. We shall prove that $S_\Upsilon(f(s_1)) \subseteq S_\Upsilon(f(s_2))$. Let $t_1 \in S_\Upsilon(f(s_1))$. Then, $(f(s_1), t_1) \in \Upsilon$. Since f is surjective, there exists $s_3 \in S$ such that $f(s_3) = t_1$. Whence $(f(s_1), f(s_3)) \in \Upsilon$, and so $(s_1, s_3) \in \Lambda$. Thus $s_3 \in S_\Lambda(s_1)$. Whence we have $s_3 \in S_\Lambda(s_2)$. Hence we have $(s_2, s_3) \in \Lambda$, and so $(f(s_2), f(s_3)) \in \Upsilon$. Thus $t_1 = f(s_3) \in S_\Upsilon(f(s_2))$. Then we have $S_\Upsilon(f(s_1)) \subseteq S_\Upsilon(f(s_2))$. Therefore we get $f(s_1) \in PS_\Upsilon(f(s_2))$.

Conversely, it is easy to verify that $s_1 \in PS_\Lambda(s_2)$ whenever $f(s_1) \in PS_\Upsilon(f(s_2))$ for all $s_1, s_2 \in S$.

(2) Let X be a non-empty subset of S. We verify firstly that $f(\overline{\Lambda}(X)) = \overline{\Upsilon}(f(X))$. Let $t_1 \in f(\overline{\Lambda}(X))$. Then there exists $s_1 \in \overline{\Lambda}(X)$ such that $f(s_1) = t_1$. Therefore, $PS_\Lambda(s_1) \cap X \neq \emptyset$. Thus there exists $s_2 \in S$ such that $s_2 \in PS_\Lambda(s_1)$ and $s_2 \in X$. By item (1), we obtain that $f(s_2) \in PS_\Upsilon(f(s_1))$ and $f(s_2) \in f(X)$. Then, $PS_\Upsilon(f(s_1)) \cap f(X) \neq \emptyset$, and so $t_1 = f(s_1) \in \overline{\Upsilon}(f(X))$. Thus we have $f(\overline{\Lambda}(X)) \subseteq \overline{\Upsilon}(f(X))$.

On the other hand, we let $t_2 \in \overline{\Upsilon}(f(X))$. Then there exists $s_3 \in S$ such that $f(s_3) = t_2$. Hence we get that $PS_\Upsilon(f(s_3)) \cap f(X) \neq \emptyset$. Thus there exists $s_4 \in X$ such that $f(s_4) \in f(X)$ and $f(s_4) \in PS_\Upsilon(f(s_3))$. By the argument (1), we get that $s_4 \in PS_\Lambda(s_3)$, and so $PS_\Lambda(s_3) \cap X \neq \emptyset$. Hence $s_3 \in \overline{\Lambda}(X)$, and so $t_2 = f(s_3) \in f(\overline{\Lambda}(X))$. Thus $\overline{\Upsilon}(f(X)) \subseteq f(\overline{\Lambda}(X))$. Therefore, $f(\overline{\Lambda}(X)) = \overline{\Upsilon}(f(X))$.

(3) Let X be a non-empty subset of S. Suppose that $t_1 \in f(\underline{\Lambda}(X))$. Then there exists $s_1 \in \underline{\Lambda}(X)$ such that $f(s_1) = t_1$. Thus we get $PS_\Lambda(s_1) \subseteq X$. We shall prove that $PS_\Upsilon(t_1) \subseteq f(X)$. Let $t_2 \in PS_\Upsilon(t_1)$. Then there exist $s_2 \in S$ such that $f(s_2) = t_2$. Thus we have $f(s_2) \in PS_\Upsilon(f(s_1))$. By the argument (1), we obtain that $s_2 \in PS_\Lambda(s_1)$, and so $s_2 \in X$. Hence $t_2 = f(s_2) \in f(X)$, and thus, $PS_\Upsilon(t_1) \subseteq f(X)$. Therefore we have $t_1 \in \underline{\Upsilon}(f(X))$. As a consequence, $f(\underline{\Lambda}(X)) \subseteq \underline{\Upsilon}(f(X))$.

(4) Let X be a non-empty subset of S. We only need to prove that $\underline{\Upsilon}(f(X)) \subseteq f(\underline{\Lambda}(X))$. Let $t_1 \in \underline{\Upsilon}(f(X))$. Then there exists $s_1 \in S$ such that $f(s_1) = t_1$. Thus $PS_\Upsilon(f(s_1)) \subseteq f(X)$. We shall show that $PS_\Lambda(s_1) \subseteq X$. Let $s_2 \in PS_\Lambda(s_1)$. Then, by the argument (1), we have $f(s_2) \in PS_\Upsilon(f(s_1))$. Hence $f(s_2) \in f(X)$. Thus there exists $s_3 \in X$ such that $f(s_3) = f(s_2)$. By the assumption, $s_2 \in X$, and so $PS_\Lambda(s_1) \subseteq X$. Hence $s_1 \in \underline{\Lambda}(X)$, and so $t_1 = f(s_1) \in f(\underline{\Lambda}(X))$. Thus $\underline{\Upsilon}(f(X)) \subseteq f(\underline{\Lambda}(X))$.

By the argument (3), we get $f(\underline{\Lambda}(X)) \subseteq \underline{\Upsilon}(f(X))$. Consequently, $f(\underline{\Lambda}(X)) = \underline{\Upsilon}(f(X))$.

(5) The proof is straightforward, so we omit it. ■

Proposition 11. *Let f be an isomorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If Υ is complete, then Λ is complete.*

Proof: Let s_1, s_2 be two elements in S. Suppose that $s_3 \in PS_\Lambda(s_1 s_2)$. Then, by Proposition 10 (1), we get that $f(s_3) \in PS_\Upsilon(f(s_1 s_2))$. Since f is a homomorphism and Υ is complete, we have

$$\begin{aligned} f(s_3) &\in PS_\Upsilon(f(s_1 s_2)) \\ &= PS_\Upsilon(f(s_1) f(s_2)) \\ &= (PS_\Upsilon(f(s_1)))(PS_\Upsilon(f(s_2))). \end{aligned}$$

Thus there exist $t_1 \in PS_\Upsilon(f(s_1))$, $t_2 \in PS_\Upsilon(f(s_2))$ such that $f(s_3) = t_1 t_2$. Since f is surjective, there exist $s_4, s_5 \in S$ such that $f(s_4) = t_1$ and $f(s_5) = t_2$. Since

$$f(s_4) f(s_5) = f(s_3) \in (PS_\Upsilon(f(s_1)))(PS_\Upsilon(f(s_2))),$$

we have $f(s_4) \in PS_\Upsilon(f(s_1))$ and $f(s_5) \in PS_\Upsilon(f(s_2))$. By Proposition 10 (1), $s_4 \in PS_\Lambda(s_1)$ and $s_5 \in PS_\Lambda(s_2)$. Since f is a homomorphism, $f(s_3) = f(s_4) f(s_5) = f(s_4 s_5)$. By the assumption, we obtain that $s_3 = s_4 s_5$. Thus we get that $s_3 \in PS_\Lambda(s_1) PS_\Lambda(s_2)$. Therefore we obtain that $PS_\Lambda(s_1 s_2) \subseteq PS_\Lambda(s_1) PS_\Lambda(s_2)$.

On the other hand, by Propositions 7 and 10 (5), we obtain that $PS_\Lambda(s_1) PS_\Lambda(s_2) \subseteq PS_\Lambda(s_1 s_2)$. Thus $PS_\Lambda(s_1) PS_\Lambda(s_2) = PS_\Lambda(s_1 s_2)$. Hence we get $\mathcal{PS}_\Lambda(S)$ is Λ-complete. Thus Λ is complete. ■

Theorem 7. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type PCR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup if and only if $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation semigroup.*

Proof: Suppose that $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup. Then, by Proposition 10 (2), we obtain that

$$\begin{aligned} (\overline{\Upsilon}(f(X)))(\overline{\Upsilon}(f(X))) &= (f(\overline{\Lambda}(X)))(f(\overline{\Lambda}(X))) \\ &= f((\overline{\Lambda}(X))(\overline{\Lambda}(X))) \\ &\subseteq f(\overline{\Lambda}(X)) \\ &= \overline{\Upsilon}(f(X)). \end{aligned}$$

Hence $\overline{\Upsilon}(f(X))$ is a subsemigroup of T. Thus we get $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation semigroup.

Conversely, we suppose that $s_1 \in (\overline{\Lambda}(X))(\overline{\Lambda}(X))$. From Proposition 10 (2), it follows that

$$\begin{aligned} f(s_1) &\in f((\overline{\Lambda}(X))(\overline{\Lambda}(X))) \\ &= (f(\overline{\Lambda}(X)))(f(\overline{\Lambda}(X))) \\ &= (\overline{\Upsilon}(f(X)))(\overline{\Upsilon}(f(X))) \\ &\subseteq \overline{\Upsilon}(f(X)) \\ &= f(\overline{\Lambda}(X)). \end{aligned}$$

Thus there exists $s_2 \in \overline{\Lambda}(X)$ such that $f(s_1) = f(s_2)$. Hence we have $PS_\Lambda(s_2) \cap X \neq \emptyset$. By the assumption, we obtain that $PS_\Lambda(s_1) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Lambda}(X)$. Hence $(\overline{\Lambda}(X))(\overline{\Lambda}(X)) \subseteq \overline{\Lambda}(X)$. Thus $\overline{\Lambda}(X)$ is a subsemigroup of S. Therefore, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation semigroup. ■

Theorem 8. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type PCR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation semigroup if and only if $\underline{\Upsilon}(f(X))$ is a $\mathcal{PS}_\Upsilon(T)$-lower approximation semigroup.*

Proof: By Proposition 10 (4) and using the similar method in the proof of Theorem 7, we can prove that the statement holds. ■

The following corollary is immediate consequences of Theorems 7 and 8.

Corollary 4. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type PCR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough semigroup if and only if $\Upsilon(f(X))$ is a $\mathcal{PS}_\Upsilon(T)$-rough semigroup.*

Theorem 9. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type PCR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal if and only if $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation ideal.*

Proof: Suppose that $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal. Then we have $S\overline{\Lambda}(X) \subseteq \overline{\Lambda}(X)$. Whence we have $f(S\overline{\Lambda}(X)) \subseteq f(\overline{\Lambda}(X))$. By Proposition 10 (2), we obtain that

$$T\overline{\Upsilon}(f(X)) = f(S\overline{\Lambda}(X)) \subseteq f(\overline{\Lambda}(X)) = \overline{\Upsilon}(f(X)).$$

Hence $\overline{\Upsilon}(f(X))$ is a left ideal of T. Similarly, we can prove that $\overline{\Upsilon}(f(X))$ is a right ideal of T. Thus $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation ideal.

Conversely, we suppose that $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation ideal. Then, $T\overline{\Upsilon}(f(X)) \subseteq \overline{\Upsilon}(f(X))$. Now, let $s_1 \in S\overline{\Lambda}(X)$. From Proposition 10 (2), it follows that

$$f(s_1) \in f(S\overline{\Lambda}(X)) = T\overline{\Upsilon}(f(X)) \subseteq \overline{\Upsilon}(f(X)) = f(\overline{\Lambda}(X)).$$

Thus there exists $s_2 \in \overline{\Lambda}(X)$ such that $f(s_1) = f(s_2)$, and so $PS_\Lambda(s_2) \cap X \neq \emptyset$. By the assumption, we have that $PS_\Lambda(s_1) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Lambda}(X)$. Thus we get $S\overline{\Lambda}(X) \subseteq \overline{\Lambda}(X)$. Then, $\overline{\Lambda}(X)$ is a left ideal of S. Similarly, we can check that $\overline{\Lambda}(X)$ is a right ideal of S. Therefore, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation ideal. ■

Theorem 10. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type PCR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation ideal if and only if $\underline{\Upsilon}(f(X))$ is a $\mathcal{PS}_\Upsilon(T)$-lower approximation ideal.*

Proof: By Proposition 10 (4) and using the similar method in the proof of Theorem 9, we can prove that the statement holds. ■

The following corollary is immediate consequences of Theorems 9 and 10.

Corollary 5. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type PCR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\Lambda(X)$ is a $\mathcal{PS}_\Lambda(S)$-rough ideal if and only if $\Upsilon(f(X))$ is a $\mathcal{PS}_\Upsilon(T)$-rough ideal.*

Theorem 11. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type CR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal if and only if $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation completely prime ideal.*

Proof: Assume that $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal. Let $t_1, t_2 \in T$ be such that $t_1t_2 \in \overline{\Upsilon}(f(X))$. Thus there exist $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$. Hence we have $PS_\Upsilon(f(s_1)f(s_2)) \cap f(X) \neq \emptyset$. Since Υ is complete,

$$(PS_\Upsilon(f(s_1)))(PS_\Upsilon(f(s_2))) \cap f(X) \neq \emptyset.$$

Then there exists $f(s_3) \in PS_\Upsilon(f(s_1))$ and exists $f(s_4) \in PS_\Upsilon(f(s_2))$ such that $f(s_3)f(s_4) \in f(X)$, and so $f(s_3s_4) \in f(X)$. Then there exists $s_5 \in X$ such that $f(s_3s_4) = f(s_5)$. By Proposition 10 (1), we obtain that $s_3 \in PS_\Lambda(s_1)$ and $s_4 \in PS_\Lambda(s_2)$. From Proposition 7 and Proposition 10 (5), it follows that $s_3s_4 \in PS_\Lambda(s_1s_2)$. By the assumption, we get that $s_5 = s_3s_4 = PS_\Lambda(s_1s_2)$. Thus $PS_\Lambda(s_1s_2) \cap X \neq \emptyset$, and so $s_1s_2 \in \overline{\Lambda}(X)$. Since $\overline{\Lambda}(X)$ is a completely prime ideal of S, we have that $s_1 \in \overline{\Lambda}(X)$ or $s_2 \in \overline{\Lambda}(X)$. Hence we have $f(s_1) \in f(\overline{\Lambda}(X))$ or $f(s_2) \in f(\overline{\Lambda}(X))$. From Proposition 10 (2), we get $f(s_1) \in \overline{\Upsilon}(f(X))$ or $f(s_2) \in \overline{\Upsilon}(f(X))$, which yields $t_1 \in \overline{\Upsilon}(f(X))$ or $t_2 \in \overline{\Upsilon}(f(X))$. Thus $\overline{\Upsilon}(f(X))$ is a completely prime ideal of T. Therefore, $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Upsilon(T)$-upper approximation completely prime ideal.

Conversely, we suppose that $\overline{\Upsilon}(f(X))$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal. Let s_6, s_7 be two elements in S such that $s_6s_7 \in \overline{\Lambda}(X)$. Then, $f(s_6s_7) \in f(\overline{\Lambda}(X))$. By Proposition 10 (2), we obtain that

$$f(s_6)f(s_7) = f(s_6s_7) \in f(\overline{\Lambda}(X)) = \overline{\Upsilon}(f(X)).$$

Thus $f(s_6) \in \overline{\Upsilon}(f(X))$ or $f(s_7) \in \overline{\Upsilon}(f(X))$. Now, we consider the following two cases.

Case 1. If $f(s_6) \in \overline{\Upsilon}(f(X))$, then we have that $f(s_6) \in f(\overline{\Lambda}(X))$ since Proposition 10 (2). Thus there exists $s_8 \in \overline{\Lambda}(X)$ such that $f(s_6) = f(s_8)$. Whence we get $PS_\Lambda(s_8) \cap X \neq \emptyset$. Thus $PS_\Lambda(s_6) \cap X \neq \emptyset$ since f is injective. Therefore, $s_6 \in \overline{\Lambda}(X)$.

Case 2. If $f(s_7) \in \overline{\Upsilon}(f(X))$, then $s_7 \in \overline{\Lambda}(X)$ since the proof is similar to that the first case.

Consequently, $\overline{\Lambda}(X)$ is an $\mathcal{PS}_\Lambda(S)$-upper approximation completely prime ideal. ■

Theorem 12. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type CR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If X is a non-empty subset of S and f is injective, then $\underline{\Lambda}(X)$ is a $\mathcal{PS}_\Lambda(S)$-lower approximation completely prime ideal if and only if $\underline{\Upsilon}(f(X))$ is a $\mathcal{PS}_\Upsilon(T)$-lower approximation completely prime ideal.*

Proof: By Proposition 10 (4) and using the similar method in the proof of Theorem 11, we can prove that the statement holds. ■

The following corollary is immediate consequences of Theorems 11 and 12.

Corollary 6. *Let f be an epimorphism from S in $(S, \mathcal{PS}_\Lambda(S))$ to T in $(T, \mathcal{PS}_\Upsilon(T))$ type CR, where the binary relation $\Lambda := \{(s_1, s_2) \in S \times S : (f(s_1), f(s_2)) \in \Upsilon\}$. If*

X is a non-empty subset of S and f is injective, then $\Lambda(X)$ is a $\mathcal{PS}_{\Lambda}(S)$-rough completely prime ideal if and only if $\Upsilon(f(X))$ is a $\mathcal{PS}_{\Upsilon}(T)$-rough completely prime ideal.

VI. Discussions and Conclusions

In this section we discuss approximation forms of this research and models in [1], [4], [7], [13], [23].

Firstly, concepts of generalizations of rough sets in general have been established as the following the diagram.

$$[1] \longrightarrow [23] \longrightarrow \text{Our rough sets}$$

Based on this point, if the equivalence property of a relation is put in the Mareay's rough set [23], then the Mareay's rough set is a generalization of the Pawlak's rough set [1]. Moreover, if our rough set is considered under the single universe and the equal condition in Definition 5, then such the rough set is a generalization of the Mareay's rough set.

Secondly, we discuss main results in a semigroup of this work (Sections IV and V), Kuroki [4], Xiao and Zhang [7], and Wang and Zhan [13] by using TABLES IV, V and VI below.

In the following TABLES IV, V and VI, the symbol ✓ denotes two statements as the following.

(1) The sufficient condition (briefly, SC) of an upper approximation semigroup (briefly, UAS) (resp. a lower approximation semigroup (briefly, LAS) and a rough semigroup (briefly, RS)) is provided in [4], [7], [13], or this work. Similarly, if sufficient conditions of an upper approximation ideal (briefly, UAI) (resp. a lower approximation ideal (briefly, LAI) and a rough ideal (briefly, RI)) and an upper approximation completely prime ideal (briefly, UAC) (resp. a lower approximation completely prime ideal (briefly, LAC) and a rough completely prime ideal (briefly, RC)) are provided.

(2) The relationship between the UAS (resp. LAS and RS) and the homomorphic image of the UAS (resp. LAS and RS) is demonstrated under homomorphism problems (briefly, HP) in [4], [7], [13], or this work. Similarly, if UAI (resp. LAI and RI) and UAC (resp. LAC and RC) are examined under HP.

TABLE IV
THE RESULTS OF UPPER APPROXIMATIONS IN SEMIGROUPS

	[4]	[7]	[13]	Our Model
UAS (SC)	✓		✓	✓
UAI (SC)	✓		✓	✓
UAC (SC)		✓	✓	✓
UAS (HP)				✓
UAI (HP)		✓		✓
UAC (HP)		✓		✓

TABLE V
THE RESULTS OF LOWER APPROXIMATIONS IN SEMIGROUPS

	[4]	[7]	[13]	Our Model
LAS (SC)	✓		✓	✓
LAI (SC)	✓		✓	✓
LAC (SC)		✓	✓	✓
LAS (HP)				✓
LAI (HP)		✓		✓
LAC (HP)		✓		✓

TABLE VI
THE RESULTS OF ROUGH SETS IN SEMIGROUPS

	[4]	[7]	[13]	Our Model
RS (SC)				✓
RI (SC)				✓
RC (SC)				✓
RS (HP)				✓
RI (HP)				✓
RC (HP)				✓

From TABLES IV, V and VI, we observe that sufficient conditions are completely obtained in this research (Section IV). Furthermore, connections under homomorphism problems are entirely verified in this work (Section V).

From the Mareay's rough set induced by a binary relation on the single universe, a generalization of the Mareay's rough set was constructed in an approximation space based on portions of successor classes induced by a binary relation between two universes, and a corresponding example was gave. Moreover, interesting algebraic properties were investigated. Under a preorder and compatible relation, approximation processings in semigroups were applied from the novel generalized rough set. As discussed above, it indicates that sufficient conditions of rough semigroups, rough ideals and rough completely prime ideals are fully obtained, and associations under homomorphism problems are ideally checked. The novel generalized rough set can be applied in a semigroup. However, when we consider other algebraic systems, the corresponding issues need to be further examined.

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