Some Transformation Properties of the Incomplete Beta Function and Its Partial Derivatives

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***Abstract*—In this paper, some transformation properties of the incomplete Beta function $B(z;x,y)$ are obtained. Based on the transformation properties, we extend the definitions of the partial derivatives of the incomplete Beta function $B_{p,q}(z;x,y)$ to the whole complex plane on x,y and z. Furthermore, we give some representations of $B_{p,q}(z;x,y)$ for complex numbers x,y,z. Moreover, numerical examples show the transformation formulas can improve the speed and precision of calculating $B(z;x,y)$ and $B_{p,q}(z;x,y)$.**

***Index Terms*—incomplete Beta function, hypergeometric function, Pochhammer symbol, neutrix limit.**

I. Introduction

THE Beta function was generalized to the incomplete Beta function by I.S.Gradshteyn et al. in [1]. The incomplete Beta function $B(z;x,y)$ is defined by

$$B(z;x,y)=\int_0^z t^{x-1}(1-t)^{y-1}dt, x,y>0;0<z<1. \tag{1}$$

where the incomplete Beta function reduces to the usual Beta function when $z=1$, *i.e.*, $B(1;x,y)=B(x,y)$. Some scholars have considered the partial derivatives of $B(x,y)$ in [2,3]. In addition, some scholars also considered the partial derivatives of $B(z;x,y)$. Noted that $B_{p,q}(z;x,y)=\frac{\partial^{p+q}}{\partial x^p\partial y^q}B(z;x,y)(p,q=0,1,\ldots.)$. For example, the definition of $B_{p,q}(z;x,y)$ was extended for negative values of x and y by E. Őzçağ in [4,5]. Furthermore, the authors showed that $B_{p,q}(z;x,y)$ existed for $p,q=0,1,2,\cdots$ and all real numbers x,y and $0<z<1$ in [4]. The authors also obtained some closed forms of $B_{p,0}(z;-n,m)$ and $B_{0,1}(z;-n,m)$ for $n,m,p=0,1,2,\cdots$ in [5]. However, the most effective method of extending the definitions of $B_{p,q}(x,y)$ and $B_{p,q}(z;x,y)$ was referred to use neutrix calculus in [4-11]. For example, it was proved in [5] that the neutrix limit

$$B_{p,q}(z;x,y)=N-\lim_{\varepsilon\to 0}\int_\varepsilon^z t^{x-1}(1-t)^{y-1}\ln^p t\ln^q(1-t)dt \tag{2}$$

existed for all real numbers x,y and $p,q=0,1,2,\cdots,0<z<1$. Moreover, the definition of $B_{p,q}(z;x,y)$ was extended to $0<|z|<1$ from $0<z<1$ in [12].

In this paper, using transformation properties of the incomplete Beta function, we extend the definitions of $B_{p,q}(z;x,y)$ to the whole complex plane on x,y and z. Furthermore, we give its representations.

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The structure of this paper is organized as follows. In Section 2, we obtain the transformation properties of the incomplete Beta function $B(z;x,y)$ with respect to z on the entire complex plane. In Section 3, the partial derivatives of the incomplete Beta function is obtained for all complex z. In Section 4, numerical examples are given to verify the results in Section 2 and Section 3. The conclusion is given in the last section of the paper.

II. Transformation properties of the incomplete Beta function

In this paper, we assume that $x,y,z\in C$ and $p,q\in N$, where C and N are the complex set and the set of nonnegative integers, respectively. Moreover, $(x)_n$ is a Pochhammer symbol, *i.e.*, $(x)_n=x(x+1)(x+2)\cdots(x+n-1)$, and $a_{j,q}(x)=\frac{d^q}{dx^q}(x)_j$.

First (1) can be rewritten as

$$B(z;x,y)=z^x\int_0^1 t^{x-1}(1-zt)^{y-1}dt. \tag{3}$$

Since a power function z^x is analytic at the point of z on the set $\{z|z\notin(-\infty,0]\}$ and $(1-zt)^{y-1}$ is analytic at the point of z on the set $\{z|z\notin[\frac{1}{t},\infty)\}$ for $t\in[0,1]$, $B(z;x,y)$ is analytic at the point of z on the set $D=\{z|Imz\neq 0\}\cup(0,1)$. It is well known that the incomplete Beta function and the hypergeometric function exist the following relationship:

$$B(z;x,y)=\frac{z^x}{x}\ {}_2F_1(1-y,x;x+1;z), \tag{4}$$

where ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function, which is defined by

$$\begin{aligned}&{}_2F_1(a,b;c;z)\\&=\tfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt\\&=\sum_{n=0}^{\infty}\tfrac{(a)_n(b)_n z^n}{(c)_n n!}.\end{aligned} \tag{5}$$

For the hypergeometric function, there exist the following transformation formulas: $z\longrightarrow 1-z,\frac{1}{z},\frac{z}{z-1},\frac{1}{1-z},\frac{z}{z-2}$. Similarly, we also obtain the following transformation formulas for $B(z;x,y)$.

Theorem 2.1 1) If z satisfies $|1-z|<1$, then

$$B(z;x,y)=B(y,x)-B(1-z;y,x). \tag{6}$$

2) If z satisfies $|z|>1$ and $Imz\neq 0$, then

$$\begin{aligned}B(z;x,y)&=B(x,y)-Hy(y,z)\\&\quad\cdot\left(B(1-x-y,y)-B(\tfrac{1}{z};1-x-y,y)\right),\end{aligned} \tag{7}$$

where

$$Hy(y,z)=\begin{cases}(-1)^{-y}, & Imz>0,\\(-1)^{y}, & Imz<0\ \ or\ \ z>1.\end{cases} \tag{8}$$

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3) If z satisfies $\left|\frac{z-1}{z}\right| < 1$ and $Rez > 1$, then

$$B(z;x,y) = B(x,y) - Hy(y,z)B(\tfrac{z-1}{z}; y, 1-x-y). \tag{9}$$

4) If z satisfies $\left|\frac{z}{z-1}\right| < 1$, then

$$B(z;x,y) = Hx(x,z)B(\tfrac{z}{z-1}; x, 1-x-y), \tag{10}$$

where

$$Hx(x,z) = \begin{cases} (-1)^{-x}, & Imz < 0, \\ (-1)^{x}, & Imz > 0 \quad or \quad z < -1. \end{cases} \tag{11}$$

5) If z satisfies $\left|\frac{1}{1-z}\right| < 1$, then

$$\begin{aligned} &B(z;x,y) \\ &= Hx(x,z)\left(B(1-x-y,x) - B(\tfrac{1}{1-z}; 1-x-y, x)\right). \end{aligned} \tag{12}$$

Proof. 1) For $|1-z| < 1$, there is

$$\begin{aligned} B(z;x,y) &= N\text{-}\lim_{\varepsilon\to 0}\int_\varepsilon^z t^{x-1}(1-t)^{y-1}dt \\ &= N\text{-}\lim_{\varepsilon\to 0}\int_{1-z}^{1-\varepsilon} t^{y-1}(1-t)^{x-1}dt \\ &= N\text{-}\lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} t^{y-1}(1-t)^{x-1}dt \\ &\quad -N\text{-}\lim_{\varepsilon\to 0}\int_\varepsilon^{1-z} t^{y-1}(1-t)^{x-1}dt \\ &= B(y,x) - B(1-z; y, x). \end{aligned} \tag{13}$$

Thus, we can obtain that the formula (6) holds.
2) For $|z| > 1$, (1) can be rewritten as

$$\begin{aligned} &B(z;x,y) \\ &= \begin{cases} B(i,x,y) + \int_i^z t^{x-1}(1-t)^{y-1}dt, Imz > 0, \\ B(-i;x,y) + \int_{-i}^z t^{x-1}(1-t)^{y-1}dt, Imz < 0 \\ \qquad\qquad or\ z > 1. \end{cases} \end{aligned} \tag{14}$$

where the path of the second integral in (14) does not across the real axes.
By using the variable substitution, we have

$$\begin{cases} \int_i^z t^{x-1}(1-t)^{y-1}dt \\ = (-1)^{-y}\int_{-i}^{1/z} t^{-x-y}(1-t)^{y-1}dt, \quad Imz > 0, \\ \int_{-i}^z t^{x-1}(1-t)^{y-1}dt \\ = (-1)^{y}\int_{i}^{1/z} t^{-x-y}(1-t)^{y-1}dt, \quad Imz < 0\ or\ z > 1. \end{cases} \tag{15}$$

Substituting (15) to (14), we get

$$\begin{aligned} &B(z;x,y) \\ &= \begin{cases} (-1)^{-y}\left(B(\tfrac{1}{z}; 1-x-y, y) - B(-i, 1-x-y, y)\right) \\ +B(i,x,y), \qquad Imz > 0, \\ (-1)^{y}\left(B(\tfrac{1}{z}; 1-x-y, y) - B(i, 1-x-y, y)\right) \\ +B(-i,x,y), \qquad Imz < 0 \quad or \quad z > 1. \end{cases} \end{aligned} \tag{16}$$

Let $z \to 1$ in (16), we have

$$\begin{aligned} &B(x,y) \\ &= \begin{cases} (-1)^{-y}\left(B(1-x-y, y) - B(-i, 1-x-y, y)\right) \\ +B(i,x,y), \qquad Imz > 0, \\ (-1)^{y}\left(B(1-x-y, y) - B(i, 1-x-y, y)\right) \\ +B(-i,x,y), \qquad Imz < 0 \quad or \quad z > 1. \end{cases} \end{aligned} \tag{17}$$

By (16) and (17), we obtain that (7) holds.

Similarly, the results of 3)-5) in Theorem 2.1 can be obtained.

Theorem 2.2 1) If z satisfies $\left|\frac{z}{z-2}\right| < 1$, then

$$B(z;x,y) = z^x(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(1-y)_l(\frac{z}{z-2})^l}{l!}C_l(x), \tag{18}$$

for $x \neq 0, -1, -2, \cdots$, where

$$C_l(x) = \sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\frac{(-2)^j}{j+x}. \tag{19}$$

2) If z satisfies $\left|\frac{z}{z-2}\right| < 1$, then

$$\begin{aligned} &B(z;-m,y) \\ &= -\frac{1}{m!}\sum_{j=0}^{m-1}\frac{(m-j-1)!(1-y)_j z^{j-m}}{(1-z)^{1+j-y}} \\ &+\frac{(1-y)_m \ln z}{m!(1-z)^{m+1-y}} - \frac{(1-y)_{m+1}}{m!}z(\tfrac{2-z}{2})^{y-1} \\ &\cdot\left(\ln z\sum_{l=0}^{\infty}\frac{(1-y)_{2l}(\frac{z}{z-2})^{2l}}{(2l+1)!} + \sum_{l=0}^{\infty}\frac{(1-y)_l(\frac{z}{z-2})^l}{(l+2)!}\sum_{j=0}^{[\frac{l}{2}]}\frac{1}{2j+1}\right), \end{aligned} \tag{20}$$

for $m = 0, 1, 2, \cdots$.

Proof. 1) By (4) and the following results of [13]

$$\begin{aligned} &{}_2F_1(a,b;c;z) \\ &= (1-\tfrac{z}{2})^{-a}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}{}_2F_1(-n,b;c;2)(\tfrac{z}{z-2})^n, \end{aligned} \tag{21}$$

we see that formula (18) holds for $x \neq 0, -1, -2, \cdots$.

2) Repeated using the following recursive formula

$$B(z;x,y) = \frac{z^x(1-z)^{y-1}}{x} + \frac{y-1}{x}B(z;x+1,y-1) \tag{22}$$

we have

$$\begin{aligned} B(z;x,y) &= \sum_{j=0}^{L-1}\frac{(-1)^j(1-y)_j(1-z)^{y-1-j}z^{x+j}}{(x)_{j+1}} \\ &+\frac{(-1)^L(1-y)_L}{(x)_L}B(z;x+L,y-L). \end{aligned} \tag{23}$$

By (2), we have

$$\begin{aligned} B(z;0,y) &= N\text{-}\lim_{\varepsilon\to 0}\int_\varepsilon^z t^{-1}(1-t)^{y-1}dt \\ &= N\text{-}\lim_{\varepsilon\to 0}\left[(1-z)^{y-1}\ln z - (1-\varepsilon)^{y-1}\ln\varepsilon\right] \\ &+N\text{-}\lim_{\varepsilon\to 0}\left[(y-1)\int_\varepsilon^z(1-t)^{y-2}\ln t dt\right] \\ &= (1-z)^{y-1}\ln z + (y-1)B_{1,0}(z;1,y-1) \end{aligned} \tag{24}$$

Setting $x = -m, L = m$ in (23) and using (24), we obtain

$$\begin{aligned} B(z;-m,y) &= -\frac{1}{m!}\sum_{j=0}^{m-1}\frac{(m-j-1)!(1-y)_j z^{j-m}}{(1-z)^{1+j-y}} \\ &-\frac{(1-y)_{m+1}}{m!}B_{1,0}(z;1,y-m-1) \\ &+\frac{(1-y)_m\ln z}{m!(1-z)^{m+1-y}}. \end{aligned} \tag{25}$$

Moreover, the following formulas hold:

$$\sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\frac{(-2)^j}{j+1} = \frac{1-(-1)^{l+1}}{2(l+1)}, \tag{26}$$

and

$$\sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\frac{(-2)^j}{(j+1)^2} = \frac{1}{l+1}\sum_{j=0}^{[\frac{l}{2}]}\frac{1}{2j+1}. \tag{27}$$

Combing (18), (26) and (27), we have

$$\begin{aligned} &B_{1,0}(z;1,y) \\ &= z(\tfrac{2-z}{2})^{y-1}\ln z\sum_{l=0}^{\infty}\frac{(1-y)_l(\frac{z}{z-2})^l}{l!}\sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\frac{(-2)^j}{j+1} \\ &-z(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(1-y)_l(\frac{z}{z-2})^l}{l!}\sum_{j=0}^{l}\begin{pmatrix} l \\ j \end{pmatrix}\frac{(-2)^j}{(j+1)^2} \\ &= z(\tfrac{2-z}{2})^{y-1}\ln z\sum_{l=0}^{\infty}\frac{(1-y)_{2l}(\frac{z}{z-2})^{2l}}{(2l+1)!} \\ &-z(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(1-y)_l(\frac{z}{z-2})^l}{(l+1)!}\sum_{j=0}^{[\frac{l}{2}]}\frac{1}{2j+1}, \end{aligned} \tag{28}$$

Inserting (28) into (25), we see that (20) holds.

In the following section, we will give the represention of the partial derivatives of the incomplete Beta function on all complex z.

III. The partial derivatives of the incomplete Beta function on all complex z

For $B_{p,q}(z;x,y)$ on all complex z, we have the following theorems by Theorem 2.1 and Leibniz derivation rule.

Theorem 3.1 If z satisfies $|1-z|<|z|<1$, then we have the following result:

$$B_{p,q}(z;x,y)=B_{q,p}(y,x)-B_{q,p}(1-z;y,x). \tag{29}$$

Theorem 3.2 1) If z satisfies $|z|>1$, then we have the following results:

$$\begin{aligned}&B_{p,q}(z;x,y)\\&=B_{p,q}(x,y)-\sum_{k=0}^{q}\binom{q}{k}Hy_{q-k}(y,z)\\&\cdot\sum_{v=0}^{k}\binom{k}{v}(-1)^{p-v}\\&\cdot\left(B_{p+v,k-v}(1-x-y,y)-B_{p+v,k-v}(\tfrac{1}{z};1-x-y,y)\right),\end{aligned} \tag{30}$$

where

$$Hy_q(y,z)=\begin{cases}(-1)^{-y}(-\pi i)^q, & Imz>0 \;\; or \;\; z<-1,\\(-1)^{y}(\pi i)^q, & Imz<0 \;\; or \;\; z>1.\end{cases} \tag{31}$$

2) If z satisfies $\left|\frac{z-1}{z}\right|<1$, then we have the following result:

$$\begin{aligned}B_{p,q}(z;x,y)&=B_{p,q}(x,y)-\sum_{k=0}^{q}\binom{q}{k}\\&\cdot Hy_{q-k}(y,z)\sum_{u=0}^{k}\binom{k}{u}(-1)^{p-u}\\&\cdot B_{k-u,p+u}(\tfrac{z-1}{z};y,1-x-y),\end{aligned} \tag{32}$$

where $Hy_q(y,z)$ is defined by (31).

Theorem 3.3 1) If z satisfies $\left|\frac{z}{z-1}\right|<1$, then we have the following result:

$$\begin{aligned}B_{p,q}(z;x,y)&=\sum_{k=0}^{p}\binom{p}{k}Hx_{p-k}(x,z)\sum_{u=0}^{k}\binom{k}{u}\\&\cdot(-1)^{q-u}B_{k-u,q+u}(\tfrac{z}{z-1};x,1-x-y),\end{aligned} \tag{33}$$

where

$$Hx_p(x,z)=\begin{cases}(-1)^{-x}(-\pi i)^p, & Imz<0 \;\; or \;\; z>0,\\(-1)^{x}(\pi i)^p, & Imz>0 \;\; or \;\; z<0.\end{cases} \tag{34}$$

2) If z satisfies $\left|\frac{1}{1-z}\right|<1$, then we have the following result:

$$\begin{aligned}&B_{p,q}(z;x,y)\\&=\sum_{k=0}^{p}\binom{p}{k}Hx_{p-k}(x,z)\sum_{u=0}^{k}\binom{k}{u}(-1)^{q-u}\\&\cdot(B_{q+u,k-u}(1-x-y,x)\\&-B_{q+u,k-u}(\tfrac{1}{1-z};1-x-y,x)),\end{aligned} \tag{35}$$

where $Hx_p(x,z)$ is defined by (34).

In the following, we obtain the following theorem by Theorem 2.2 and Leibniz derivation rule.

Theorem 3.4 1)If z satisfies $\left|\frac{z}{z-2}\right|<1$, we have the following result:

$$\begin{aligned}B_{p,q}(z;x,y)&=z^x(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(\frac{z}{z-2})^lQ_q(z,y,l)}{l!}\\&\cdot\sum_{k=0}^{p}\binom{p}{k}C_l^{(k)}(x)\ln^{p-k}z\end{aligned} \tag{36}$$

for $x\neq 0,-1,-2,\cdots$, where

$$Q_q(z,y,l)=\sum_{j=0}^{q}\binom{q}{j}(-1)^ja_{l,j}(1-y)\ln^{q-j}\frac{2-z}{2}, \tag{37}$$

and

$$C_l(x)=C_l^{(0)}(x)=\sum_{j=0}^{l}\binom{l}{j}\frac{(-2)^j}{j+x}, \tag{38}$$

$$C_l^{(k)}(x)=(-1)^kk!\sum_{j=0}^{l}\binom{l}{j}\frac{(-2)^j}{(j+x)^{k+1}}. \tag{39}$$

2) If z satisfies $\left|\frac{z}{z-2}\right|<1$, then we have the following result:

$$\begin{aligned}&B_{p,q}(z;-m,y)\\&=(-2)^m\frac{\ln^{p+1}z}{p+1}z^{-m}(\tfrac{2-z}{2})^{y-1}\sum_{l=m}^{\infty}\binom{l}{m}\\&\cdot\frac{(\frac{z}{z-2})^lQ_q(z,y,l)}{l!}+z^{-m}(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(\frac{z}{z-2})^lQ_q(z,y,l)}{l!}\\&\cdot\sum_{k=0}^{p}\binom{p}{k}\ln^{p-k}zC_l^{(k)}(-m).\end{aligned} \tag{40}$$

for $m=0,1,2,\cdots$, where

$$C_l^{(k)}(-m)=(-1)^kk!\sum_{j=0,j\neq m}^{l}\binom{l}{j}\frac{(-2)^j}{(j-m)^{k+1}}. \tag{41}$$

and $Q_q(z,y,l)$ is defined by (37).

Proof. 1) By Leibniz derivation rule for (18), we obtain that (36) holds.

2) By the neutrix limit for (36), we obtain

$$\begin{aligned}&B_{p,q}(z;-m,y)\\&=N-\lim_{\varepsilon\to0}z^{\varepsilon-m}(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(\frac{z}{z-2})^lQ_q(z,y,l)}{l!}\\&\cdot\sum_{k=0}^{p}\binom{p}{k}\ln^{p-k}zC_l^{(k)}(\varepsilon-m)\\&=z^{-m}(\tfrac{2-z}{2})^{y-1}\sum_{l=0}^{\infty}\frac{(\frac{z}{z-2})^lQ_q(z,y,l)}{l!}\\&\cdot\sum_{k=0}^{p}\binom{p}{k}\ln^{p-k}zC_l^{(k)}(-m)\\&+(\tfrac{-2}{z})^m(\tfrac{2-z}{2})^{y-1}\sum_{l=m}^{\infty}\binom{l}{m}\frac{(\frac{z}{z-2})^lQ_q(z,y,l)}{l!}\\&\cdot\sum_{k=0}^{p}\binom{p}{k}(-1)^kk!\ln^{p-k}zN-\lim_{\varepsilon\to0}\frac{z^\varepsilon}{\varepsilon^{k+1}},\end{aligned} \tag{42}$$

Since

$$N-\lim_{\varepsilon\to0}\frac{z^\varepsilon}{\varepsilon^{k+1}}=\frac{\ln^{k+1}z}{(k+1)!} \tag{43}$$

and

$$\sum_{k=0}^{p}\binom{p}{k}(-1)^kk!\ln^{p-k}z\frac{\ln^{k+1}z}{(k+1)!}=\frac{\ln^{p+1}z}{p+1} \tag{44}$$

Substituting (43) and (44) into (42), we obtain that (40) holds.

Remark 3.1 From Theorem 3.4, we notice that the computation of $a_{j,q}(x)$ and $C_l^{(k)}(x)$ are important in process of computing $B_{p,q}(z;x,y)$. $a_{n,l}(x) = \frac{d^l}{dx^l}(x)_n$ can be calculated by the following formulas:

$$a_{n,l}(x) = (x+n-1)a_{n-1,l}(x) + l a_{n-1,l-1}(x), \tag{45}$$

and

$$a_{n,0}(x) = (x)_n, a_{n,l}(x) = \left\{ \begin{array}{l} 0, n < l, \\ l!, n = l, \end{array} \right. . \tag{46}$$

for $n, l = 1, 2, \ldots$.

Moreover, $C_l(x)$ and $C_l^{(k)}(x)$ can be calculated by the following recursive formulas:

$$C_{l+1}(x) = \frac{l}{l+1+x} C_{l-1}(x) + \frac{1-x}{l+1+x} C_l(x), \tag{47}$$

$$\begin{aligned} C_{l+1}^{(k)}(x) &= \frac{l}{l+1+x} C_{l-1}^{(k)}(x) + \frac{1-x}{l+1+x} C_l^{(k)}(x) \\ &\quad - \frac{k}{l+1+x}\left(C_l^{(k-1)}(x) + C_{l+1}^{(k-1)}(x) \right), \end{aligned} \tag{48}$$

and

$$\begin{array}{c} C_{-1}^{(0)}(x) = C_{-1}(x) = 0, C_0^{(0)}(x) = C_0(x) = \frac{1}{x}, \\ C_{-1}^{(k)}(x) = 0, C_0^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}. \end{array} \tag{49}$$

for $x \neq 0, -1, -2, \ldots$.

For $C_l^{(k)}(-m)$, we have the following formulas:

$$\begin{array}{c} C_{l+1}^{(k)}(-m) = C_l^{(k)}(-m) - 2C_l^{(k)}(1-m), \\ C_0^{(k)}(-m) = \left\{ \begin{array}{ll} \frac{(-1)^k k!}{(-m)^{k+1}}, & m = 1, 2, \ldots, \\ 0, & m = 0, \end{array} \right. , \\ C_l^{(k)}(0) = (-1)^k k! \sum\limits_{j=1}^{l} \left(\begin{array}{c} l \\ j \end{array} \right) \frac{(-2)^j}{j^{k+1}}. \end{array} \tag{50}$$

for $l, m = 0, 1, 2, \ldots$.

Moreover, (40) has also a different representation. In fact, by Leibniz derivation rule for (23), we have

$$\begin{aligned} & B_{p,q}(z;x,y) \\ & = \sum_{j=0}^{L-1} (-1)^j (1-z)^{y-1-j} z^{x+j} Q(q,z,y,j) \\ & \cdot \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) A_{j+1,k}(x) \ln^{p-k} z \\ & + (-1)^L \sum_{u=0}^{q} \left(\begin{array}{c} q \\ u \end{array} \right) (-1)^u a_{L,u}(1-y) \\ & \cdot \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) A_{L,k}(x) B_{p-k,q-u}(z;x+L,y-L). \end{aligned} \tag{51}$$

where

$$Q(q,z,y,j) = \sum_{l=0}^{q} \left(\begin{array}{c} q \\ l \end{array} \right) (-1)^l a_{j,l}(1-y) \ln^{q-l}(1-z), \tag{52}$$

and

$$A_{j,k}(x) = \frac{d^k}{dx^k}\left[\frac{1}{(x)_j} \right]. \tag{53}$$

Using the neutrix limit, we obtain

$$\begin{aligned} & B_{p,q}(z;0,y) \\ & = N\!-\!\lim_{\varepsilon \to 0} \int_\varepsilon^z t^{-1}(1-t)^{y-1} \ln^p t \ln^q(1-t) dt \\ & = N\!-\!\lim_{\varepsilon \to 0} \frac{(1-z)^{y-1} \ln^{p+1} z \ln^q(1-z)}{p+1} \\ & - N\!-\!\lim_{\varepsilon \to 0} \frac{(1-\varepsilon)^{y-1} \ln^{p+1} \varepsilon \ln^q(1-\varepsilon)}{p+1} \\ & + N\!-\!\lim_{\varepsilon \to 0} \frac{y-1}{p+1} \int_\varepsilon^z (1-t)^{y-2} \ln^{p+1} t \ln^q(1-t) dt \\ & + N\!-\!\lim_{\varepsilon \to 0} \frac{q}{p+1} \int_\varepsilon^z (1-t)^{y-2} \ln^{p+1} t \ln^{q-1}(1-t) dt \\ & = \frac{(1-z)^{y-1} \ln^{p+1} z \ln^q(1-z)}{p+1} + \frac{y-1}{p+1} B_{p,q}(z;1,y-1) \\ & + \frac{q}{p+1} B_{p,q-1}(z;1,y-1) \end{aligned} \tag{54}$$

Setting $x = -m, L = m$ in (51) and using (54), we can obtain the following theorem.

Theorem 3.5 If z satisfies $\left| \frac{z}{z-2} \right| < 1$, then we have the following result

$$\begin{aligned} & B_{p,q}(z;-m,y) \\ & = \sum_{j=0}^{m-1} \frac{(-1)^j z^{j-m} Q(q,z,y,j)}{(1-z)^{1+j-y}} \sum_{k=0}^{p} A_{j+1,k}(-m) \ln^{p+1-k} z \\ & + \frac{(-1)^m Q(q,z,y,m)}{(p+1)(1-z)^{m+1-y}} \sum_{k=0}^{p} \left(\begin{array}{c} p+1 \\ k \end{array} \right) A_{m,k}(-m) \\ & - \frac{(-1)^m (1-y)_{m+1}}{p+1} \sum_{k=0}^{p} \left(\begin{array}{c} p+1 \\ k \end{array} \right) A_{m,k}(-m) \\ & \cdot B_{p-k+1,q}(z;1,y-m-1) + \sum_{k=0}^{p} \left(\begin{array}{c} p \\ k \end{array} \right) \frac{A_{m,k}(-m)}{p+1-k} \\ & \cdot \sum_{l=1}^{q} \left(\begin{array}{c} q \\ l \end{array} \right) (-1)^{l-m} B_{p-k+1,q-l}(z;1,y-m-1) \\ & \cdot \left((y-m-1) a_{m,l}(1-y) - l a_{m,l+1}(1-y) \right), \end{aligned} \tag{55}$$

for $m = 0, 1, 2, \cdots$, where $B_{p,q}(z;1,y-m-1)$ is calculated by (36), $A_{n,p}(x) = \frac{d^p}{dx^p}\left[\frac{1}{(x)_n} \right]$ is calculated by the recursive formulas:

$$A_{n,p}(x) = \frac{1}{x+n-1} \left(A_{n-1,p}(x) - p A_{n,p-1}(x) \right),$$

and

$$A_{n,0}(x) = \frac{1}{(x)_n}$$

In the following section, we give some numerical examples to verify the results of Section 2 and Section 3.

IV. Numerical Examples

Using the following transformation formulas

$$z \longrightarrow 1-z, \frac{1}{z}, \frac{z}{z-1}, \frac{1}{1-z}, \frac{z}{z-2}$$

we can obtain high precision and fast calculation for the hypergeometric function ${}_2F_1(a,b;c;z)$. By (4), $B(z,x,y)$ can be directly calculated by using the hypergeometric function ${}_2F_1(a,b;c;z)$. However, using transform formulas of Theorem 2.1 and 2.2, $B(z,x,y)$ can be calculated more effectively. Some mathematical softwares have internal functions for calculating $B(z;x,y)$. For example, we illustrate the validity of the numerical calculation by the following numerical results of Table I and Table II in Mathematica.

Table I The comparison of numerical results for $B(z,x,y)(1)$

z, x, y	*algorithm*	T_{16}, r_{16}
$\frac{23}{24} + \frac{i}{20}, \frac{1}{2}, \frac{-12}{7}$	$sys1$	$0.0625, 1.3 \times 10^{-15}$
	$sys2$	$0.0156, 0.0 \times 10^{-16}$
	(6)	$0.0, 0.0 \times 10^{-21}$
$\frac{22}{3} + \frac{5i}{4}, \frac{5}{2}, \frac{-17}{5}$	$sys1$	$0.0937, 4.0 \times 10^{-15}$
	$sys2$	$0.0156, 0.0 \times 10^{-16}$
	(9)	$0.0, 2.0 \times 10^{-19}$
$\frac{-7}{6} + \frac{i}{9}, \frac{16}{3}, \frac{-22}{5}$	$sys1$	$0.0625, 6.9 \times 10^{-11}$
	$sys2$	$0.0156, 0.0 \times 10^{-16}$
	(10)	$0.0, 0.0 \times 10^{-20}$
$\frac{1}{2} + \frac{\sqrt{3}i}{2}, \frac{5}{7}, \frac{-14}{5}$	$sys1$	$0.0625, 1.9 \times 10^{-14}$
	$sys2$	$0.0156, 0.0 \times 10^{-16}$
	(18)	$0.0, 4.0 \times 10^{-17}$

and

Table II The comparison of numerical results for $B(z,x,y)(2)$

z,x,y	*algorithm*	T_{32}, r_{32}
$\frac{23}{24}+\frac{i}{20},\frac{1}{2},\frac{-12}{7}$	$sys1$	$0.2968, 0.0\times10^{-32}$
	$sys2$	$0.0156, 0.0\times10^{-32}$
	(6)	$0.0156, 0.0\times10^{-41}$
$\frac{22}{3}+\frac{5i}{4},\frac{5}{2},\frac{-17}{5}$	$sys1$	$0.3593, 1.3\times10^{-32}$
	$sys2$	$0.0156, 0.0\times10^{-32}$
	(9)	$0.0156, 2.0\times10^{-38}$
$\frac{-7}{6}+\frac{i}{9},\frac{16}{3},\frac{-22}{5}$	$sys1$	$0.1093, 0.0\times10^{-32}$
	$sys2$	$0.0156, 0.0\times10^{-32}$
	(10)	$0.0, 0.0\times10^{-40}$
$\frac{1}{2}+\frac{\sqrt{3}i}{2},\frac{5}{7},\frac{-14}{5}$	$sys1$	$0.1718, 0\times10^{-32}$
	$sys2$	$0.0156, 0.0\times10^{-32}$
	(18)	$0.0156, 0.0\times10^{-36}$

Where $sys1$ represents the call of numerical integral function of Mathematica to calculate the formula (1) for $x > 0$, $sys2$ represents the call of internal Beta function of Mathematica to calculate the formula (1). In this section, T_P and r_p represent the running time(unit:second) and the error with the precision P, respectively.

Seen from Table I and Table II , the calculation accuracy of using the formulas (6), (9), (10) and (18) is best. Moreover, $sys2$ and formulas (6),(9),(10),(18) have almost the same computation speed. According to Theorem 2.1 and 2.2, the internal functions of calculating $B(z;x,y)$ are written, calculation efficiency will be better.

In the following, we consider the calculation of $B_{p,q}(z;x,y)$. Due to there is no specific command which is used to calculate $B_{p,q}(z;x,y)$ in the mathematical software. The symbol derivation of function $B(z;x,y)$ seems to be able to get $B_{p,q}(z;x,y)$. However, it is rather time consuming. Especially, when $p+q$ is bigger, it cannot give calculation of $B_{p,q}(z;x,y)$. When (2) is integrable, it is the integral representation of $B_{p,q}(z;x,y)$. Thus, the transform formulas of Theorem 3.1-3.4 can be compared with the numerical integration of (2). Here, we give the comparison of numerical results for $B_{p,q}(z;x,y)(p=2,q=2)$ in Table III and Table IV.

Table III The comparison of numerical results for $B_{2,2}(z;x,y)(1)$

z,x,y	*algorithm*	T_{16}, r_{16}
$\frac{115+6i}{120},\frac{-3}{2},\frac{-12}{7}$	$sys3$	$0.0937, 7.5\times10^{-15}$
	(29)	$0.0156, 0.0\times10^{-16}$
$\frac{88+15i}{12},\frac{-3}{2},\frac{-17}{5}$	$sys3$	$0.1562, 4.5\times10^{-14}$
	(30)	$0.0468, 0.0\times10^{-15}$
$\frac{2i-21}{18},\frac{-5}{3},\frac{-22}{5}$	$sys3$	$0.0781, 7.0\times10^{-16}$
	(33)	$0.0312, 4.8\times10^{-15}$
$\frac{1+\sqrt{3}i}{2},\frac{-12}{7},\frac{-14}{5}$	$sys3$	$0.0781, 7.0\times10^{-15}$
	(36)	$0.0468, 6.0\times10^{-14}$

and

Table IV The comparison of numerical results for $B_{2,2}(z;x,y)(2)$

z,x,y	*algorithm*	T_{32}, r_{32}
$\frac{115+6i}{120},\frac{-3}{2},\frac{-12}{7}$	$sys3$	$0.5468, 0.0\times10^{-32}$
	(29)	$0.0156, 0.0\times10^{-36}$
$\frac{88+15i}{12},\frac{-3}{2},\frac{-17}{5}$	$sys3$	$0.7812, 0.0\times10^{-32}$
	(30)	$0.0468, 0.0\times10^{-35}$
$\frac{2i-21}{18},\frac{-5}{3},\frac{-22}{5}$	$sys3$	$0.1562, 0.0\times10^{-32}$
	(33)	$0.0312, 6.8\times10^{-37}$
$\frac{1+\sqrt{3}i}{2},\frac{-12}{7},\frac{-14}{5}$	$sys3$	$0.3593, 0.0\times10^{-32}$
	(36)	$0.0781, 0.0\times10^{-35}$

Where $sys3$ is used to represent numerical integral function of (2) for $x > -q$ in Mathematica.

Seen from the Table III and Table IV, the speed and precision of calculating $B_{2,2}(z;x,y)$ is high by using (29),(30),(33) and (36). Especially with the improvement of specific precision, more obvious advantage can be seen.

V. Conclusion

In this paper, some transformation properties of the incomplete Beta function are obtained. Based on the transformation properties, we extend the definitions of the partial derivatives of the incomplete Beta function $B_{p,q}(z;x,y)$ to the whole complex plane on x,y and z. Furthermore, we give some representations of $B_{p,q}(z;x,y)$. Finally, numerical examples show the transformation formulas of Section 2 and Section 3 can improve the speed and precision of calculating $B(z;x,y)$ and $B_{p,q}(z;x,y)$.

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