Relationships Among Various Definitions of Two-Dimensional Quaternion Linear Canonical Transforms

Mawardi Bahri

***Abstract*—The quaternion linear canonical transforms (QLCT) is a nontrivial generalization of the linear canonical transform (LCT) using quaternion algebra. Due to the non-commutative property of quaternion multiplication there are different definitions for the QLCT. We establish the relationships among different types of the QLCTs.**

***Index Terms*—quaternion linear canonical transform, quaternion Fourier transform**

I. Introduction

The linear canonical transform (LCT) attracts increasing research interests recently as new analysis tool in many field of signal processing, image processing and optics. It can be regarded as a generalization of many mathematical transforms such as the Fourier transform, Laplace transform, the fractional Fourier transform, the Fresnel transform and the other transforms. With intensive research of the LCT, many useful properties of this transform have been found including shift, modulation, convolution, and correlation and uncertainty principle and so on (see, e.g. [7], [11], [16], [18], [19], [21], [22], [23]). Therefore, it is worthwhile and interesting to extend the properties of the LCT to new integral transform using quaternion algebra. This extension is not straightforward, mainly due to the inherent property of non-commutativity of quaternion multiplications. As it is shown in [2], [5], [14], [15], [24] the LCT has been investigated within the context of quaternion algebra and is a so-called the quaternion linear canonical transform (QLCT). Based on the quaternion Fourier transform (QFT) definitions [3], [4], [6], [8], [10], [12], [13], there exist different definitions of the QLCT. Further, they also have been successfully established several important properties of various types of QLCT such as shift, modulation, inversion formula and the uncertainty principle, which are generalizations of the corresponding properties of the LCT with some modifications. An important issue regarding the QLCT is to study relations of some definitions of the QLCT. In [17], the author considered relationship among various definitions of 2-D quaternion Fourier transforms (QFTs). As a generalized form of the QFT, it is possible to obtain relations of various definitions of the QLCTs using relation between the QFT and the QLCT. We have in mind to find out in which sense the properties of various definitions of the QFTs can be established in the QLCT definitions. In the present paper, our purpose is to treat in detail some relationships among different types of the QLCTs.

The paper is organized as follows: Section II presents notations and some useful properties of quaternions and decomposition of quaternion signal. Various definitions of the QLCT and properties of their kernel functions are presented in Section III. The relationships among three different definitions of type I QLCTs are studied in Section IV. Section V provides the relationships among three different definitions of type II QLCTs. Some conclusions are drawn in Section VI.

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Mawardi Bahri is with the Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia (corresponding author: e-mail: mawardibahri@gmail.com)

II. Preliminaries

In the present section we summarize some basic facts about quaternions and decomposition of quaternion signal, which will be needed throughout in the paper.

A. Quaternion Algebra

Let $\mathbb{H}$ be the set of quaternions over $\mathbb{R}$. Every element of $\mathbb{H}$ can be written in the form

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \ ; \ q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \quad (1)$$

which obeys the following multiplication rules:

$$\begin{aligned} &\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \\ &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \quad (2)$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is called the *scalar* part of q denoted by $Sc(q)$ and $\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is called the *vector* (or *pure*) part of q. The vector part of q is conventionally denoted by $\mathbf{q}$.

Like complex numbers, the quaternion conjugate of q is given by

$$\bar{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \quad (3)$$

and satisfies an anti-involution, i.e.

$$\overline{qp} = \bar{p}\bar{q}. \quad (4)$$

From (3) we obtain the norm or modulus of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \quad (5)$$

It is not difficult to see that

$$|qp| = |q||p|, \qquad \forall p, q \in \mathbb{H}. \quad (6)$$

Applying the conjugate (3) and modulus of q, we obtain the inverse of $q \in \mathbb{H} \setminus \{0\}$ in the form

$$q^{-1} = \frac{\bar{q}}{|q|^2}. \quad (7)$$

A quaternion number q may be defined as a complex number with complex and imaginary parts.

$$q = a + \mathbf{j}b, \quad a = q_0 + \mathbf{i}q_1, \quad b = q_2 + \mathbf{i}q_3. \tag{8}$$

Equation (8) is known as the Cayley-Dickson form.

B. Decomposition of Quaternion Signal

The non-commutativity of quaternion multiplication causes difficulty in some applications of quaternions. One of the most effective method of solving this problem is the decompositions of quaternions which assist in making simplifications. There are many ways to decompose quaternion signal as described below (see [1], [17]). According to (8) every 2-D quaternion signal $f(\boldsymbol{x})$ can be decomposed into a symplectic form as follows:

$$\begin{aligned} f(\boldsymbol{x}) &= f_0(\boldsymbol{x}) + \mu_1 f_1(\boldsymbol{x}) + \mu_2 f_2(\boldsymbol{x}) + \mu_3 f_3(\boldsymbol{x}) \\ &= (f_0(\boldsymbol{x}) + \mu_1 f_1(\boldsymbol{x})) + (f_2(\boldsymbol{x}) + \mu_1 f_3(\boldsymbol{x}))\mu_2 \\ &= f_s(\boldsymbol{x}) + f_p(\boldsymbol{x})\mu_2, \quad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, \end{aligned} \tag{9}$$

where $\mu_3 = \mu_1\mu_2$ with $\{\mu_1, \mu_2, \mu_3\}$ being quaternionic roots of $-1(\mu_1^2 = \mu_2^2 = \mu_3^2 = -1)$. Here $f_s(\boldsymbol{x}) = f_0(\boldsymbol{x}) + \mu_1 f_1(\boldsymbol{x})$ is known as the simplex part and $f_p(\boldsymbol{x}) = f_2(\boldsymbol{x}) + \mu_1 f_3(\boldsymbol{x})$ is called the perplex part.

Any 2-D quaternion signal $f(\boldsymbol{x})$ also can be split into even (e) and odd (o) parts along the x_1-and x_2-axis as

$$f(\boldsymbol{x}) = f_{ee}(\boldsymbol{x}) + f_{eo}(\boldsymbol{x}) + f_{oe}(\boldsymbol{x}) + f_{oo}(\boldsymbol{x}). \tag{10}$$

Here f_{ee} denotes the part of f that is even with respect to x_1 and x_2, f_{oe} denotes the part that is odd with respect to x_1 and even with respect to x_2, and so on. The decomposition (10) can be written uniquely as

$$\begin{aligned} & f_{ee}(\boldsymbol{x}) \\ &= \frac{1}{4}[f(\boldsymbol{x}) + f(-x_1, x_2) + f(x_1, -x_2) + f(-x_1, -x_2)] \\ & f_{eo}(\boldsymbol{x}) \\ &= \frac{1}{4}[f(\boldsymbol{x}) + f(-x_1, x_2) - f(x_1, -x_2) - f(-x_1, -x_2)] \\ & f_{oe}(\boldsymbol{x}) \\ &= \frac{1}{4}[f(\boldsymbol{x}) - f(-x_1, x_2) + f(x_1, -x_2) - f(-x_1, -x_2)] \\ & f_{oo}(\boldsymbol{x}) \\ &= \frac{1}{4}[f(\boldsymbol{x}) - f(-x_1, x_2) - f(x_1, -x_2) + f(-x_1, -x_2)]. \end{aligned} \tag{11}$$

Besides the even and odd parts, any quaternion signal can be decomposed into sum of the partial even and odd parts as

$$\begin{aligned} f(\boldsymbol{x}) &= f_{e1}(\boldsymbol{x}) + f_{o1}(\boldsymbol{x}) \\ &= f_{e2}(\boldsymbol{x}) + f_{o2}(\boldsymbol{x}), \end{aligned} \tag{12}$$

where

$$\begin{aligned} f_{e1}(\boldsymbol{x}) &= f_{ee}(\boldsymbol{x}) + f_{eo}(\boldsymbol{x}) = \frac{1}{2}[f(\boldsymbol{x}) + f(-x_1, x_2)] \\ f_{o1}(\boldsymbol{x}) &= f_{oe}(\boldsymbol{x}) + f_{oo}(\boldsymbol{x}) = \frac{1}{2}[f(\boldsymbol{x}) - f(-x_1, x_2)] \end{aligned} \tag{13}$$

and

$$\begin{aligned} f_{e2}(\boldsymbol{x}) &= f_{ee}(\boldsymbol{x}) + f_{oe}(\boldsymbol{x}) = \frac{1}{2}[f(\boldsymbol{x}) + f(x_1, -x_2)] \\ f_{o2}(\boldsymbol{x}) &= f_{eo}(\boldsymbol{x}) + f_{oo}(\boldsymbol{x}) = \frac{1}{2}[f(\boldsymbol{x}) - f(x_1, -x_2)]. \end{aligned} \tag{14}$$

Here f_{e1} and f_{o1} have an even and odd symmetry along the x_1-axis, respectively. Similarly, f_{e2} and f_{o2} individually have an even and odd symmetry along the x_2-axis.

III. Quaternion Linear Canonical Transform (QLCT)

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In this section we discuss various definitions of the QLCT and properties of their kernel functions.

A. Various Definitions of QLCT

In [1], the author presents 8 different possible definitions of the quaternion Fourier transform (QFT). Based on these, we introduce 6 different definitions of the QLCT. They are constructed using the QLCT kernel functions.

Definition 1 (Single-axis (type I), Left-sided, Right-sided and Two-sided QLCTs)**.** *Suppose that $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ are real matrix parameters such that $det(A_1) = det(A_2) = 1$. The left-sided, right-sided and two-sided type I QLCTs of a quaternion signal $f \in L^1(\mathbb{R}^2; \mathbb{H})$ are defined by, respectively,*

$$L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} K^{\mu_1}_{A_1}(x_1, \omega_1) K^{\mu_1}_{A_2}(x_2, \omega_2) f(\boldsymbol{x})\, d\boldsymbol{x} \tag{15}$$

$$L^{I,(r),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) K^{\mu_1}_{A_1}(x_1, \omega_1) K^{\mu_1}_{A_2}(x_2, \omega_2)\, d\boldsymbol{x} \tag{16}$$

$$L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} K^{\mu_1}_{A_1}(x_1, \omega_1) f(\boldsymbol{x}) K^{\mu_1}_{A_2}(x_2, \omega_2)\, d\boldsymbol{x}, \tag{17}$$

where the kernel functions of the QLCT above are given by

$$K^{\mu_1}_{A_1}(x_1, \omega_1) = \begin{cases} \frac{1}{\sqrt{2\pi b_1}} e^{\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2 - \frac{\pi}{2}\right)}, & b_1 \neq 0 \\ \sqrt{d_1}\, e^{\mu_1\left(\frac{c_1 d_1}{2}\right)\omega_1^2}, \ b_1 = 0, \end{cases} \tag{18}$$

and

$$K^{\mu_2}_{A_2}(x_2, \omega_2) = \begin{cases} \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mu_2}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 - \frac{\pi}{2}\right)}, & b_2 \neq 0 \\ \sqrt{d_2}\, e^{\mu_1\left(\frac{c_2 d_2}{2}\right)\omega_2^2}, \ b_2 = 0. \end{cases} \tag{19}$$

From the above definition we obtain the following remark:

1) It is noted that for $b_i = 0$ or $b_1 b_2 = 0, i = 1, 2$ the QLCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Therefore, we shall consider the QLCT definitions for $b_1 b_2 \neq 0$.
2) As a special case, when $A_1 = A_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$ for $i = 1, 2$, the two-sided QLCT definition (17) reduces to the two-sided QFT definition. That

is

$$
\begin{aligned}
&L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2}\frac{e^{\mu_1\frac{\pi}{4}}}{\sqrt{2\pi}}e^{\mu_1\omega_1x_1}\,f(\boldsymbol{x})\frac{e^{\mu_1\frac{\pi}{4}}}{\sqrt{2\pi}}\,e^{\mu_1\omega_2x_2}\,d\boldsymbol{x}\\
&=\frac{e^{\mu_1\frac{\pi}{4}}}{\sqrt{2\pi}}\mathcal{F}_q\{f\}(\boldsymbol{\omega})\frac{e^{\mu_1\frac{\pi}{4}}}{\sqrt{2\pi}},
\end{aligned}\tag{20}
$$

where $\mathcal{F}_q\{f\}$ is the two-sided QFT defined in [8], [13].

Next, from the symplectic decomposition of the 2-D quaternion signal (9) and the definition of the left-sided type I QLCT (15) we immediately get

$$
\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1\omega_1+\frac{d_2}{b_1}\omega_1^2-\frac{\pi}{2}\right)}\\
&\times\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mu_1}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2^2-\frac{\pi}{2}\right)}(f_s(\boldsymbol{x})+f_p(\boldsymbol{x})\,\mu_2)\,d\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1\omega_1+\frac{d_2}{b_1}\omega_1^2-\frac{\pi}{2}\right)}\\
&\times\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mu_1}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2^2-\frac{\pi}{2}\right)}f_s(\boldsymbol{x})\,d\boldsymbol{x}\\
&+\int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}}e^{\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1\omega_1+\frac{d_2}{b_1}\omega_1^2-\frac{\pi}{2}\right)}\\
&\times\frac{1}{\sqrt{2\pi b_2}}e^{\frac{\mu_1}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2^2-\frac{\pi}{2}\right)}f_p(\boldsymbol{x})\mu_2\,d\boldsymbol{x}\\
&=L^{I,(l),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega})+L^{I,(l),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega})\mu_2.
\end{aligned}\tag{21}
$$

In a similar manner,

$$
L^{I,(r),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})=L^{I,(r),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega})+\mu_2L^{I,(r),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega}).\tag{22}
$$

Definition 2 (Factored (Type II), Left-sided, Right-sided and Two-sided QLCTs)**.** *Let* $A_1=(a_1,b_1,c_1,d_1)$ *and* $A_2=(a_2,b_2,c_2,d_2)$ *be matrix parameters satisfying* $det(A_1)=det(A_2)=1$*. The left-sided, right-sided and two-sided type II QLCTs of a quaternion signal* $f\in L^1(\mathbb{R}^2;\mathbb{H})$ *are defined by, respectively,*

$$
\begin{aligned}
&L^{II,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_2}_{A_2}(x_2,\omega_2)f(\boldsymbol{x})\,d\boldsymbol{x}
\end{aligned}\tag{23}
$$

$$
\begin{aligned}
&L^{II,(r),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2}f(\boldsymbol{x})K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_2}_{A_2}(x_2,\omega_2)\,d\boldsymbol{x}
\end{aligned}\tag{24}
$$

$$
\begin{aligned}
&L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)f(\boldsymbol{x})K^{\mu_2}_{A_2}(x_2,\omega_2)\,d\boldsymbol{x}.
\end{aligned}\tag{25}
$$

B. Useful Properties of QLCT Kernel

The following proposition summarize some useful properties of the kernel functions $K_{A_1}(x_1,\omega_1)$ and $K_{A_2}(x_2,\omega_2)$ of the QLCT, which will be used in the next section.

Proposition 1. *Let the kernel functions* $K_{A_1}(x_1,\omega_1)$ *and* $K_{A_2}(x_2,\omega_2)$ *be defined by* (18) *and* (19)*. Then we get*

$$
\begin{aligned}
K^{\mu_1}_{A_1}(-x_1,\omega_1)&=K^{\mu_1}_{A_1}(x_1,-\omega_1)\\
K^{\mu_2}_{A_2}(-x_2,\omega_2)&=K^{\mu_2}_{A_2}(x_2,-\omega_2);\\
K^{\mu_1}_{A_1}(-x_1,-\omega_1)&=K^{\mu_1}_{A_1}(x_1,\omega_1)\\
K^{\mu_2}_{A_2}(-x_2,-\omega_2)&=K^{\mu_2}_{A_2}(x_2,\omega_2)\\
\overline{K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_2}_{A_2}(x_2,\omega_2)}&=K^{\mu_2}_{A_2^{-1}}(x_2,\omega_2)K^{\mu_1}_{A_1^{-1}}(x_1,\omega_1).
\end{aligned}
$$

Proof: The proof of the above proposition follows directly from the equations (18) and (19). Details are left to the reader. ■

We obtain the following important result which hold for all types of the QLCTs. For simplicity we only consider the left-sided type I QLCT.

Lemma 1. *If quaternion signal* $f(\boldsymbol{x})$ *is even or odd, then its left-sided QLCT is also even or odd, i.e.,*

$$L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})=L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{ee}\}(\boldsymbol{\omega})\tag{26}$$

$$L^{I,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})=L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{oe}\}(\boldsymbol{\omega})\tag{27}$$

$$L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})=L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{eo}\}(\boldsymbol{\omega})\tag{28}$$

$$L^{I,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})=L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{oo}\}(\boldsymbol{\omega}).\tag{29}$$

Proof: We only prove the first assertion in (26), with the other being similar. Truly, we have from (11)

$$
\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{ee}\}(\boldsymbol{\omega})\\
&=\frac{1}{4}\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)\\
&\quad\times\Big[f(\boldsymbol{x})+f(-x_1,x_2)+f(x_1,-x_2)+f(-x_1,-x_2)\Big]\,d\boldsymbol{x}\\
&=\frac{1}{4}\Big[\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)f(\boldsymbol{x})\,d\boldsymbol{x}\\
&\quad+\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)f(-x_1,x_2)\,d\boldsymbol{x}\\
&\quad+\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)f(x_1,-x_2)\,d\boldsymbol{x}\\
&\quad+\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)f(-x_1,-x_2)\,d\boldsymbol{x}\Big]
\end{aligned}
$$

Applying Proposition 1 we get

$$
\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{ee}\}(\boldsymbol{\omega})\\
&=\frac{1}{4}\Big[\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)f(\boldsymbol{x})\,d\boldsymbol{x}\\
&+\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,-\omega_1)K^{\mu_1}_{A_2}(x_2,\omega_2)f(x_1,x_2)\,d\boldsymbol{x}\\
&+\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,\omega_1)K^{\mu_1}_{A_2}(x_2,-\omega_2)f(x_1,x_2)\,d\boldsymbol{x}\\
&+\int_{\mathbb{R}^2}K^{\mu_1}_{A_1}(x_1,-\omega_1)K^{\mu_1}_{A_2}(x_2,-\omega_2)f(x_1,x_2)\,d\boldsymbol{x}\Big]\\
&=\frac{1}{4}\Big[L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})+L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(-\omega_1,\omega_2)\\
&+L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\omega_1,-\omega_2)+L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}((-\omega_1,-\omega_2)\Big]\\
&=L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}),
\end{aligned}
$$

where $L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}$ is even part of the QLCT of f. The proof is complete. ■

TABLE I
RELATIONSHIP BETWEEN LEFT-SIDED AND RIGHT-SIDED TYPE I QLCTS.

	Right-Sided QLCT
Left-Sided QLCT	$L^{I,(l),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega})$
	$\mu_2 L^{I,(l),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{p,A_1^*,A_2^*}\{f\}(\boldsymbol{\omega})\mu_2$

IV. RELATIONSHIPS AMONG DIFFERENT DEFINITIONS OF TYPE I QLCTS

In this section we investigate the relationships among three different definitions of type I QLCTs. We will see that three QLCTs do not lead to the same results.

A. Relationship Between Left-Sided and Right-Sided Type I QLCTs

The following result describes the relationship between the left-sided and right-sided type I QLCTs as follows (see Table I).

Proposition 2. *If quaternion signal $f(\boldsymbol{x})$ is uniquely decomposed into symplectic form, then its type I QLCT satisfies the following relations:*

$$L^{I,(l),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega}), \tag{30}$$

and

$$\mu_2 L^{I,(l),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{p,A_1^*,A_2^*}\{f\}(\boldsymbol{\omega})\mu_2, \tag{31}$$

where $A_1^* = (a_1, -b_1, c_1, d_1)$ and $A_2^* = (a_2, -b_2, c_2, d_2)$.

Proof: The proof of (30) is based on the commutativity of simplex part to kernel of the QLCT, so we omit it. For (31), simple computations show that

$$\begin{aligned}
&\mu_2 L^{I,(r),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&= \int_{\mathbb{R}^2} \mu_2 f_p(\boldsymbol{x}) \frac{1}{\sqrt{2\pi b_1}} e^{\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2 - \frac{\pi}{2}\right)}\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mu_1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 - \frac{\pi}{2}\right)}\, d\boldsymbol{x}\\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2 - \frac{\pi}{2}\right)}\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{-\frac{\mu_1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 - \frac{\pi}{2}\right)} f_p(\boldsymbol{x})\mu_2\, d\boldsymbol{x}\\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{-2\pi b_1}} e^{-\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2 + \frac{\pi}{2}\right)}\\
&\quad \times \frac{1}{\sqrt{-2\pi b_2}} e^{-\frac{\mu_1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 + \frac{\pi}{2}\right)} f_p(\boldsymbol{x})\mu_2\, d\boldsymbol{x}\\
&= L^{I,(l),\mathbb{H}}_{p,A_1^*,A_2^*}\{f\}(\boldsymbol{\omega})\mu_2.
\end{aligned}$$

This proves the proposition. ■

From Proposition 2 it seems that the simplex part of the left-sided and right-sided type I QLCTs are the same, while the perplex parts of the two transforms is not the same. In this case the matrix parameter $A_i = (a_i, b_i, c_i, d_i)$ becomes $A_i^* = (a_i, -b_i, c_i, d_i)$ for $i = 1, 2$.

B. Relationship Between Left-Sided and Two-Sided QLCTs

In order to study the relationship between left-sided and two-sided type I QLCTs. We first discuss the even and odd parts of the left-sided and two-sided type I QLCTs. For a 2-D quaternion signal $f(\boldsymbol{x})$ we may define (compare to [17])

$$\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})\\
&= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \cos(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} \cos(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}) f(\boldsymbol{x})\, d\boldsymbol{x}\\
&= C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})
\end{aligned} \tag{32}$$

$$\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})\\
&= \frac{\mu_1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \cos(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} \sin(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}) f(\boldsymbol{x})\, d\boldsymbol{x}\\
&= \mu_1 C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})
\end{aligned} \tag{33}$$

$$\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\\
&= \frac{\mu_1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \sin(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} \cos(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}) f(\boldsymbol{x})\, d\boldsymbol{x}\\
&= \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})
\end{aligned} \tag{34}$$

$$\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\\
&= \frac{\mu_1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \sin(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})\\
&\quad \times \frac{\mu_1}{\sqrt{2\pi b_2}} \sin(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}) f(\boldsymbol{x})\, d\boldsymbol{x}\\
&= -S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega}).
\end{aligned} \tag{35}$$

Here the symbol $C_{A_1,x_1}C_{A_1,x_2}$ performs the cosine transforms associated with the LCT in the x_1 and x_2 directions, $C_{A_1,x_1}S_{A_1,x_2}$ denotes the cosine transform associated with the LCT in the x_1 direction and sine transform associated with the LCT in the x_2 direction and so on.

Theorem 1. *The relationship between the even and odd parts of left-sided and two-sided QLCTs is given by*

$$\begin{aligned}
L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) &= L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})\\
L^{I,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) &= L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\\
L^{I,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) &= -\mu_1 L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})\mu_1\\
L^{I,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) &= -\mu_1 L^{I,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1,
\end{aligned} \tag{36}$$

Proof: Direct application of Euler's formula to the kernel of the left-sided type I QLCT we easily get

$$\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mu_1\left(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4}\right)}\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{\mu_1\left(\frac{a_2}{2b_2}x_2^2 - \frac{2}{2b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}\right)} f(\boldsymbol{x})\, d\boldsymbol{x}\\
&= \int_{\mathbb{R}^2} \Bigg(\frac{1}{\sqrt{2\pi b_1}} \cos(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})\\
&\quad + \frac{\mu_1}{\sqrt{2\pi b_1}} \sin(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})
\end{aligned}$$

TABLE II
RELATIONSHIP BETWEEN LEFT-SIDED AND TWO SIDED TYPE I QLCTS.

	Two-Sided QLCT
Left-Sided QLCT	$L^{I,(t),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}) = L^{I,(l),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega})$
	$L^{I,(t),\mathbb{H}}_{A_1,A_2,o_2}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(l),\mathbb{H}}_{A_1,o_2}\{f\}(\boldsymbol{\omega})\mu_1$

$$\times \frac{1}{\sqrt{2\pi b_2}}\cos(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}) + \\ + \frac{\mu_1}{\sqrt{2\pi b_2}}\sin(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4})\Big) f(\boldsymbol{x})\, d\boldsymbol{x}$$

Simplifying the above identity yields

$$L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) + L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) \\ + L^{I,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) + L^{I,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}).$$

Performing similar reasoning as mentioned above, we obtain the even and odd parts of the two-sided QLCT as follows.

$$L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \tag{37}$$
$$L^{I,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1 \tag{38}$$
$$L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \tag{39}$$
$$L^{I,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = \mu_1 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1. \tag{40}$$

From equations (32), (34), (37) and (39) it is obvious that

$$L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}), \\ L^{I,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}). \tag{41}$$

If we multiply (33) by $-\mu_1$ on the left side and by μ_1 on the right side, we get

$$-\mu_1 L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})\mu_1 = -\mu_1[\mu_1 C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})]\mu_1 \\ = C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1 \\ \overset{(38)}{=} L^{I,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}). \tag{42}$$

Similar to (42) we can easily get

$$-\mu_1 L^{I,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})(-\mu_1) = L^{I,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}).$$

This finishes the proof of the theorem. ■

Next, we establish the relationship between the partial even and odd parts of the left-sided and right-sided type I QLCTs as shown in Table II.

Proposition 3. *If 2-D quaternion signal $f(\boldsymbol{x})$ is split into the partial even and odd parts, i.e., $f(\boldsymbol{x}) = f_{e2}(\boldsymbol{x}) + f_{o2}(\boldsymbol{x})$. The following relations are satisfied:*

$$L^{I,(t),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}) = L^{I,(l),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}), \\ L^{I,(t),\mathbb{H}}_{A_1,A_2,o_2}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(l),\mathbb{H}}_{A_1,o_2}\{f\}(\boldsymbol{\omega})\mu_1.$$

Moreover, the two-sided type I QLCT of 2-D quaternion signal $f(\boldsymbol{x})$ can be represented using the left-sided type I QLCT in the form

$$L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(l),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) - \mu_1 L^{I,(l),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})\mu_1. \tag{43}$$

Proof: It follows that

$$L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \\ = L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{e2} + f_{o2}\}(\boldsymbol{\omega}) \\ = L^{(l),\mathbb{H}}_{A_1,A_2}\{f_{e2}\}(\boldsymbol{\omega}) + L^{I,(l),\mathbb{H}}_{A_1,A_2}\{f_{o2}\}(\boldsymbol{\omega}) \\ = L^{I,(l),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) + L^{I,(l),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega}) \\ = [L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) + L^{I,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})] \\ + [L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) + L^{I,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})] \\ = [C_{A_1,x_1}C_{A_1,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}C_{A_1,x_2}(\boldsymbol{\omega})] \\ + [\mu_1 C_{A_1,x_1}S_{A_1,x_2}(\boldsymbol{\omega}) - S_{A_1,x_1}S_{A_1,x_2}(\boldsymbol{\omega})]. \tag{44}$$

Following the steps of (44) one can conclude that

$$L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \\ = L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f_{e2} + f_{o2}\}(\boldsymbol{\omega}) \\ = L^{I,(t),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) + L^{I,(t),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega}) \\ = [C_{A_1,x_1}C_{A_1,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}C_{A_1,x_2}(\boldsymbol{\omega})] \\ + [C_{A_1,x_1}S_{A_1,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}S_{A_1,x_2}(\boldsymbol{\omega})\mu_1]. \tag{45}$$

Comparing equation (44) with equation (45), we finally arrive at

$$L^{I,(t),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) = L^{I,(l),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) \\ L^{I,(t),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(l),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})\mu_1.$$

The proof is complete. ■

It is worth noting here if $f(\boldsymbol{x}) = f_{e1}(\boldsymbol{x}) + f_{o1}(\boldsymbol{x})$, then there is no relationship between left-sided and two-sided type I QLCTs.

C. Relationship Between Right-Sided and Two-Sided Type I QLCTs

In section we will present the basic connection between the right-sided and two-sided type I QLCTs. For the purpose, we first need define the even and odd parts of the right-sided type I QLCT of the quaternion signal $f(\boldsymbol{x})$ as

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) \\ = \frac{1}{\sqrt{2\pi b_1}}\int_{\mathbb{R}^2} f(\boldsymbol{x})\cos(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4}) \\ \times \frac{1}{\sqrt{2\pi b_2}}\cos(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4})\, d\boldsymbol{x} \\ = C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \tag{46}$$

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) \\ = \frac{1}{\sqrt{2\pi b_1}}\int_{\mathbb{R}^2} f(\boldsymbol{x})\cos(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4}) \\ \times \mu_1\frac{1}{\sqrt{2\pi b_2}}\sin(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4})\, d\boldsymbol{x} \\ = C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1 \tag{47}$$

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) \\ = \frac{1}{\sqrt{2\pi b_1}}\int_{\mathbb{R}^2} f(\boldsymbol{x})\mu_1\sin(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4}) \\ \times \frac{1}{\sqrt{2\pi b_2}}\cos(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4})\, d\boldsymbol{x} \\ = S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})\mu_1 \tag{48}$$

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) \\ = \frac{-\mu_1}{\sqrt{2\pi b_1}}\int_{\mathbb{R}^2} \sin(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2 - \frac{\pi}{4})$$

TABLE III
RELATIONSHIP BETWEEN RIGT-SIDED AND TWO SIDED TYPE I QLCTS.

	Two-Sided QLCT
Right-Sided QLCT	$L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})$
	$L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})$
	$L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\mu_1$
	$L^{I,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1$

$$\times \frac{-\mu_1}{\sqrt{2\pi b_2}} \sin(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2 - \frac{\pi}{4}) f(\boldsymbol{x})\, d\boldsymbol{x}$$
$$= -S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega}). \tag{49}$$

This leads to the following result (see Table III).

Theorem 2. *The even and odd parts of the right-sided and two-sided type I QLCTs is related by*

$$\begin{aligned}
&L^{I,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})\\
&L^{I,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})\\
&L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\mu_1\\
&L^{I,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1.
\end{aligned} \tag{50}$$

Proof:
Comparing equations (37), (38), (46) and (47) yields

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}),$$

and

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}).$$

If we multiply (48) by $-\mu_1$ on the left side and by μ_1 the right side, we get

$$\begin{aligned}
-\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\mu_1 &= -\mu_1[S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})\mu_1]\mu_1\\
&= \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})\\
&= L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}).
\end{aligned}$$

Analogously we can get

$$-\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1 = L^{I,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}).$$

This finishes the proof of the theorem. ∎

Proposition 4. *If 2-D quaternion signal $f(\boldsymbol{x})$ is split into the partial even and odd parts, i.e.,*

$$f(\boldsymbol{x}) = f_{e1}(\boldsymbol{x}) + f_{o1}(\boldsymbol{x}) \tag{51}$$

then the following relations hold:

$$L^{I,(t),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}),$$

and

$$L^{I,(t),\mathbb{H}}_{A_1,A_2,o_2}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,o_2}\{f\}(\boldsymbol{\omega})\mu_1.$$

Furthermore, the two-sided type I QLCT of signal $f(\boldsymbol{x})$ can be represented in the partial even and odd parts of the right-sided type I QLCT as

$$L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) - \mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})\mu_1. \tag{52}$$

Proof: A straightforward calculation gives

$$\begin{aligned}
&L^{I,(r),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&= L^{I,(r),\mathbb{H}}_{A_1,A_2}\{f_{e1} + f_{o1}\}(\boldsymbol{\omega})\\
&= L^{I,(r),\mathbb{H}}_{A_1,A_2}\{f_{e1}\}(\boldsymbol{\omega}) + L^{I,(r),\mathbb{H}}_{A_1,A_2}\{f_{o1}\}(\boldsymbol{\omega})\\
&= L^{I,(r),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega}) + L^{I,(r),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega})\\
&= [L^{I,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) + L^{I,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})]\\
&\quad + [L^{I,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) + L^{I,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})]\\
&= [C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1]\\
&\quad + [C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1 - S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})].
\end{aligned} \tag{53}$$

Similar to (53), we easily get

$$\begin{aligned}
&L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&= L^{I,(t),\mathbb{H}}_{A_1,A_2}\{f_{e1} + f_{o1}\}(\boldsymbol{\omega})\\
&= L^{I,(t),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega}) + L^{I,(t),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega})\\
&= [C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})]\\
&\quad + [\mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1].
\end{aligned} \tag{54}$$

Comparing equation (53) with equation (54), we obtain

$$\begin{aligned}
&L^{I,(t),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega}) = L^{I,(r),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega})\\
&L^{I,(t),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega})\mu_1.
\end{aligned}$$

This gives the desired result. ∎

It is not difficult to check that there is no relationship between the right-sided and two-sided type I QLCTs if $f(\boldsymbol{x}) = f_{e2}(\boldsymbol{x}) + f_{o2}(\boldsymbol{x})$.

V. RELATIONSHIPS AMONG DIFFERENT DEFINITIONS OF TYPE II QLCTS

In this section we study the relationships among three different definitions of type II QLCTs. For this discussion we first start by establishing the relationship between the left-sided and right-sided type II QLCTs.

A. Relationship Between Left-Sided and Right-Sided Type II QLCTs

The following proposition describes connection between the left-sided and right-sided type II QLCTs.

Proposition 5. *If quaternion signal $f(\boldsymbol{x})$ is decomposed into symplectic form, then its left-sided and right-sided type II QLCTs are related by*

$$L^{II,(l),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{II,(r),\mathbb{H}}_{s,A_1,A_2}\{f\}(\boldsymbol{\omega}), \tag{55}$$

and

$$L^{II,(l),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega}) = L^{II,(r),\mathbb{H}}_{p,A_1^*,A_2}\{f\}(\boldsymbol{\omega}), \tag{56}$$

where $A_1^* = (a_1, -b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$.

Proof: Equation (55) can be proved using the similar argument as in the first term of (30). Simple computations show that

$$\begin{aligned}
&\mu_2 L^{I,(r),\mathbb{H}}_{p,A_1,A_2}\{f\}(\boldsymbol{\omega})\\
&= \int_{\mathbb{R}^2} \mu_2 f_p(\boldsymbol{x}) \frac{1}{\sqrt{2\pi b_1}} e^{\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_2}{b_1}\omega_1^2 - \frac{\pi}{2}\right)}\\
&\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mu_2}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 - \frac{\pi}{2}\right)}\, d\boldsymbol{x}\\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_2}{b_1}\omega_1^2 - \frac{\pi}{2}\right)}
\end{aligned}$$

$$\times \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mu_2}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 - \frac{\pi}{2}\right)} f_p(\boldsymbol{x})\mu_2\, d\boldsymbol{x}$$
$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{-2\pi b_1}} e^{-\frac{\mu_1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2 + \frac{\pi}{2}\right)}$$
$$\times \frac{1}{\sqrt{2\pi b_2}} e^{\frac{\mu_2}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2 - \frac{\pi}{2}\right)} f_p(\boldsymbol{x})\mu_2\, d\boldsymbol{x}$$
$$= L^{II,(l),\mathbb{H}}_{p,A_1^*,A_2^*}\{f\}(\boldsymbol{\omega})\mu_2.$$

This proves the proposition. ∎

B. Relationship Between Left-Sided and Two-Sided Type II QLCTs

In this subsection we discuss the even and odd parts of the left-sided and two-sided type II QLCTs. Using the same steps in the previous section we get for a 2-D quaternion signal $f(\boldsymbol{x})$

$$L^{II,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \quad (57)$$
$$L^{II,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = \mu_2 C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega}) \quad (58)$$
$$L^{II,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \quad (59)$$
$$L^{II,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = \mu_3 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega}). \quad (60)$$

Proposition 6. *The relationship between the even and odd parts of the left-sided and two-sided type II QLCTs is given by*

$$L^{II,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{II,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})$$
$$L^{II,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = L^{II,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{II,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})\mu_1$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = \mu_2 L^{II,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_2. \quad (61)$$

Proof: We first observe that

$$L^{II,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \quad (62)$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2 \quad (63)$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \quad (64)$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = \mu_1 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2. \quad (65)$$

From equations (57), (59), (62) and (64) it is obvious that

$$L^{II,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})$$
$$L^{II,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = L^{I,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}). \quad (66)$$

Multiplying (58) by $-\mu_2$ on the left side and by μ_2 on the right side yields

$$-\mu_2 L^{II,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})\mu_1 = -\mu_2[\mu_2 C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})]\mu_2$$
$$= C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2$$
$$\stackrel{(63)}{=} L^{II,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}). \quad (67)$$

From (60) we get

$$\mu_2 L^{II,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1 = \mu_2[\mu_3 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})]\mu_2$$
$$= L^{II,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}).$$

This finishes the proof of the proposition. ∎

Proposition 7. *If 2-D quaternion signal $f(\boldsymbol{x})$ is split into the partial even and odd parts, i.e., $f(\boldsymbol{x}) = f_{e2}(\boldsymbol{x}) + f_{o2}(\boldsymbol{x})$. The following relations are satisfied:*

$$L^{II,(t),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}) = L^{II,(l),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}),$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,o_2}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{II,(l),\mathbb{H}}_{A_1,o_2}\{f\}(\boldsymbol{\omega})\mu_1.$$

Moreover, the two-sided type II QLCT of 2-D quaternion signal $f(\boldsymbol{x})$ can be represented using the left-sided Type II QLCT in the form

$$L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$$
$$= L^{II,(l),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) - \mu_2 L^{II,(l),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})\mu_2. \quad (68)$$

Proof: It follows that

$$L^{II,(l),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$$
$$= L^{II,(l),\mathbb{H}}_{A_1,A_2}\{f_{e2} + f_{o2}\}(\boldsymbol{\omega})$$
$$= L^{II,(l),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) + L^{II,(l),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})$$
$$= [L^{II,(l),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) + L^{II,(l),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})]$$
$$+ [L^{II,(l),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) + L^{II,(l),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})]$$
$$= [C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})]$$
$$+ [\mu_2 C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega}) + \mu_3 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})]. \quad (69)$$

Analogously we get

$$L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$$
$$= L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f_{e2} + f_{o2}\}(\boldsymbol{\omega})$$
$$= L^{II,(t),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) + L^{II,(t),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})$$
$$= [C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})]$$
$$+ [C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2 + \mu_1 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2]. \quad (70)$$

Comparing (69) with (70) gives

$$L^{II,(t),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) = L^{II,(l),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega})$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega}) = -\mu_2 L^{II,(l),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})\mu_2.$$

The proof is complete. ∎

C. Relationship Between Right-Sided and Two-Sided Type II QLCTs

In section we will derive the relationship between the right-sided and two-sided type II QLCTs. For a 2-D quaternion signal $f(\boldsymbol{x})$ we get the even and odd parts of the right-sided type II QLCT as

$$L^{II,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) \quad (71)$$
$$L^{II,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2 \quad (72)$$
$$L^{(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})\mu_1 \quad (73)$$
$$L^{II,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_3. \quad (74)$$

This gives the following result.

Proposition 8. *The even and odd parts of right-sided and two-sided type II QLCTs is related by*

$$L^{II,,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{II,,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})$$
$$L^{II,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = L^{II,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{II,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\mu_1$$
$$L^{II,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{II,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1. \quad (75)$$

Proof:

From equations (62),(63), (71) and (72) we get

$$L^{I,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) = L^{II,(t),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega})$$

$$L^{II,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}) = L^{II,(t),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega}).$$

If we multiply (73) by $-\mu_1$ on the left side and by μ_1 the right side, we get

$$\begin{aligned}-\mu_1 L^{II,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega})\mu_1 &= -\mu_1[S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})\mu_1]\mu_1\\ &= \mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega})\\ &= L^{II,(t),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}).\end{aligned}$$

Analogously we can get

$$L^{II,(t),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega}) = -\mu_1 L^{II,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})\mu_1.$$

This finishes the proof of the theorem. ■

Proposition 9. *If 2-D quaternion signal $f(\boldsymbol{x})$ is split into the partial even and odd parts, i.e.,*

$$f(\boldsymbol{x}) = f_{e1}(\boldsymbol{x}) + f_{o1}(\boldsymbol{x}) \tag{76}$$

then the following relations hold:

$$\begin{aligned}L^{II,(t),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega}) &= L^{II,(r),\mathbb{H}}_{A_1,A_2,e_2}\{f\}(\boldsymbol{\omega})\\ L^{II,(t),\mathbb{H}}_{A_1,A_2,o_2}\{f\}(\boldsymbol{\omega}) &= -\mu_1 L^{II,(r),\mathbb{H}}_{A_1,o_2}\{f\}(\boldsymbol{\omega})\mu_1.\end{aligned}$$

Furthermore, the two-sided type II QLCT of signal $f(\boldsymbol{x})$ can be represented in the partial even and odd parts of the right-sided type II QLCT as

$$\begin{aligned}&L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\ &= L^{II,(r),\mathbb{H}}_{A_1,A_2,e2}\{f\}(\boldsymbol{\omega}) - \mu_1 L^{II,(r),\mathbb{H}}_{A_1,A_2,o2}\{f\}(\boldsymbol{\omega})\mu_1.\end{aligned} \tag{77}$$

Proof: A straightforward calculation gives

$$\begin{aligned}&L^{II,(r),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\ &= L^{II,(r),\mathbb{H}}_{A_1,A_2}\{f_{e1}+f_{o1}\}(\boldsymbol{\omega})\\ &= L^{II,(r),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega}) + L^{II,(r),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega})\\ &= [L^{II,(r),\mathbb{H}}_{A_1,A_2,ee}\{f\}(\boldsymbol{\omega}) + L^{II,(r),\mathbb{H}}_{A_1,A_2,eo}\{f\}(\boldsymbol{\omega})]\\ &\quad + [L^{II,(r),\mathbb{H}}_{A_1,A_2,oe}\{f\}(\boldsymbol{\omega}) + L^{II,(r),\mathbb{H}}_{A_1,A_2,oo}\{f\}(\boldsymbol{\omega})]\\ &= [C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2]\\ &\quad + [C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_1 + S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_3].\end{aligned} \tag{78}$$

In the same way,

$$\begin{aligned}&L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\\ &= L^{II,(t),\mathbb{H}}_{A_1,A_2}\{f_{e1}+f_{o1}\}(\boldsymbol{\omega})\\ &= L^{II,(t),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega}) + L^{II,(t),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega})\\ &= [C_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + C_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2]\\ &\quad + [\mu_1 S_{A_1,x_1}C_{A_2,x_2}(\boldsymbol{\omega}) + \mu_1 S_{A_1,x_1}S_{A_2,x_2}(\boldsymbol{\omega})\mu_2].\end{aligned} \tag{79}$$

Making a comparison between (78) and (79) we obtain

$$\begin{aligned}L^{I,(t),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega}) &= L^{II,(r),\mathbb{H}}_{A_1,A_2,e1}\{f\}(\boldsymbol{\omega})\\ L^{II,(t),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega}) &= -\mu_1 L^{I,(r),\mathbb{H}}_{A_1,A_2,o1}\{f\}(\boldsymbol{\omega})\mu_1.\end{aligned}$$

This gives the desired result. ■

VI. Conclusion

Due to the non-commutative property of quaternion multiplication, there are different definitions of the QLCT. They are constructed using the QLCT kernel function. In this work, we have discussed that the simplex part of the left-sided and right-sided type I QLCTs are the same, while the perplex parts of the two transform are not the same.

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