Resistance Distance and Kirchhoff Index of Two Edge-subdivision Corona Graphs

Qun Liu [a,b]

Abstract—**In this work we first obtain the group inverse of the edge-subdivision-vertex corona and edge-subdivision-edge corona in terms of the group inverse of the factor graphs. Then the resistance distance and Kirchhoff index of these graphs can be derived from the resistance distance and Kirchhoff index of the factor graphs.**

Index Terms—**Kirchhoff index; Resistance distance; Edge-subdivision-vertex corona; Edge-subdivision-edge corona**

I. Introduction

LET $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = diag(d_1, d_2, \cdots d_{|V(G)|})$ the diagonal matrix with all vertex degrees of G as its diagonal entries. For a graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where D_G is the diagonal matrix of vertex degrees of G. We use $\mu_1(G) \geq u_2(G) \geq \cdots \geq \mu_n(G) = 0$ to denote the eigenvalues of L_G. The $\{1\}$-inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite $\{1\}$-inverse [1]. We use $M^{(1)}$ to denote any $\{1\}$-inverse of a matrix M, and let $(M)_{uv}$ denote the (u, v)-entry of M.

Klein and Randić[2] introduced a new distance function named resistance distance on the basis of electrical network theory. The resistance distance between any two vertices u and v in G is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The Kirchhoff index of G is the sum of resistance distances between all pairs of vertices of G. Let $r_{uv}(G)$ denote the resistance distance between u and v in G and $Kf(G)$ denote the Kirchhoff index of G. The resistance distance and the Kirchhoff index have attracted extensive attention due to its wide applications in physics, chemistry and others. Up till now, many results on the resistance distance and the Kirchhoff index are obtained. See ([4], [5], [7]), [13]-[19]) and the references therein to know more.

The computation of resistance distance and Kirchhoff index is a hot topic in mathematics, computer science and so on. However, the computation of the effective resistances is difficult, as they are highly sensitive to small perturbations on the network, so this has prompted researchers try to find some techniques to compute the resistance distance and Kirchhoff index of a given graph and obtained its closed formula. The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. The set of such new vertices is denoted by $I(G)$. In [6], a new graph operation: edge-subdivision-vertex and edge-subdivision-edge corona are introduced, and their A-spectra(resp., L-spectra) are investigated. This paper considers the resistance distance and Kirchhoff index of the graph operations below, which come from [6].

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Q. Liu is with (a) School of Computer Science, Fudan University, Shanghai 200433, China and (b) School of Mathematics and Statistics, Hexi University, Gansu, Zhangye, 734000, P.R. China. Email: liuqun@fudan.edu.cn.

Definition 1 [6] The edge-subdivision-vertex corona of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from G_1 and $|E(G_1)|$ copies of $S(G_2)$ with each edge of G_1 corresponding to one copy of $S(G_2)$ and all vertex-disjoint, by joining end-vertex of the ith edge of $E(G_1)$ to each vertex of $V(G_2)$ in the ith copy of $S(G_2)$.

Definition 2 [6] The edge-subdivision-edge corona of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \forall G_2$, is the graph obtained from G_1 and $|E(G_1)|$ copies of $S(G_2)$ with each edge of G_1 corresponding to one copy of $S(G_2)$ and all vertex-disjoint, by joining end-vertex of the ith edge of $E(G_1)$ to each vertex of $I(G_2)$ in the ith copy of $S(G_2)$.

Bu et al. investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs [7]. Liu et al. [8] gave the resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs. Liu et al. [9] gave the resistance distance and Kirchhoff index of corona and neighborhood corona of two graphs. Lu et al. [10] computed the resistance distance and Kirchhoff index of two corona graphs. Motivated by the results, in this paper, we further explore the generalized inverse of the edge-subdivision-vertex corona and edge-subdivision-edge corona in terms of the generalized inverse of the factor graphs. Thus the effective resistances and Kirchhoff index of the edge-subdivision-vertex corona and edge-subdivision-edge corona can be derived from the resistance distance and Kirchhoff index of the factor graphs.

II. Preliminaries

For a square matrix M, the group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if $rank(M) = rank(M^2)$ ([1],[11]). If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M since M is symmetric [11]. It is known that resistance distances in a connected graph G can be obtained from any $\{1\}$-inverse of G ([4]).

Lemma 2.1 ([11]) Let G be a connected graph. Then

$$\begin{aligned} r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} \\ &= (L_G^{\#})_{uu} + (L_G^{\#})_{vv} - 2(L_G^{\#})_{uv}. \end{aligned}$$

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Let 1_n denote the column vector of dimension n with all the entries equal one. We will often use 1 to denote an all-one column vector if the dimension can be read from the context.

Lemma 2.2 ([7]) For any graph, we have $L_G^{\#}1 = 0$.

Lemma 2.3 ([12]) Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a nonsingular matrix. If A and D are nonsingular, then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned}$$

where $S = D - CA^{-1}B$.

For a square matrix M, let $tr(M)$ denote the trace of M.

Lemma 2.5 ([16]) Let G be a connected graph on n vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^{\#}).$$

Lemma 2.6 ([15]) Let G be an r-regular graph with n vertices and m edges, $\mu_G(x)$ denote the Laplacian characteristic polynomial of G, $l(G)$ be the line graph of G. Then

$$\mu_{l(G)}(x) = (x - 2r)^{m-n}\mu_G(x).$$

Lemma 2.7([3]) Let G be a connected graph of order n with edge set E. Then

$$\sum_{u<v, uv\in E} r_{uv}(G) = n - 1.$$

For a vertex i of a graph G, let $T(i)$ denote the set of all neighbors of i in G.

Lemma 2.8 ([17]) Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^T H^{\#} & D^{-1} + D^{-1}B^T H^{\#} B D^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

III. Resistance distance and Kirchhoff index of edge-subdivision-vertex corona for graphs

In this section, we focus on determing the resistance distance and Kirchhoff index of edge-subdivision-vertex corona whenever G_1 is an r_1-regular graph.

Theorem 3.1 Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges and G_2 an r_2-regular graphs on n_2 vertices and m_2 edges. Then $G = G_1 \vee G_2$ have the resistance distance and Kirchhoff index

(i) For any $i, j \in V(G_1)$, we have

$$\begin{aligned} r_{ij}(G) &= \frac{2}{n_2+2}\left(L_1^{\#}\right)_{ii} + \frac{2}{n_2+2}\left(L_1^{\#}\right)_{jj} \\ &\quad - \frac{4}{n_2+2}\left(L_1^{\#}\right)_{ij} = \frac{2}{n_2+2} r_{ij}(G_1). \end{aligned}$$

(ii) For any $i, j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G) &= \left((2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}\right)_{ii} + \left((2I_{n_2} + \frac{1}{2}L_2)^{-1}\right. \\ &\quad \left.\otimes I_{m_1}\right)_{jj} - 2\left((2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}\right)_{ij}. \end{aligned}$$

(iii) For any $i \in V(G_1), j \in V(G_2)$, we have

$$\begin{aligned} r_{ij}(G) &= \frac{2}{n_2+2}\left(L_1^{\#}\right)_{ii} + \left((2I_{n_2} + \frac{1}{2}L_2)^{-1}\right. \\ &\quad \left.\otimes I_{m_1}\right)_{jj} - \frac{4}{n_2+2}\left(L_1^{\#}\right)_{ij}. \end{aligned}$$

(iv) $Kf(L_G)$

$$\begin{aligned} &= (n_1 + n_2 + 2m_1)\left(\frac{2 + r_1(n_2 + m_2)}{n_1(n_2+2)} Kf(G_1)\right. \\ &\quad + m_1 \sum_{i=1}^{n_2} \frac{1}{\frac{1}{2}\mu_i(G_2) + 2} + (r_2+2)m_1 \sum_{i=1}^{n_1} \frac{1}{4 + \mu_i(G_2)} \\ &\quad \left. + \frac{m_1(r_2+2)(n_2 - m_2)}{4 + 2r_2} - \frac{(n_1 - 1)(n_2 + m_2)}{2(n_2+2)}\right) \\ &\quad - \frac{6m_1 m_2 + m_2(r_2+2)}{4}, \end{aligned}$$

where $\mu_i(G_2)$ is the Laplacian eigenvalues of G_2.

Proof Let $R_i (i = 1, 2)$ be the incidence matrix of G_i. Then with a proper labeling of vertices, the Laplacian matrix of $G_1 \vee G_2$ can be written as

$L(G_1 \vee G_2)$

$$= \begin{pmatrix} L_1 + r_1 n_2 I_{n_1} & -1_{n_2}^T \otimes R_1 & 0_{n_1 \times m_1 m_2} \\ -1_{n_2} \otimes R_1^T & (2 + r_2)I_{n_2} \otimes I_{m_1} & -R_2 \otimes I_{m_1} \\ 0_{m_1 m_2 \times n_1} & -R_2^T \otimes I_{m_1} & 2I_{m_1 m_2} \end{pmatrix}.$$

Let $A = L_1 + r_1 n_2 I_{n_1}$, $B = \begin{pmatrix} -1_{n_2}^T \otimes R_1 & 0_{n_1 \times m_1 m_2} \end{pmatrix}$, $B^T = \begin{pmatrix} -1_{n_2} \otimes R_1^T \\ 0_{m_1 m_2 \times n_1} \end{pmatrix}$, and

$$D = \begin{pmatrix} (2 + r_2)I_{n_2} \otimes I_{m_1} & -R_2 \otimes I_{m_1} \\ -R_2^T \otimes I_{m_1} & 2I_{m_1 m_2} \end{pmatrix}.$$

First we compute the D^{-1}. By Lemma 2.3, we have $A_1 - B_1 D_1^{-1} C_1$

$$\begin{aligned} &= (2 + r_2)I_{n_2} \otimes I_{m_1} - \tfrac{1}{2}(R_2 \otimes I_{m_1})(R_2^T \otimes I_{m_1}) \\ &= [(2 + r_2)I_{n_2} - \tfrac{1}{2}(r_2 I_{n_2} + A(G_2))] \otimes I_{m_1} \\ &= (2I_{n_2} + \tfrac{1}{2}L_2) \otimes I_{m_1}, \end{aligned}$$

so $(A_1 - B_1 D_1^{-1} C_1)^{-1} = (2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}$.

By Lemma 2.3, we have

$$\begin{aligned} S &= (D_1 - C_1 A_1^{-1} B_1) \\ &= 2I_{m_1 m_2} - \tfrac{1}{r_2+2}(R_2^T \otimes I_{m_1})(I_{n_2} \otimes I_{m_1})(R_2 \otimes I_{m_1}) \\ &= 2I_{m_1 m_2} - \tfrac{1}{r_2+2}(R_2^T R_2 \otimes I_{m_1}) \\ &= (2I_{m_1 m_2} - \tfrac{1}{r_2+2}(2I_{m_2} + A(l(G_2))) \otimes I_{m_1} \\ &= \tfrac{1}{r_2+2}(4I_{m_2} + L_{l(G_2)}) \otimes I_{m_1}. \end{aligned}$$

So $S^{-1} = (r_2 + 2)(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$.

By Lemma 2.3, we have $-A_1^{-1} B_1 S^{-1}$

$$\begin{aligned} &= -\tfrac{1}{r_2+2}(I_{n_2} \otimes I_{m_1})(-R_2 \otimes I_{m_1})(r_2 + 2) \\ &\quad ((4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}) \\ &= R_2(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}. \end{aligned}$$

Similarly, $-S^{-1}C_1A_1^{-1} = (4I_{m_2} + L_{l(G_2)})^{-1}R_2^T \otimes I_{m_1}$.

Let $V = (2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}$, $T = (r_2+2)(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, $M = R_2(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$.

So

$$D^{-1} = \begin{pmatrix} V & M \\ M^T & T \end{pmatrix}.$$

Now we are ready to calculate H.
Let $P = (2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}$, $Q = (r_2+2)(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, $M = R_2(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, then

$$\begin{aligned} H &= L_1 + r_1n_2I_{n_1} - \begin{pmatrix} -1_{n_2}^T \otimes R_1 & 0 \end{pmatrix} \begin{pmatrix} V & M \\ M^T & T \end{pmatrix} \begin{pmatrix} -1_{n_2} \otimes R_1^T \\ 0 \end{pmatrix}, \\ &= L_1 + r_1n_2I_{n_1} - \tfrac{n_2}{2}R_1R_1^T = \tfrac{n_2+2}{2}L_1, \end{aligned}$$

By Lemma 2.8, we have $H^{\#} = \frac{2}{n_2+2}L_1^{\#}$.

Next according to Lemma 2.8, we calculate $-H^{\#}BD^{-1}$ and $-D^{-1}B^TH^{\#}$.
$-H^{\#}BD^{-1}$

$$\begin{aligned} &= -\tfrac{2}{n_2+2}L_1^{\#}\begin{pmatrix} -1_{n_2}^T \otimes R_1 & 0 \end{pmatrix}\begin{pmatrix} P & M \\ M^T & Q \end{pmatrix} \\ &= \tfrac{2}{n_2+2}L_1^{\#}\begin{pmatrix} \frac{1}{2}1_{n_2}^T \otimes R_1 & 1_{n_2}^TM \end{pmatrix}. \end{aligned}$$

Note that $1_{n_2}^TR_2 = 2 \cdot 1_{m_2}^T$, then $1_{n_2}^TR_2(4I_{m_2} + L_{l(G_2)})^{-1} \otimes R_1 = 2 \cdot 1_{m_2}^T(4I_{m_2} + L_{l(G_2)})^{-1} \otimes R_1 = \frac{1}{2}1_{m_2}^T \otimes R_1$, so

$$-H^{\#}BD^{-1} = \tfrac{1}{n_2+2}L_1^{\#}\begin{pmatrix} 1_{n_2}^T \otimes R_1 & 1_{m_2}^T \otimes R_1 \end{pmatrix}$$

and
$-D^{-1}B^TH^{\#}$

$$\begin{aligned} &= -\begin{pmatrix} P & M \\ M^T & Q \end{pmatrix}\begin{pmatrix} -1_{n_2} \otimes R_1^T \\ 0 \end{pmatrix}\tfrac{1}{n_2+2}L_1^{\#} \\ &= \tfrac{1}{n_2+2}\begin{pmatrix} 1_{n_2} \otimes R_1^T \\ 1_{m_2} \otimes R_1^T \end{pmatrix}L_1^{\#}. \end{aligned}$$

We are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.
Let $1_{n_2} \otimes R_1^T = H$, $K = 1_{m_2}^T \otimes R_1$, then
$D^{-1}B^TH^{\#}BD^{-1}$

$$\begin{aligned} &= \begin{pmatrix} \frac{1}{2}H \\ \frac{1}{2}K^T \end{pmatrix}\tfrac{1}{n_2+2}L_1^{\#}\begin{pmatrix} H^T & K \end{pmatrix} \\ &= \tfrac{1}{2(n_2+2)}\begin{pmatrix} HL_1^{\#}H^T & HL_1^{\#}K \\ K^TL_1^{\#}H^T & K^TL_1^{\#}K \end{pmatrix}. \end{aligned}$$

Let $1_{n_2} \otimes R_1^T = H$, $K = 1_{m_2}^T \otimes R_1$, then based on Lemma 2.3 and 2.7, the following matrix
$N =$

$$\begin{pmatrix} \frac{2}{n_2+2}L_1^{\#} & \frac{1}{n_2+2}L_1^{\#}H^T & \frac{1}{n_2+2}L_1^{\#}K \\ \frac{1}{n_2+2}HL_1^{\#} & P + \frac{1}{2(n_2+2)}HL_1^{\#}H^T & M + \frac{1}{2(n_2+2)}HL_1^{\#}K \\ \frac{1}{n_2+2}K^TL_1^{\#} & M^T + \frac{1}{2(n_2+2)}K^TL_1^{\#}H^T & Q + \frac{1}{2(n_2+2)}K^TL_1^{\#}H^T \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of $L_{G_1 \vee G_2}$, where $P = (2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}$, $Q = (r_2+2)(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, $M = R_2(4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$. Let N be Equation (3.1).

For any $i, j \in V(G_1)$, by Lemma 2.1 and the Equation (3.1), we have
$r_{ij}(G_1 \vee G_2)$

$$\begin{aligned} &= \frac{2}{n_2+2}\left(L_1^{\#}\right)_{ii} + \frac{2}{n_2+2}\left(L_1^{\#}\right)_{jj} \\ &\quad -\frac{4}{n_2+2}\left(L_1^{\#}\right)_{ij} = \frac{2}{n_2+2}r_{ij}(G_1). \end{aligned}$$

For any $i, j \in V(G_2)$, by Lemma 2.1 and the Equation (3.1), we have
$r_{ij}(G_1 \vee G_2)$

$$\begin{aligned} &= \left((2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}\right)_{ii} + \Big((2I_{n_2} + \frac{1}{2}L_2)^{-1} \\ &\quad \otimes I_{m_1}\Big)_{jj} - 2\left((2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}\right)_{ij}. \end{aligned}$$

For any $i \in V(G_1), j \in V(G_2)$, by Lemma 2.1 and the Equation (3.1), we have
$r_{ij}(G_1 \vee G_2)$

$$\begin{aligned} &= \frac{2}{n_2+2}\left(L_1^{\#}\right)_{ii} + \left((2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}\right)_{jj} \\ &\quad -\frac{4}{n_2+2}\left(L_1^{\#}\right)_{ij}. \end{aligned}$$

By Lemma 2.5, we have
$Kf(L_{G_1 \vee G_2})$

$$\begin{aligned} &= (n_1 + n_2 + 2m_1)tr(N) - 1^TN1 \\ &= (n_1 + n_2 + 2m_1)\Bigg(\frac{2}{n_2+2}tr\left(L_1^{\#}\right) \\ &\quad +tr\left((2I_{n_2} + \frac{1}{2}L_2)^{-1} \otimes I_{m_1}\right) \\ &\quad +(r_2+2)tr\left((4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}\right) \\ &\quad +\frac{1}{2(n_2+2)}tr\left((1_{n_2} \otimes R_1^T)L_1^{\#}(1_{n_2}^T \otimes R_1)\right) \\ &\quad +\frac{1}{2(n_2+2)}tr\left((1_{m_2} \otimes R_1^T)L_1^{\#}(1_{m_2}^T \otimes R_1)\right)\Bigg) - 1^TN1. \end{aligned}$$

Note that the eigenvalues of $(2I_{n_2} + \frac{1}{2}L_2)$ are $\frac{1}{2}\mu_1(G_2) + 2, \frac{1}{2}\mu_2(G_2) + 2, ..., \frac{1}{2}\mu_{n_2}(G_2) + 2$. Then

$$\begin{aligned} tr((2I_{n_2} + \tfrac{1}{2}L_2)^{-1} \otimes I_{m_1})^{-1} &= m_1\textstyle\sum_{i=1}^{n_2}(\frac{1}{2}\mu_i(G_2) + 2)^{-1} \\ &= m_1\textstyle\sum_{i=1}^{n_2}\frac{1}{\frac{1}{2}\mu_i(G_2)+2}. \end{aligned}$$

By Lemma 2.6, then

$$\begin{aligned} tr((4I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}) &= m_1\textstyle\sum_{i=1}^{n_1}\frac{1}{4+\mu_i(G_2)} + \\ &\quad \tfrac{m_1(n_2-m_2)}{4+2r_2}. \end{aligned}$$

By Lemma 2.7, we have
$tr\left((1_{n_2} \otimes R_1^T)L_1^{\#}(1_{n_2}^T \otimes R_1)\right)$

$$\begin{aligned} &= n_2tr\left(R_1^TL_1^{\#}R_1\right) \\ &= n_2\textstyle\sum_{i<j, ij \in E(G)}\left(L_{ii}^{\#} + L_{jj}^{\#} + 2L_{ij}^{\#}\right) \\ &= n_2\textstyle\sum_{i<j, ij \in E(G)}\left(2L_{ii}^{\#} + 2L_{jj}^{\#} - r_{ij}(G_1)\right) \\ &= 2n_2r_1tr(L_{G_1}^{\#}) - n_2(n_1 - 1). \end{aligned}$$

Similarly,

$$\begin{aligned} &tr\left((1_{m_2} \otimes R_1^T)L_1^{\#}(1_{m_2}^T \otimes R_1)\right) \\ &= 2m_2r_1tr(L_{G_1}^{\#}) - m_2(n_1 - 1). \end{aligned}$$

So

$$
\begin{aligned}
&Kf(L_{G_1\vee G_2})\\
=\ & (n_1+n_2+2m_1)tr(N)-1^TN1\\
=\ & (n_1+n_2+2m_1)\Bigg(\frac{2}{n_1(n_2+2)}Kf(G_1)\\
&+m_1\sum_{i=1}^{n_2}\frac{1}{\frac{1}{2}\mu_i(G_2)+2}\\
&+(r_2+2)tr\left((4I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1}\right)\\
&+\frac{1}{2(n_2+2)}tr\left((1_{n_2}\otimes R_1^T)L_1^{\#}(1_{n_2}^T\otimes R_1)\right)\\
&+\frac{1}{2(n_2+2)}tr\left((1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)\right)\Bigg)\\
&-1^TN1\\
=\ & (n_1+n_2+2m_1)\Bigg(\frac{2}{n_1(n_2+2)}Kf(G_1)\\
&+m_1\sum_{i=1}^{n_2}\frac{1}{\frac{1}{2}\mu_i(G_2)+2}\\
&+(r_2+2)\left(m_1\sum_{i=1}^{n_1}\frac{1}{4+\mu_i(G_2)})+\frac{m_1(n_2-m_2)}{4+2r_2}\right)\\
&+\frac{1}{2(n_2+2)}\left(2n_2r_1tr(L_{G_1}^{\#})-n_2(n_1-1)\right)\\
&+\frac{1}{2(n_2+2)}\left(2m_2r_1tr(L_{G_1}^{\#})-m_2(n_1-1)\right)\Bigg)\\
&-1^TN1.
\end{aligned}
$$

Next, we calculate the $1^T(L_{G_1\vee G_2}^{(1)})1$. Since $L_G^{\#}1=0$, then $1^T(L_{G_1\vee G_2}^{(1)})1$

$$
\begin{aligned}
=\ & 1^T((2I_{n_2}+\frac{1}{2}L_2)^{-1}\otimes I_{m_1})1+1^T((r_2+2)\\
&(4I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1})1\\
&+1^T(R_2(4I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1})1\\
&+1^T((4I_{m_2}+L_{l(G_2)})^{-1}R_2^T\otimes I_{m_1})1\\
&+\frac{1}{2(n_2+2)}1^T(1_{n_2}\otimes R_1^T)L_1^{\#}(1_{n_2}^T\otimes R_1)1\\
&+\frac{1}{2(n_2+2)}1^T(1_{n_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1\\
&+\frac{1}{2(n_2+2)}1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{n_2}^T\otimes R_1)1\\
&+\frac{1}{2(n_2+2)}1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1.
\end{aligned}
$$

Let $T=1_{m_1n_2}^T((2I_{n_2}+\frac{1}{2}L_2)^{-1}\otimes I_{m_1})1_{m_1n_2}$, $F=((2I_{n_2}+\frac{1}{2}L_2)\otimes I_{m_1})$, then

$$
\begin{aligned}
T\ =\ & \begin{pmatrix}1_{n_2}^T & 1_{n_2}^T & \cdots & 1_{n_2}^T\end{pmatrix}\\
&\begin{pmatrix}F^{-1} & & & \\ & F^{-1} & & \\ & & \ddots & \\ & & & F^{-1}\end{pmatrix}\begin{pmatrix}1_{n_2}\\ 1_{n_2}\\ \cdots \\ 1_{n_2}\end{pmatrix}\\
=\ & m_11_{n_2}^T(2I_{n_2}+\tfrac{1}{2}L_2)^{-1}1_{n_2}=\tfrac{m_1n_2}{2}.
\end{aligned}
$$

Similarly,
$1^T(4I_{m_2}+L_{l(G_2)})^{-1}1=\frac{m_2}{4}$, $1^T(R_2(4I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1})1=1^T((4I_{m_2}+L_{l(G_2)})^{-1}R_2^T\otimes I_{m_1})1=\frac{m_1m_2}{2}$.

By direct computation,

$1^T(1_{n_2}\otimes R_1^T)L_1^{\#}(1_{n_2}^T\otimes R_1)1=n_2^2\pi^TL_1^{\#}\pi=n_2^2r_1^21^TL_1^{\#}1=0$, $1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1=m_2^2\pi^TL_1^{\#}\pi=0$,

$1^T(1_{n_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1=1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{n_2}^T\otimes R_1)1=n_2m_2\pi^TL_1^{\#}\pi=0$.

So

$$
\begin{aligned}
1^T(L_{G_1\vee G_2}^{(1)})1\ =\ & \frac{m_1m_2}{2}+\frac{m_2(r_2+2)}{4}+m_1m_2\\
=\ & \frac{6m_1m_2+m_2(r_2+2)}{4}.
\end{aligned}
$$

Lemma 2.5 implies that

$$Kf(G)=(n_1+n_2+2m_1)tr(N)-1^TN1.$$

Then plugging $tr(L_{G_1\vee G_2}^{(1)})$ and $1^T(L_{G_1\vee G_2}^{(1)})1$ into the equation above, we obtain the required result.

IV. Resistance distance and Kirchhoff index of edge-subdivision-edge corona for graphs

In this section, we focus on determing the resistance distance and Kirchhoff index of edge-subdivision-edge corona whenever G_1 is an r_1-regular graph.

Theorem 4.1 Let G_1 be an r_1-regular graph with n_1 vertices and m_1 edges and G_2 an r_2-regular graphs with n_2 vertices and m_2 edges. Then $G_1\forall G_2$ have the resistance distance and Kirchhoff index

(i) For any $i,j\in V(G_1)$, we have
$r_{ij}(G_1\forall G_2)$

$$
\begin{aligned}
=\ & \frac{2}{m_2+2}(L_1^{\#})_{ii}+\frac{2}{m_2+2}(L_1^{\#})_{jj}-\frac{4}{m_2+2}(L_1^{\#})_{ij}\\
=\ & \frac{2}{m_2+2}r_{ij}(G_1).
\end{aligned}
$$

(ii) For any $i,j\in V(G_2)$, we have
$r_{ij}(G_1\forall G_2)$

$$
\begin{aligned}
=\ & (I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{ii}+(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{jj}\\
&-2(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1})_{ij}.
\end{aligned}
$$

(iii) For any $i\in V(G_1), j\in V(G_2)$, we have
$r_{ij}(G_1\forall G_2)$

$$
\begin{aligned}
=\ & \frac{2}{m_2+2}\left(L_1^{\#}\right)_{ii}+\left(I_{n_1}\otimes(L_{G_2}+I_{n_2})^{-1}\right)_{jj}\\
&-\frac{4}{m_2+2}\left(L_1^{\#}\right)_{ij}.
\end{aligned}
$$

(iv) $Kf(G_1\forall G_2)$

$$
\begin{aligned}
=\ & (n_1+n_2+2m_1)\Bigg(\frac{4r_2+2n_1r_1^3+m_1r_2}{2n_1r_2(m_2+2)}Kf(G_1)\\
&+m_1\sum_{i=1}^{n_2}\frac{1}{\frac{1}{4}\mu_i(G_2)+\frac{r_2}{2}}+r_2m_1\sum_{i=1}^{n_2}\frac{1}{2r_2+\mu_i(G_2)}\\
&+\frac{m_1(n_2-m_2)}{4}-\frac{(n_1-1)(n_1r_1^2+m_2r_2)}{2r_2(m_2+2)}\Bigg)\\
&-\frac{4m_1n_2+m_2r_2+4m_1m_2}{2r_2}.
\end{aligned}
$$

Proof Let $R_i (i=1,2)$ be the incidence matrix of G_i. Then with a proper labeling of vertices, the Laplacian matrix of $G_1\forall G_2$ can be written as

$L(G_1 \forall G_2) =$

$$\begin{pmatrix} L_1 + r_1 m_2 I_{n_1} & 0_{n_1 \times m_1 n_2} & -1_{m_2}^T \otimes R_1 \\ 0_{m_1 n_2 \times n_1} & r_2 I_{n_2} \otimes I_{m_1} & -R_2 \otimes I_{m_1} \\ -1_{m_2} \otimes R_1^T & -R_2^T \otimes I_{m_1} & 4I_{m_1 m_2} \end{pmatrix}.$$

Let $A = L_1 + r_1 m_1 I_{n_1}$, $B = \begin{pmatrix} 0_{n_1 \times m_1 n_2} & -1_{m_2}^T \otimes R_1 \end{pmatrix}$, $B^T = \begin{pmatrix} 0_{m_1 n_2 \times n_1} \\ -1_{m_2} \otimes R_1^T \end{pmatrix}$ and

$$D = \begin{pmatrix} r_2 I_{n_2} \otimes I_{m_1} & -R_2 \otimes I_{m_1} \\ -R_2^T \otimes I_{m_1} & 4I_{m_1 m_2} \end{pmatrix}.$$

First we compute the D^{-1}. By Lemma 2.3, we have
$A_1 - B_1 D_1^{-1} C_1$

$$\begin{array}{rl} = & r_2 I_{n_2} \otimes I_{m_1} - \frac{1}{4}(R_2 \otimes I_{m_1})(R_2^T \otimes I_{m_1}) \\ = & (r_2 I_{n_2} - \frac{1}{4}(r_2 I_{n_2} + A(G_2))) \otimes I_{m_1} \\ = & (\frac{r_2}{2} I_{n_2} + \frac{1}{4} L_2) \otimes I_{m_1}, \end{array}$$

so $(A_1 - B_1 D_1^{-1} C_1)^{-1} = (\frac{r_2}{2} I_{n_2} + \frac{1}{4} L_2)^{-1} \otimes I_{m_1}$.
By Lemma 2.3, we have

$$\begin{array}{rcl} S & = & D_1 - C_1 A_1^{-1} B_1 \\ & = & 4I_{m_1 m_2} - \frac{1}{r_2}(R_2^T \otimes I_{m_1})(I_{n_2} \otimes I_{m_1})(R_2 \otimes I_{m_1}) \\ & = & 4I_{m_1 m_2} - \frac{1}{r_2}(R_2^T R_2 \otimes I_{m_1}) \\ & = & 4I_{m_1 m_2} - \frac{1}{r_2}((2I_{m_2} + A(l(G_2)) \otimes I_{m_1}) \\ & = & \frac{1}{r_2}(2r_2 I_{m_2} + L_{l(G_2)}) \otimes I_{m_1}, \end{array}$$

so $S^{-1} = r_2(2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$.
By Lemma 2.3, we have

$$\begin{array}{rcl} -A_1^{-1} B_1 S^{-1} & = & -\frac{1}{r_2}(I_{n_2} \otimes I_{m_1})(-R_2 \otimes I_{m_1}) \\ & & r_2((2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}) \\ & = & R_2(2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}. \end{array}$$

Similarly, $-S^{-1} C_1 A_1^{-1} = (2r_2 I_{m_2} + L_{l(G_2)})^{-1} R_2^T \otimes I_{m_1}$.

Let $P = (\frac{r_2}{2} I_{n_2} + \frac{1}{4} L_2)^{-1} \otimes I_{m_1}$, $Q = r_2(2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, $M = R_2(2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, then

$$D^{-1} = \begin{pmatrix} P & M \\ M^T & Q \end{pmatrix}.$$

Now we are ready to calculate H.

$$\begin{array}{rcl} H & = & L_1 + r_1 m_2 I_{n_1} - \begin{pmatrix} 0_{n_1 \times m_1 n_2} & -1_{m_2}^T \otimes R_1 \end{pmatrix} \\ & & \begin{pmatrix} P & M \\ M^T & Q \end{pmatrix} \begin{pmatrix} 0_{m_1 n_2 \times n_1} \\ -1_{m_2} \otimes R_1^T \end{pmatrix} \\ & = & L_1 + r_1 m_2 I_{n_1} - \frac{m_2}{2} R_1 R_1^T = \frac{m_2+2}{2} L_1. \end{array}$$

By Lemma 2.8, we have $H^\# = \frac{2}{m_2+2} L_1^\#$.
Next according to Lemma 2.8, we calculate $-H^\# B D^{-1}$ and $-D^{-1} B^T H^\#$.
Note that $R(G)\mathbf{1} = \pi$, where $\pi = (d_1, d_2, ..., d_n)^T$, then
$-H^\# B D^{-1}$

$$\begin{array}{rl} = & -\frac{2}{m_2+2} L_1^\# \begin{pmatrix} 0 & -1_{m_2}^T \otimes R_1 \end{pmatrix} \begin{pmatrix} P & M \\ M^T & Q \end{pmatrix} \\ = & \frac{2}{m_2+2} L_1^\# \begin{pmatrix} \frac{1}{2} 1_{m_2}^T R_2^T \otimes R_1 & \frac{1}{2} 1_{m_2}^T \otimes R_1 \end{pmatrix} \\ = & \frac{1}{m_2+2} L_1^\# \begin{pmatrix} \pi^T \otimes R_1 & 1_{m_2}^T \otimes R_1 \end{pmatrix} \end{array}$$

and

$-D^{-1} B^T H^\#$

$$\begin{array}{rl} = & -\begin{pmatrix} P & M \\ M^T & Q \end{pmatrix} \begin{pmatrix} 0 \\ -1_{m_2} \otimes R_1^T \end{pmatrix} \frac{1}{m_2+2} L_1^\# \\ = & \frac{1}{m_2+2} \begin{pmatrix} \pi \otimes R_1^T \\ 1_{m_2} \otimes R_1^T \end{pmatrix} L_1^\#. \end{array}$$

We are ready to compute the $D^{-1} B^T H^\# B D^{-1}$. Let $W = \pi \otimes R_1^T$, $R = 1_{m_2} \otimes R_1^T$, then
$D^{-1} B^T H^\# B D^{-1}$

$$\begin{array}{rl} = & \begin{pmatrix} \frac{1}{2r_2} \pi \otimes R_1^T \\ \frac{1}{2}(1_{m_2} \otimes R_1^T) \end{pmatrix} \frac{1}{m_2+2} L_1^\# \begin{pmatrix} \pi^T \otimes R_1 & 1_{m_2}^T \otimes R_1 \end{pmatrix} \\ = & \frac{1}{m_2+2} \begin{pmatrix} \frac{1}{2r_2} W L_1^\# W^T & \frac{1}{2r_2} W L_1^\# R^T \\ \frac{1}{2} R L_1^\# W^T & \frac{1}{2} R L_1^\# R^T \end{pmatrix}. \end{array}$$

Based on Lemmas 2.3 and 2.8, the following matrix
$N =$

$$\begin{pmatrix} \frac{2}{m_2+2} L_1^\# & \frac{1}{m_2+2} L_1^\# W & \frac{1}{m_2+2} L_1^\# R^T \\ \frac{1}{m_2+2} W^T & P + \frac{1}{2r_2(m_2+2)} W L_1^\# W^T & M + \frac{1}{2r_2(m_2+2)} W L_1^\# R^T \\ \frac{1}{m_2+2} R & M^T + \frac{1}{2(m_2+2)} R L_1^\# W^T & Q + \frac{1}{2(m_2+2)} R L_1^\# R^T \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of $L_{G_1 \forall G_2}$, where $P = (\frac{r_2}{2} I_{n_2} + \frac{1}{4} L_2)^{-1} \otimes I_{m_1}$, $Q = r_2(2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$, $M = R_2(2r_2 I_{m_2} + L_{l(G_2)})^{-1} \otimes I_{m_1}$. Let the above N be the Equation (4.1).
For any $i, j \in V(G_1)$, by Lemma 2.1 and the Equation (4.1), we have
$r_{ij}(G_1 \forall G_2)$

$$\begin{array}{rl} = & \dfrac{2}{m_2+2}(L_1^\#)_{ii} + \dfrac{2}{m_2+2}(L_1^\#)_{jj} - \dfrac{4}{m_2+2}(L_1^\#)_{ij} \\ = & \dfrac{2}{m_2+2} r_{ij}(G_1). \end{array}$$

For any $i, j \in V(G_2)$, by Lemma 2.1 and the Equation (4.1), we have
$r_{ij}(G_1 \forall G_2)$

$$\begin{array}{rl} = & \left((\dfrac{r_2}{2} I_{n_2} + \dfrac{1}{4} L_2)^{-1} \otimes I_{m_1}\right)_{ii} + \\ & \left((\dfrac{r_2}{2} I_{n_2} + \dfrac{1}{4} L_2)^{-1} \otimes I_{m_1}\right)_{jj} \\ & -2\left((\dfrac{r_2}{2} I_{n_2} + \dfrac{1}{4} L_2)^{-1} \otimes I_{m_1}\right)_{ij}. \end{array}$$

For any $i \in V(G_1), j \in V(G_2)$, by Lemma 2.1 and the Equation (4.1), we have
$r_{ij}(G_1 \forall G_2)$

$$\begin{array}{rl} = & \dfrac{2}{m_2+2}\left(L_1^\#\right)_{ii} + \left((\dfrac{r_2}{2} I_{n_2} + \dfrac{1}{4} L_2)^{-1} \otimes I_{m_1}\right)_{jj} \\ & -\dfrac{4}{m_2+2}\left(L_1^\#\right)_{ij}. \end{array}$$

By Lemma 2.5, we have

$$
\begin{aligned}
&Kf(L_{G_1\forall G_2})\\
&= (n_1+n_2+2m_1)tr(N)-1^TN1\\
&= (n_1+n_2+2m_1)\left(\frac{2}{m_2+2}tr(L_1^{\#})\right.\\
&\quad +tr\left((\frac{r_2}{2}I_{n_2}+\frac{1}{4}L_2)^{-1}\otimes I_{m_1}\right)\\
&\quad + r_2tr\left((2r_2I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1}\right)\\
&\quad + \frac{1}{2r_2(m_2+2)}tr\left((\pi\otimes R_1^T)L_1^{\#}(\pi^T\otimes R_1)\right)\\
&\quad \left.+\frac{1}{2(m_2+2)}tr\left((1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1))\right)\right)-1^TN1.
\end{aligned}
$$

Note that the Laplacian eigenvalues of $(\frac{r_2}{2}I_{n_2}+\frac{1}{4}L_2)$ are $\frac{1}{4}\mu_1(G_2)+\frac{r_2}{2}, \frac{1}{4}\mu_2(G_2)+\frac{r_2}{2},...,\frac{1}{4}\mu_{n_2}(G_2)+\frac{r_2}{2}$. Then

$$
\begin{aligned}
&tr((\tfrac{r_2}{2}I_{n_2}+\tfrac{1}{4}L_2)^{-1}\otimes I_{m_1})^{-1}\\
&= m_1\sum_{i=1}^{n_2}(\tfrac{1}{4}\mu_i(G_2)+\tfrac{r_2}{2})\\
&= m_1\sum_{i=1}^{n_2}\frac{1}{\frac{1}{4}\mu_i(G_2)+\frac{r_2}{2}}.
\end{aligned}
$$

By Lemma 2.6, then

$$
\begin{aligned}
&tr((2r_2I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1})=\\
&m_1\sum_{i=1}^{n_2}\frac{1}{2r_2+\mu_i(G_2)}+\frac{m_1(n_2-m_2)}{4r_2}..
\end{aligned}
$$

By direct computation and Lemma 2.7, $tr(\pi\otimes R_1^T)L_1^{\#}(\pi^T\otimes R_1)$

$$
\begin{aligned}
&= (\textstyle\sum_{i=1}^{n_1}d_i^2)tr(R_1^TL_1^{\#}R_1)\\
&= (\textstyle\sum_{i=1}^{n_1}d_i^2)\sum_{i<j,ij\in E(G)}\left(L_{ii}^{\#}+L_{jj}^{\#}+2L_{ij}^{\#}\right)\\
&= (\textstyle\sum_{i=1}^{n_1}d_i^2)\sum_{i<j,ij\in E(G)}\left(2L_{ii}^{\#}+2L_{jj}^{\#}-r_{ij}(G_1)\right).\\
&= 2(\textstyle\sum_{i=1}^{n_1}d_i^2)tr(D_{G_1}L_{G_1}^{\#})-(\sum_{i=1}^{n_1}d_i^2)(n_1-1)\\
&= 2n_1r_1^3tr(L_{G_1}^{\#})-n_1r_1^2(n_1-1),
\end{aligned}
$$

and

$$
\begin{aligned}
&tr(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)\\
&= m_2tr(R_1^TL_1^{\#}R_1)\\
&= m_2\textstyle\sum_{i<j,ij\in E(G)}\left[L_{ii}^{\#}+L_{jj}^{\#}+2L_{ij}^{\#}\right]\\
&= m_2tr(2L_{ii}^{\#}+2L_{jj}^{\#}-r_{ij}(G_1)) .\\
&= m_2(tr(D_{G_1}L_{G_1}^{\#})-(n_1-1))\\
&= m_2(r_1tr(L_{G_1}^{\#})-(n_1-1)).
\end{aligned}
$$

Next, we calculate the $1^T(L^{(1)}_{G_1\vee G_2})1$. Since $L_G^{\#}1=0$, then $1^T(L^{(1)}_{G_1\forall G_2})1$

$$
\begin{aligned}
&= 1^T((\frac{r_2}{2}I_{n_2}+\frac{1}{4}L_2)^{-1}\otimes I_{m_1})1+r_21^T(2r_2I_{m_2}\\
&\quad +L_{l(G_2)})^{-1}\otimes I_{m_1}1\\
&\quad +1^T(R_2(2r_2I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1})1\\
&\quad +1^T((2r_2I_{m_2}+L_{l(G_2)}R_2^T)^{-1}\otimes I_{m_1})1\\
&\quad +\frac{1}{2r_2(n_2+2)}1^T(\pi\otimes R_1^T)L_1^{\#}(\pi^T\otimes R_1)1\\
&\quad +\frac{1}{2r_2(n_2+2)}1^T(\pi\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1\\
&\quad +\frac{1}{2(m_2+2)}1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(\pi^T\otimes R_1)1\\
&\quad +\frac{1}{2(m_2+2)}1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1.
\end{aligned}
$$

Similarly, $1^T((\frac{r_2}{2}I_{n_2}+\frac{1}{4}L_2)^{-1}\otimes I_{m_1})^{-1}1=\frac{2m_1n_2}{r_2}$, $1^T(2r_2I_{m_2}+L_{l(G_2)})^{-1}1=\frac{m_2}{2r_2}$, $1^T(R_2(2r_2I_{m_2}+L_{l(G_2)})^{-1}\otimes I_{m_1})1=1^T((2r_2I_{m_2}+L_{l(G_2)})^{-1}R_2^T\otimes I_{m_1})1=\frac{m_1m_2}{r_2}$.
By direct computation, $1^T(\pi\otimes R_1^T)L_1^{\#}(\pi^T\otimes R_1)1=n_1^21^TR_1^TL_1^{\#}R_11=n_1^2r_2^21^TL_1^{\#}1=0$.
Similarly, $1^T(\pi\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1=1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(\pi^T\otimes R_1)1=1^T(1_{m_2}\otimes R_1^T)L_1^{\#}(1_{m_2}^T\otimes R_1)1=0$.

So

$$
\begin{aligned}
1^T(G_1\forall G_2)1 &= \frac{2m_1n_2}{r_2}+\frac{m_2}{2}+\frac{2m_1m_2}{r_2}\\
&= \frac{4m_1n_2+m_2r_2+4m_1m_2}{2r_2}.
\end{aligned}
$$

Lemma 2.5 implies that

$$Kf(G_1\forall G_2)=(n_1+n_2+2m_1)tr(N)-1^TN1.$$

Then plugging $tr(L^{(1)}_{G_1\forall G_2})$ and $1^T(L^{(1)}_{G_1\forall G_2})1$ into the equation above, we obtain the required result.

V. Conclusion

In this paper, we give the closed-form formulas for resistance distance and Kirchhoff index of the edge-subdivision-vertex and edge-subdivision-edge corona. This method is a general method. The resistance distance and Kirchhoff index of the the edge-subdivision-vertex corona and edge-subdivision-edge corona can obtain in terms of the resistance distance and Kirchhoff index of the factor graph.

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