Decoupled Modular Regularized VMS-POD for Darcy-Brinkman Equations

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***Abstract*—We extend the post-processing implementation of a projection based variational multiscale (VMS) method with proper orthogonal decomposition (POD) to flows governed by double diffusive convection. In the method, the stabilization terms are added to momentum equation, heat and mass transfer equations as a completely decoupled separate steps. The theoretical analyses are presented. The results are verified with numerical tests on a benchmark problem.**

***Index Terms*—post-process, variational multiscale, proper orthogonal decomposition, double-diffusive, reduced order models.**

I. Introduction

In this study, the Darcy-Brinkman equations with double diffusive convection is considered. The dimensionless form is given as:

$$\begin{aligned} \mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}u + (\mathbf{u}\cdot\nabla)\mathbf{u} + Da^{-1}\mathbf{u} \\ +\nabla p = (\beta_T T + \beta_C C)\,\mathbf{g} \text{ in } (0,\tau]\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 \text{ in } (0,\tau]\times\Omega, \\ \mathbf{u} = \mathbf{0} \text{ in } (0,\tau]\times\partial\Omega, \\ T_t + \mathbf{u}\cdot\nabla T = \gamma\Delta T \text{ in } (0,\tau]\times\partial\Omega, \\ C_t + \mathbf{u}\cdot\nabla C = D_c\Delta C \text{ in } (0,\tau]\times\partial\Omega, \\ T, C = 0 \text{ on } \Gamma_D, \\ \nabla T\cdot n = \nabla C\cdot n = 0 \text{ on } \Gamma_N, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0,\ T(0,\mathbf{x}) = T_0,\ C(0,\mathbf{x}) = C_0 \text{ in } \Omega, \end{aligned} \tag{1}$$

where $\mathbf{u}(t,\mathbf{x})$, $p(t,\mathbf{x})$, $T(t,\mathbf{x})$, $C(t,\mathbf{x})$ are the fluid velocity, the pressure, the temperature, and the concentration fields, respectively. Let $\Omega\subset\mathbb{R}^d, d\in\{2,3\}$ be a confined porous enclosure with polygonal boundary $\partial\Omega$ and Γ_N be a regular open subset of the boundary and $\Gamma_D = \partial\Omega\setminus\Gamma_N$. The initial velocity, temperature and concentration fields are given as $\mathbf{u}_0$, T_0, C_0. The parameters in (1) are the kinematic viscosity $\nu > 0$, inversely proportional to Re, the thermal diffusivity $\gamma > 0$, the velocity deformation tensor $\mathbb{D}\mathbf{u} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$, the mass diffusivity $D_c > 0$, the Darcy number Da, and the gravitational acceleration vector $\mathbf{g}$. The solutal and the thermal expansion coefficients are β_C, and β_T, respectively. The dimensionless parameters are the Prandtl number Pr, the Darcy number Da, the buoyancy ratio N, the Lewis number Le, the Schmidt number Sc, and the thermal and solutal Grashof numbers Gr_T and Gr_C, respectively. Here H is the cavity height, k the permeability, and ΔT and ΔC are the temperature and the concentration differences, respectively.

Double diffusive convection represents a form of convection driven by two different potentials with different diffusion rates. It is very important in many applications such as oceanography, meteorology and geology. The physical model is formed by forcing of momentum with both heat and mass transfer. Darcy terms defines the porosity of domain.

The basic challenges of the Darcy-Brinkman scheme come from the Navier Stokes equations (NSE). This is due to the complex behaviour of the flow at high Re and the absence of the analytical solution of NSE. Combining momentum equation with mass and heat transfer equations makes the problem more difficult. Thus, solving Darcy Brinkman equations accurately and efficiently remains a challenge for the computational fluid dynamics community. Furthermore, the use of full order methods lead to large degrees of freedom. This causes complex algebraic systems and high computational time. To address this issue, model order reduction techniques are used.

In this study, we use Galerkin based proper orthogonal decomposition (POD) method. The idea is find most energetic structure in the sytem which represent the snapshots. This idea can be found in Karhunen Lóeve expansion [1] , principal component analysis [2] and singular value decomposition [3]. Optimal POD basis functions are obtained by using finite element solution. Since, POD uses only most energetic structure in the system, it decreases the computational cost, process time and complexity of system. Thanks to the significant advantages, POD has been found efficient method for different multhpysics problems [4], [5], [6]. Hence the application model order reduction with POD methodology to Darcy-Brinkman scheme is significant.

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In this system, heat transfer is expressed with Rayleigh number (Ra) which is defined as ratio of buoyancy term to viscous term. The magnitude of the Ra indicates whether the flow is laminar or turbulent. For high Ra, the instability occurs due to the emergence of convection cells. Thus, the behaviour of the flow becomes turbulent. For such case, the VMS method can be used to eliminate the oscillation and stabilize the convective terms. Recent works [7], [8], [9], [10] show VMS-POD increase numerical accuracy.

The basic points of VMS are separation of scales as resolved small scale and resolved large scale, and eliminate the oscillations in small scales by using projection. Separation of scales is a challenge in the method. Selection of POD basis functions in descending order are a remedy to this difficulty, i.e. small and large scales are decomposed naturally in POD method.

Many complex flows are solved by legacy codes, so it may be difficult to implement a new method in these flows. For such cases, the post-processing methods are easily added to legacy codes. Hence, the main objective of our study is application the post-processing VMS-POD idea by adding a separate, uncoupled and modular stabilization step for POD solution of Darcy-Brinkman system in each time step.

This work is arranged as follows. Section 2 presents the continuous variational formulation of the double diffusive Darcy-Brinkman system (1) and its discretization, and here the VMS-POD variational formulation is defined. Section 3 is devoted to the numerical analysis of the VMS-POD formulation. Finally, Section 4 concludes the work with a summary.

II. Full order model for the double diffusive Darcy-Brinkman system

Throughout the work standard notations for Sobolev spaces and their norms will be used. The norm in $(H^k(\Omega))^d$ is denoted by $\|\cdot\|_k$ and the norms in Lebesgue spaces $(L^p(\Omega))^d$, $1 \le p < \infty$, $p \neq 2$ by $\|\cdot\|_{L^p}$. The space $L^2(\Omega)$ is equipped with the norm and inner product $\|\cdot\|$ and $(\cdot,\cdot)$, respectively, and for these we drop the subscripts. The continuous velocity, pressure, temperature and concentration spaces are denoted by

$$\begin{aligned} \mathbf{X} &:= (\mathbf{H}_0^1(\Omega))^d, Q := L_0^2(\Omega), \\ W &:= \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_D\}, \\ \Psi &:= \{\Phi \in H^1(\Omega) : \Phi = 0 \text{ on } \Gamma_D\}, \end{aligned}$$

and the divergence free space given as

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0, \ \forall q \in Q\}.$$

We denote the dual space of $\mathbf{X}$ by $\mathbf{H}^{-1}$ with norm

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in X} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla \mathbf{v}\|}.$$

The following notations are utilized for discrete norms

$$\begin{aligned} |||w|||_{\infty,p} &:= \max_{0\le n\le M} \|w^n\|_p, \\ |||w|||_{m,p} &:= \left(\Delta t \sum_{n=0}^{M} \|w^n\|_p^m\right)^{1/m}. \end{aligned}$$

The variational formulation of (1) reads as follows: Find $\mathbf{u} : (0,\tau] \to \mathbf{X}$, $p : (0,\tau] \to Q$, $T : [0,\tau] \to W$ and $C : [0,\tau] \to \Psi$ satisfying

$$\begin{aligned} &(\mathbf{u}_t,\mathbf{v}) + 2\nu(\mathbb{D}\mathbf{u},\mathbb{D}\mathbf{v}) + b_1(\mathbf{u},\mathbf{u},\mathbf{v}) \\ &\quad +(Da^{-1}\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) \\ &\quad = \beta_T(\mathbf{g}T,\mathbf{v}) + \beta_C(\mathbf{g}C,\mathbf{v}), \end{aligned} \tag{2}$$

$$(T_t,S) + b_2(\mathbf{u},T,S) + \gamma(\nabla T,\nabla S) = 0, \tag{3}$$

$$(C_t,\phi) + b_3(\mathbf{u},C,\Phi) + D_c(\nabla C,\nabla\phi) = 0, \tag{4}$$

for all $(\mathbf{v},q,S,\Phi) \in (X,Q,W,\Psi)$, where

$$\begin{aligned} b_1(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})\right), \\ b_2(\mathbf{u},T,S) &:= \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)T,S) - ((\mathbf{u}\cdot\nabla)S,T)\right), \\ b_3(\mathbf{u},C,\Phi) &:= \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)C,\Phi) - ((\mathbf{u}\cdot\nabla)\Phi,C)\right), \end{aligned}$$

represent the skew-symmetric forms of the convective terms.

We consider a conforming finite element method for (2)-(4), with spaces $\mathbf{X}_h \subset \mathbf{X}$, $Q_h \subset Q$, $W_h \subset W$ and $\Psi_h \subset \Psi$. We also assume that the pair $(\mathbf{X}_h, Q_h)$ satisfies the discrete inf-sup condition. It will also be assumed for simplicity that the finite element spaces $\mathbf{X}_h$, W_h, Ψ_h are composed of piecewise polynomials of degree at most m and Q_h is composed of piecewise polynomials of degree at most $m-1$. In addition, we assume that the spaces satisfy the interpolation approximation properties. The discretely divergence free space for $(\mathbf{X}_h, Q_h)$ pairs is given by

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h : (\nabla\cdot\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h\}. \tag{5}$$

The inf-sup condition implies that the space $\mathbf{V}_h$ is a closed subspace of $\mathbf{X}_h$ and the formulation above involving $\mathbf{X}_h$ and Q_h is equivalent to the following $\mathbf{V}_h$ formulation: Find $(\mathbf{u}_h, T_h, C_h) \in (\mathbf{V}_h, W_h, \Psi_h)$ satisfying

$$\begin{aligned} &(\mathbf{u}_{h,t},\mathbf{v}_h) + 2\nu(\mathbb{D}\mathbf{u}_h,\mathbb{D}\mathbf{v}_h) \\ &\quad + b_1(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) + (Da^{-1}\mathbf{u}_h,\mathbf{v}_h) \\ &\quad = \beta_T(\mathbf{g}T_h,\mathbf{v}_h) + \beta_C(\mathbf{g}C_h,\mathbf{v}_h), \end{aligned} \tag{6}$$

$$\begin{aligned} &(T_{h,t},S_h) + b_2(\mathbf{u}_h,T_h,S_h) \\ &\quad + \gamma(\nabla T_h,\nabla S_h) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} &(C_{h,t},\Phi_h) + b_3(\mathbf{u}_h,C_h,\Phi_h) \\ &\quad + D_c(\nabla C_h,\nabla\Phi_h) = 0, \end{aligned} \tag{8}$$

for all $(\mathbf{v}_h, S_h, \Phi_h) \in (\mathbf{V}_h, W_h, \psi_h)$.

The goal of the POD is to find low dimensional bases for velocity, temperature, concentration by solving the minimization problems. The solution of the problem is obtained by using the method of snapshots. We note that all eigenvalues are sorted in descending order. Thus, the basis functions $\{\boldsymbol{\psi}_i\}_{i=1}^{r_1}$, $\{\phi_i\}_{i=1}^{r_2}$ and $\{\eta_i\}_{i=1}^{r_3}$ correspond to the first r_1, r_2 and r_3 largest eigenvalues $\{\lambda_i\}_{i=1}^{r_1}$, $\{\mu_i\}_{i=1}^{r_2}$, $\{\xi_i\}_{i=1}^{r_3}$ of the velocity, the temperature, the concentration, respectively. For simplicity, we will denote POD spaces using just r instead of r_1, r_2 and r_3. However, in the analysis, we are careful to distinguish that these parameters can be chosen independently.

Following spaces are needed for VMS-POD formulation.

$$\begin{aligned}
\mathbf{X}_r &= span\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_{r_1}\}, && (9)\\
W_r &= span\{\phi_1, \phi_2, \ldots, \phi_{r_2}\}, && (10)\\
\Psi_r &= span\{\eta_1, \eta_2, \ldots, \eta_{r_3}\}, && (11)\\
\mathbf{X}_R &= span\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_{R_1}\}, && (12)\\
W_R &= span\{\phi_1, \phi_2, \ldots, \phi_{R_2}\}, && (13)\\
\Psi_R &= span\{\eta_1, \eta_2, \ldots, \eta_{R_3}\}, && (14)
\end{aligned}$$

and

$$L_{R,\mathbf{u}} = \nabla \mathbf{X}_R,\ L_{R,T} = \nabla W_R,\ L_{R,C} = \nabla \Psi_R. \quad (15)$$

Note that by construction $\mathbf{X}_R \subseteq \mathbf{X}_r \subset \mathbf{V}_h \subset \mathbf{X}$, $W_R \subseteq W_r \subset W_h \subset W$ and $\Psi_R \subseteq \Psi_r \subset \Psi_h \subset \Psi$. The error term is decomposed by using the L^2 projection $P_{w,r}$ which is defined by

$$(w - P_{w,r}w, v_r) = 0, \quad (16)$$

for all test functions v_r which is in POD spaces. The following lemma will be used to bound the POD projection error. The proof can be found in [8].

Lemma II.1. *For true solution w^n at time t^n, we have*

$$\frac{1}{M}\sum_{n=1}^{M}\|w^n - P_{w,r}w^n\|^2 \le C\Big(h^{2m+2}|||w|||_{2,m+1}^2 + \sum_{i=r+1}^{d}\lambda_i\Big), \quad (17)$$

$$\frac{1}{M}\sum_{n=1}^{M}\|\nabla(w^n - P_{w,r}w^n)\|^2 \le C\Big((h^{2m} + \|S_{w,r}\|_2 h^{2m+2})|||w|||_{2,m+1}^2 + \varepsilon_u^2\Big), \quad (18)$$

where $\varepsilon_{w,r} = \sqrt{\sum_{i=r+1}^{d}\|\boldsymbol{\psi}_i^w\|_1^2\lambda_i^w}$.

Now, we state the POD-Galerkin (POD-G) formulation of the Darcy-Brinkman double diffusive system. Find $(\mathbf{u}_r, T_r, C_r) \in (\mathbf{X}_r, W_r, \Psi_r)$ satisfying

$$\begin{aligned}
&(\mathbf{u}_{r,t}, \mathbf{v}_r) + 2\nu(\mathbb{D}\mathbf{u}_r, \mathbb{D}\mathbf{v}_r)\\
&\quad + b_1(\mathbf{u}_r, \mathbf{u}_r, \mathbf{v}_r) + (Da^{-1}\mathbf{u}_r, \mathbf{v}_r)\\
&\quad = \beta_T(\mathbf{g}T_r, \mathbf{v}_r) + \beta_C(\mathbf{g}C_r, \mathbf{v}_r), && (19)
\end{aligned}$$

$$\begin{aligned}
&(T_{r,t}, S_r) + b_2(\mathbf{u}_r, T_r, S_r)\\
&\quad + \gamma(\nabla T_r, \nabla S_r) = 0, && (20)
\end{aligned}$$

$$\begin{aligned}
&(C_{r,t}, \Phi_r) + b_3(\mathbf{u}_r, C_r, \Phi_r)\\
&\quad + D_c(\nabla C_r, \nabla \Phi_r) = 0, && (21)
\end{aligned}$$

for all $(\mathbf{v}_r, S_r, \Phi_r) \in (\mathbf{X}_r, W_r, \Psi_r)$.

We equip this system (19)-(21) with the BDF2 temporal discretization. We consider adding the decoupled VMS stabilization from [10], [11], where in effect additional viscosity gets added to the smaller R_1, R_2, R_3 velocity, temperate and concentration modes in a post-processing step.

In order to prove an error estimate for the error between the true solution and the VMS-POD solution of the system, we use the L^2 projection operators $P_{u,r} : L^2 \to L_{R,\mathbf{u}}$, $P_{T,r} : L^2 \to L_{R,T}$, $P_{C,r} : L^2 \to L_{R,C}$. They are defined by

$$\begin{aligned}
(\mathbf{u} - P_{u,R}\mathbf{u}, \mathbf{v}_R) &= 0,\\
(T - P_{T,R}T, S_R) &= 0,\\
(C - P_{C,R}C, \zeta_R) &= 0,
\end{aligned} \quad (22)$$

for all $(\mathbf{v}_R, S_R, \zeta_R) \in (L_{R,\mathbf{u}}, L_{R,T}, L_{R,C})$ For simplicity we use $\tilde{P}_{w,R}$ instead of $I - P_{w,R}$.

Specifically, we post-process $(\mathbf{u}_r^{n+1}, T_r^{n+1}, C_r^{n+1})$ by solving the following algorithm. Let $\mathbf{g} \in L^2(0,\tau;\mathbf{H}^{-1}(\Omega))$ and initial conditions

$$(\mathbf{u}_0, T_0, C_0), (\mathbf{u}_1, T_1, C_1) \in ((L^2(\Omega))^d, L^2(\Omega), L^2(\Omega))$$

be given in $(\mathbf{X}_r, W_r, \Psi_r)$.

Algorithm II.1. *The post-processing VMS-POD approximation for double diffusive system (1) given as:* ***Step 1****: Find $(\mathbf{w}_{\mathbf{u},r}^{n+1}, w_{T,r}^{n+1}, w_{C,r}^{n+1}) \in (\mathbf{X}_r, W_r, \Psi_r)$ satisfying*

$$\begin{aligned}
&\Big(\frac{3\mathbf{w}_{\mathbf{u},r}^{n+1} - 4\mathbf{u}_r^n + \mathbf{u}_r^{n-1}}{2\Delta t}, \mathbf{v}_r\Big) + 2\nu(\mathbb{D}\mathbf{w}_{\mathbf{u},r}^{n+1}, \mathbb{D}\mathbf{v}_r)\\
&\quad + b_1(\mathbf{w}_{\mathbf{u},r}^{n+1}, \mathbf{w}_{\mathbf{u},r}^{n+1}, \mathbf{v}_r) + (Da^{-1}\mathbf{w}_{\mathbf{u},r}^{n+1}, \mathbf{v}_r)\\
&= \beta_T(\mathbf{g}w_{T,r}^{n+1}, \mathbf{v}_r) + \beta_C(\mathbf{g}w_{C,r}^{n+1}, \mathbf{v}_r), && (23)
\end{aligned}$$

$$\begin{aligned}
&\Big(\frac{3w_{T,r}^{n+1} - 4T_r^n + T_r^{n-1}}{2\Delta t}, S_r\Big) + b_2(\mathbf{w}_{\mathbf{u},r}^{n+1}, w_{T,r}^{n+1}, S_r)\\
&\quad + \gamma(\nabla w_{T,r}^{n+1}, \nabla S_r) = 0, && (24)
\end{aligned}$$

$$\begin{aligned}
&\Big(\frac{3w_{C,r}^{n+1} - 4C_r^n + C_r^{n-1}}{\Delta t}, \Phi_r\Big) + b_3(\mathbf{w}_{\mathbf{u},r}^{n+1}, w_{C,r}^{n+1}, \Phi_r)\\
&\quad + D_c(\nabla w_{C,r}^{n+1}, \nabla \Phi_r) = 0, && (25)
\end{aligned}$$

for all $(\mathbf{v}_r, S_r, \Phi_r) \in (\mathbf{X}_r, W_r, \Psi_r)$.
Step 2: *Find* $(\mathbf{u}_r^{n+1}, T_r^{n+1}, C_r^{n+1}) \in (\mathbf{X}_r, W_r, \Psi_r)$, *for all* $(\mathbf{v}_r, S_r, \Phi_r) \in (\mathbf{X}_r, W_r, \Psi_r)$:

$$\left(\frac{\mathbf{w}_{\mathbf{u},r}^{n+1} - \mathbf{u}_r^{n+1}}{\Delta t}, \mathbf{v}_r\right) = \left(\alpha_1 \widetilde{P}_{\mathbf{u},R} \nabla \frac{(\mathbf{u}_r^{n+1} + \mathbf{w}_{\mathbf{u},r}^{n+1})}{2}, \widetilde{P}_{\mathbf{u},R} \nabla \mathbf{v}_r\right), \quad (26)$$

$$\left(\frac{\mathbf{w}_{T,r}^{n+1} - T_r^{n+1}}{\Delta t}, S_r\right) = \left(\alpha_2 \widetilde{P}_{T,R} \nabla (\frac{T_r^{n+1} + \mathbf{w}_{T,r}^{n+1}}{2}), \widetilde{P}_{T,R} \nabla S_r\right), \quad (27)$$

$$\left(\frac{\mathbf{w}_{C,r}^{n+1} - C_r^{n+1}}{\Delta t}, \Phi_r\right) = \left(\alpha_3 \widetilde{P}_{C,R} \nabla (\frac{C_r^{n+1} + \mathbf{w}_{C,r}^{n+1}}{2}), \widetilde{P}_{C,R} \nabla \Phi_r\right), \quad (28)$$

where P_R *is the* L^2 *projection into* $\mathbf{X}_R$, *which is the subset of* $\mathbf{X}_r$ *that is the span of the first* R $(< r)$ *velocity modes.*

III. Numerical analysis of double diffusive Darcy-Brinkman system

This section is devoted to a derivation of the priori error estimation of (23)-(28). We first give the stability of solutions of (23)-(28).

Lemma III.1. *(Stability) The Algorithm II.1 is stable for* $\alpha_1 \le 2\nu$, $\alpha_2 \le 8\gamma$, $\alpha_3 \le 8D_c$ *in the following sense: for any* $\Delta t > 0$,

$$\begin{aligned}
&\|\mathbf{u}_r^{M+1}\|^2 + \|2\mathbf{u}_r^{M+1} - \mathbf{u}_r^M\|^2 \\
&\quad + 2\nu\Delta t \|\mathbb{D}\mathbf{w}_{\mathbf{u},r}^{M+1}\|^2 + 2Da^{-1}\Delta t \sum_{n=1}^{M} \|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2 \\
&\quad + 2\alpha_1 \Delta t \left\| \widetilde{P}_{\mathbf{u},R} \nabla \frac{(\mathbf{u}_r^{M+1} + \mathbf{w}_{\mathbf{u},r}^{M+1})}{2} \right\|^2 \\
\le\ & \|\mathbf{u}_1\|^2 + \|2\mathbf{u}_1 - \mathbf{u}_0\|^2 + \frac{\alpha_1 \Delta t}{2} \|\nabla \mathbf{u}_1\|^2 \\
&\quad + C^* \|\mathbf{g}\|_\infty^2 (\beta_T^2 \gamma^{-1} (\|T_1\|^2 + \|2T_1 - T_0\|^2 \\
&\quad + \frac{\alpha_2 \Delta t}{2} \|\nabla T_1\|^2) + \beta_C^2 D_c^{-1} (\|C_1\|^2 \\
&\quad + \|2C_1 - C_0\|^2 + \frac{\alpha_3 \Delta t}{2} \|\nabla C_1\|^2)), \qquad (29)
\end{aligned}$$

$$\begin{aligned}
&\|T_r^{M+1}\|^2 + 4\gamma\Delta t \|\nabla w_{T,r}^{M+1}\|^2 + \|2T_r^{M+1} - T_r^M\|^2 \\
&\quad + 2\alpha_2 \Delta t \|\widetilde{P}_{T,R} \nabla (\frac{T_r^{M+1} + w_{T,r}^{M+1}}{2})\|^2 \\
&\quad \le \|T_1\|^2 + \|2T_1 - T_0\|^2 + \frac{\alpha_2 \Delta t}{2} \|\nabla T_1\|^2, \quad (30)
\end{aligned}$$

$$\begin{aligned}
&\|C_r^{M+1}\|^2 + 4D_c\Delta t \|\nabla w_{C,r}^{M+1}\|^2 + \|2C_r^{M+1} - C_r^M\|^2 \\
&\quad + 2\alpha_3 \Delta t \|\widetilde{P}_{C,R} \nabla (\frac{C_r^{M+1} + w_{C,r}^{M+1}}{2})\|^2 \\
&\quad \le \|C_1\|^2 + \|2C_1 - C_0\|^2 + \frac{\alpha_3 \Delta t}{2} \|\nabla C_1\|^2, (31)
\end{aligned}$$

where $C^* = \min\{\nu^{-1}, Da\}$.

Proof: Letting $S_r = w_{T,r}^{n+1}$ in (24) and using skew symmetry property yields

$$\begin{aligned}
&(\frac{3w_{T,r}^{n+1} - 4T_r^n + T_r^{n-1}}{2\Delta t}, w_{T,r}^{n+1}) \\
&\quad + b_2(\mathbf{w}_{\mathbf{u},r}^{n+1}, w_{T,r}^{n+1}, w_{T,r}^{n+1}) \\
&\quad + \gamma(\nabla w_{T,r}^{n+1}, \nabla w_{T,r}^{n+1}) = 0 \qquad (32)
\end{aligned}$$

Using the skew symmetry property, $b_2(\mathbf{w}_{\mathbf{u},r}^{n+1}, w_{T,r}^{n+1}, w_{T,r}^{n+1}) = 0$ and the identity:

$$\begin{aligned}
\alpha(3\alpha - 4\beta + \theta) = \frac{1}{2}((\alpha^2 - \beta^2) \\
+ (2\alpha - \beta)^2 - (2\beta - \theta)^2 \\
+ (\alpha - 2\beta + \theta)^2), \qquad (33)
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{1}{4\Delta t}\|w_{T,r}^{n+1}\|^2 - \frac{1}{4\Delta t}\|T_r^n\|^2 + \gamma\|\nabla w_{T,r}^{n+1}\|^2 \\
&+ \frac{1}{4\Delta t}(\|2w_{T,r}^{n+1} - T_r^n\|^2 - \|2T_r^n - T_r^{n-1}\|^2) \\
&+ \frac{1}{4\Delta t}\|w_{T,r}^{n+1} - 2T_r^n + T_r^{n-1}\|^2 = 0. \qquad (34)
\end{aligned}$$

Setting $S_r = \dfrac{T_r^{n+1} + w_{T,r}^{n+1}}{2}$ in (27) gives

$$\begin{aligned}
&\|w_{T,r}^{n+1}\|^2 - \|T_r^{n+1}\|^2 = \\
&\quad 2\alpha_2 \Delta t \|\widetilde{P}_{T,R} \nabla (\frac{T_r^{n+1} + w_{T,r}^{n+1}}{2})\|^2. \quad (35)
\end{aligned}$$

Substituting (35) in (34), multiplying with $4\Delta t$ and adding and subtracting $\|2T_r^{n+1} - T_r^n\|^2$ in (34) gives

$$\begin{aligned}
&\|T_r^{n+1}\|^2 - \|T_r^n\|^2 + 4\gamma\Delta t\|\nabla w_{T,r}^{n+1}\|^2 \\
&+ (\|2w_{T,r}^{n+1} - T_r^n\|^2 - \|2T_r^{n+1} - T_r^n\|^2) \\
&+ (\|2T_r^{n+1} - T_r^n\|^2 - \|2T_r^n - T_r^{n-1}\|^2) \\
&+ \|w_{T,r}^{n+1} - 2T_r^n + T_r^{n-1}\|^2 \\
&+ 2\alpha_2 \Delta t \|\widetilde{P}_{T,R} \nabla (\frac{T_r^{n+1} + w_{T,r}^{n+1}}{2})\|^2 = 0. (36)
\end{aligned}$$

Using the properties of L^2 inner product and (35), we rearrange the fourth and fifth terms in the right hand side of (36). See [10] for details of the operations.

$$\begin{aligned}
&\|2w_{T,r}^{n+1} - T_r^n\|^2 - \|2T_r^{n+1} - T_r^n\|^2 \\
&= 8\alpha_2 \Delta t \|\widetilde{P}_{T,R} \nabla \frac{(w_{T,r}^{n+1} + T_r^{n+1} - T_r^n)}{2}\|^2 \\
&+ 8\alpha_2 \Delta t (\widetilde{P}_{T,R} \nabla (\frac{T_r^n}{2}), \widetilde{P}_{T,R} \nabla \frac{(w_{T,r}^{n+1} + T_r^{n+1} - T_r^n)}{2}) \\
&\qquad (37)
\end{aligned}$$

Inserting (37) in (36) and applying the Cauchy-

Schwarz and Young's inequalities gives

$$\begin{aligned}&\|T_r^{n+1}\|^2-\|T_r^n\|^2+4\gamma\Delta t\|\nabla w_{T,r}^{n+1}\|^2\\&+\|2T_r^{n+1}-T_r^n\|^2-\|2T_r^n-T_r^{n-1}\|^2\\&+2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^{n+1}+w_{T,r}^{n+1}}{2})\|^2\\&+\|w_{T,r}^{n+1}-2T_r^n+T_r^{n-1}\|^2\\&\le 2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^n}{2})\|^2. \qquad (38)\end{aligned}$$

Dropping the positive seventh term and rearranging the right hand side term of (38) yields

$$\begin{aligned}&\|T_r^{n+1}\|^2-\|T_r^n\|^2+4\gamma\Delta t\|\nabla w_{T,r}^{n+1}\|^2\\&+\|2T_r^{n+1}-T_r^n\|^2-\|2T_r^n-T_r^{n-1}\|^2\\&+2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^{n+1}+w_{T,r}^{n+1}}{2})\|^2\\&\le 2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^n+w_{T,r}^n}{2})\|^2\\&+\frac{\alpha_2\Delta t}{2}\|\widetilde{P}_{T,R}\nabla w_{T,r}^n\|^2. \qquad (39)\end{aligned}$$

Using $\|\widetilde{P}_{T,R}\|\le 1$ and $\alpha_2\le 8\gamma$, we get

$$\begin{aligned}&\|T_r^{n+1}\|^2-\|T_r^n\|^2+4\gamma\Delta t\|\nabla w_{T,r}^{n+1}\|^2\\&+\|2T_r^{n+1}-T_r^n\|^2-\|2T_r^n-T_r^{n-1}\|^2\\&+2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^{n+1}+w_{T,r}^{n+1}}{2})\|^2\\&\le 2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^n+w_{T,r}^n}{2})\|^2\\&+4\gamma\Delta t\|\nabla w_{T,r}^n\|^2. \qquad (40)\end{aligned}$$

Finally summing over the time step $n=1,\ldots,M$ yields

$$\begin{aligned}&\|T_r^{M+1}\|^2+4\gamma\Delta t\|\nabla w_{T,r}^{M+1}\|^2+\|2T_r^{M+1}-T_r^M\|^2\\&+2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_r^{M+1}+w_{T,r}^{M+1}}{2})\|^2\\&\le\|T_1\|^2+\|2T_1-T_0\|^2+4\gamma\Delta t\|\nabla w_{T,r}^1\|^2\\&+2\alpha_2\Delta t\|\widetilde{P}_{T,R}\nabla(\frac{T_1+w_{T,r}^1}{2})\|^2. \qquad (41)\end{aligned}$$

Using $\|\widetilde{P}_{T,R}\|\le 1$ and assuming $w_{T,r}^1=0$ and we get the stated result (30).

In a similar manner, setting $\Phi_r=w_{C,r}^{n+1}$ in (25) and using the assumption $\alpha_3\le 8D_c$ yields (31). Finally, choosing $\mathbf{v}_r=\mathbf{w}_{\mathbf{u},r}^{n+1}$ in (23), using the polarization identity, and multiplying both sides by $4\Delta t$ yields

$$\begin{aligned}&\|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2-\|\mathbf{u}_r^n\|^2+\|2\mathbf{w}_{\mathbf{u},r}^{n+1}-\mathbf{u}_r^n\|^2\\&\quad-\|2\mathbf{u}_r^n-\mathbf{u}_r^{n-1}\|^2+8\nu\Delta t\|\mathbb{D}\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\&\quad+\|\mathbf{w}_{\mathbf{u},r}^{n+1}-2\mathbf{u}_r^n+\mathbf{u}_r^{n-1}\|^2\\&\quad+4Da^{-1}\Delta t\|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\&=\ 4\Delta t\beta_T(\mathbf{g}w_{T,r}^{n+1},\mathbf{w}_{\mathbf{u},r}^{n+1})\\&\quad+4\Delta t\beta_C(\mathbf{g}w_{C,r}^{n+1},\mathbf{w}_{\mathbf{u},r}^{n+1}). \qquad (42)\end{aligned}$$

Note that if we let $\mathbf{v}_r=\frac{(\mathbf{u}_r^{n+1}+\mathbf{w}_{\mathbf{u},r}^{n+1})}{2}$ in (26), we have

$$\begin{aligned}&\|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2-\|\mathbf{u}_r^{n+1}\|^2=\\&\quad 2\alpha_1\Delta t\|\widetilde{P}_{\mathbf{u},R}\nabla(\frac{\mathbf{u}_r^{n+1}+\mathbf{w}_{\mathbf{u},r}^{n+1}}{2})\|^2. \qquad (43)\end{aligned}$$

Insert (43) in (42), and apply Cauchy-Schwarz, Young's inequality and Poincaré's inequality, which provides

$$\begin{aligned}&\|\mathbf{u}_r^{n+1}\|^2-\|\mathbf{u}_r^n\|^2+\|2\mathbf{w}_{\mathbf{u},r}^{n+1}-\mathbf{u}_r^n\|^2\\&\quad-\|2\mathbf{u}_r^n-\mathbf{u}_r^{n-1}\|^2+4\nu\Delta t\|\mathbb{D}\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\&\quad+\|\mathbf{w}_{\mathbf{u},r}^{n+1}-2\mathbf{u}_r^n+\mathbf{u}_r^{n-1}\|^2\\&\quad+2Da^{-1}\Delta t\|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\&\quad+2\alpha_1\Delta t\left\|\widetilde{P}_{\mathbf{u},R}\nabla(\frac{\mathbf{u}_r^{n+1}+\mathbf{w}_{\mathbf{u},r}^{n+1}}{2})\right\|^2\\&\le\ C^*\|\mathbf{g}\|_\infty^2(\beta_T^2\Delta t\|w_{T,r}^{n+1}\|^2\\&\quad+\beta_C^2\Delta t\|w_{C,r}^{n+1}\|^2) \qquad (44)\end{aligned}$$

where $C^*=\min\{\nu^{-1},Da^{-1}\}$. Using the similar argument with (40) for velocity and utilizing the assumption $\alpha_1\le 2\nu$, we obtain

$$\begin{aligned}&\|\mathbf{u}_r^{n+1}\|^2-\|\mathbf{u}_r^n\|^2+\|2\mathbf{u}_r^{n+1}-\mathbf{u}_r^n\|^2\\&\quad-\|2\mathbf{u}_r^n-\mathbf{u}_r^{n-1}\|^2+4\nu\Delta t\|\mathbb{D}\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\&\quad+2Da^{-1}\Delta t\|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\&\quad+2\alpha_1\Delta t\left\|\widetilde{P}_{\mathbf{u},R}\nabla(\frac{\mathbf{u}_r^{n+1}+\mathbf{w}_{\mathbf{u},r}^{n+1}}{2})\right\|^2\\&\le\ C^*\|\mathbf{g}\|_\infty^2(\beta_T^2\Delta t\|\nabla w_{T,r}^{n+1}\|^2\\&\quad+\beta_C^2\Delta t\|\nabla w_{C,r}^{n+1}\|^2)+4\nu\Delta t\|\mathbb{D}\mathbf{w}_{\mathbf{u},r}^n\|^2\\&\quad+2\alpha_1\Delta t\left\|\widetilde{P}_{\mathbf{u},R}\nabla(\frac{\mathbf{u}_r^n+\mathbf{w}_{\mathbf{u},r}^n}{2})\right\|^2 \qquad (45)\end{aligned}$$

Summing over the time steps and inserting (30) and

(31) yields

$$\begin{aligned}
&\|\mathbf{u}_r^{M+1}\|^2+\|2\mathbf{u}_r^{M+1}-\mathbf{u}_r^M\|^2\\
&\quad +2\nu\Delta t\|\mathbb{D}\mathbf{w}_{\mathbf{u}_T}^{M+1}\|^2+2Da^{-1}\Delta t\sum_{n=1}^{M}\|\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\\
&\quad +2\alpha_1\Delta t\left\|\widetilde{P}_{\mathbf{u},R}\nabla(\frac{\mathbf{u}_r^{M+1}+\mathbf{w}_{\mathbf{u},r}^{M+1}}{2})\right\|^2\\
&\leq \|\mathbf{u}_r^1\|^2+\|2\mathbf{u}_r^1-\mathbf{u}_r^0\|^2+2\nu\Delta t\|\mathbb{D}\mathbf{w}_{\mathbf{u},r}^1\|^2\\
&\quad +2\alpha_1\Delta t\left\|\widetilde{P}_{\mathbf{u},R}\nabla(\frac{\mathbf{u}_r^1+\mathbf{w}_{\mathbf{u},r}^1}{2})\right\|^2\\
&\quad +C^*\|\mathbf{g}\|_\infty^2(\beta_T^2\gamma^{-1}(\|T_1\|^2+\|2T_1-T_0\|^2\\
&\quad +\frac{\alpha_2\Delta t}{2}\|\nabla T_1\|^2)+\beta_C^2D_c^{-1}(\|C_1\|^2\\
&\quad +\|2C_1-C_0\|^2+\frac{\alpha_3\Delta t}{2}\|\nabla C_1\|^2)
\end{aligned} \tag{46}$$

Using the assumption of $\mathbf{w}_{\mathbf{u},r}^1=0$ and $\|\widetilde{P}_{\mathbf{u},R}\|\leq 1$, we get stated result (29). ■

The optimal asymptotic error estimation requires the following regularity assumptions for the true solution:

$$\begin{aligned}
\mathbf{u}&\in L^\infty(0,k;\mathbf{H}^{m+1}(\Omega)),\\
\mathbf{u}_{tt}&\in L^2(0,T;\mathbf{H}^1(\Omega)),\\
T,\ C&\in L^\infty(0,k;H^{m+1}(\Omega)),\\
T_{tt},\ C_{tt}&\in L^2(0,T;H^1(\Omega)),\\
p&\in L^\infty(0,k;H^m(\Omega)).
\end{aligned} \tag{47}$$

We define the discrete norms for $\mathbf{v}^n\in\mathbf{H}^p(\Omega)$, $n=0,1,2,...,M$ as the following:

$$|||\mathbf{v}|||_{\infty,p}:=\max_{0\leq n\leq M}\|\mathbf{v}^n\|_p,$$

$$|||\mathbf{v}|||_{m,p}:=(\Delta t\sum_{n=0}^{M}\|\mathbf{v}^n\|_p^m)^{1/m}.$$

Theorem III.1. *(Error Estimation) Suppose regularity assumptions (47) hold. Then for the sufficiently small Δt, the error satisfies*

$$\begin{aligned}
&\|\mathbf{u}^M-\mathbf{u}_r^M\|^2+\|T^M-T_r^M\|^2+\|C^M-C_r^M\|^2\\
&\leq K\Bigg(1+h^{2m}+(\Delta t)^2+\Big(1+\|S_{u,r}\|_2\\
&\quad +\|S_{T,r}\|_2+\|S_{C,r}\|_2+\|S_{\mathbf{u},R}\|_2\\
&\quad +\|S_{T,R}\|_2+\|S_{C,R}\|_2\Big)h^{2m+2}\\
&\quad +\sum_{i=r_1+1}^{d}(\|\boldsymbol{\psi}_i\|_1^2+1)\lambda_i+\sum_{i=r_2+1}^{d}(\|\phi_i\|_1^2+1)\mu_i\\
&\quad +\sum_{i=r_3+1}^{d}(\|\eta_i\|_1^2+1)\xi_i+\sum_{i=R_1+1}^{d}\|\boldsymbol{\psi}_i\|_1^2\lambda_i\\
&\quad +\sum_{i=R_2+1}^{d}\|\boldsymbol{\phi}_i\|_1^2\mu_i+\sum_{i=R_3+1}^{d}\|\eta_i\|_1^2\xi_i\Bigg).
\end{aligned} \tag{48}$$

Proof: We begin the proof by deriving error equations, subtracting from (2), (3), (4) to (23), (24), (25) at time t^{n+1}, respectively, then we have

$$\begin{aligned}
&\left(\mathbf{u}_t^{n+1}-\frac{3\mathbf{w}_{\mathbf{u},r}^{n+1}-4\mathbf{u}_r^n+\mathbf{u}_r^{n-1}}{2\Delta t},\mathbf{v}_r\right)\\
&+2\nu(\mathbb{D}(\mathbf{u}^{n+1}-\mathbf{w}_{\mathbf{u},r}^{n+1}),\mathbb{D}\mathbf{v}_r)\\
&+b_1(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\mathbf{v}_r)-b_1(\mathbf{w}_{\mathbf{u},r}^{n+1},\mathbf{w}_{\mathbf{u},r}^{n+1},\mathbf{v}_r)\\
&+(Da^{-1}(\mathbf{u}-\mathbf{w}_{\mathbf{u},r}^{n+1}),\mathbf{v}_r)-(p^{n+1},\nabla\cdot\mathbf{v}_r)\\
&=\beta_T(\mathbf{g}(T^{n+1}-w_{T,r}^{n+1}),\mathbf{v}_r)\\
&\quad+\beta_C(\mathbf{g}(C^{n+1}-w_{C,r}^{n+1}),\mathbf{v}_r),
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\left(T_t^{n+1}-\frac{3w_{T,r}^{n+1}-4T_r^n+T_r^{n-1}}{2\Delta t},S_r\right)\\
&+b_2(\mathbf{u}^{n+1},T^{n+1},S_r)-b_2(\mathbf{w}_{u,r}^{n+1},w_{T,r}^{n+1},S_r)\\
&+\gamma(\nabla(T^{n+1}-w_{T,r}^{n+1}),\nabla S_r)=0,
\end{aligned} \tag{50}$$

$$\begin{aligned}
&\left(C_t^{n+1}-\frac{3w_{C,r}^{n+1}-4C_r^n+C_r^{n-1}}{\Delta t},\Phi_r\right)\\
&+b_3(\mathbf{u}^{n+1},C^{n+1},\Phi_r)-b_3(\mathbf{w}_{\mathbf{u},r}^{n+1}w_{C,r}^{n+1},\Phi_r)\\
&+D_c(\nabla(C^{n+1}-w_{C,r}^{n+1}),\nabla\Phi_r)=0.
\end{aligned} \tag{51}$$

The notations that are used in the proof are defined as

$$\begin{array}{llllll}
\boldsymbol{\eta}_{\mathbf{u}}^n & := & \mathbf{u}^n-\tilde{\mathbf{U}}^n, & \boldsymbol{\phi}_{\mathbf{u},r}^n & := & \mathbf{w}_{\mathbf{u},r}^n-\tilde{\mathbf{U}}^n,\\
\boldsymbol{\theta}_{\mathbf{u},r}^n & := & \mathbf{u}_r^n-\tilde{\mathbf{U}}^n, & \mathbf{E}_{\mathbf{u},r}^n & := & \mathbf{u}^n-\mathbf{w}_{\mathbf{u},r}^n\\
\mathbf{e}_{\mathbf{u},r}^n & = & \mathbf{u}^n-\mathbf{u}_r^n, & & & \\
\eta_T^n & := & T^n-\tilde{T}^n, & \phi_{T,r}^n & := & w_{T,r}^n-\tilde{T}^n,\\
\theta_{T,r}^n & := & T_r^n-\tilde{T}^n, & E_{T,r}^n & := & T^n-w_{T,r}^n\\
e_{T,r}^n & = & T^n-T_r^n, & & & \\
\eta_C^n & := & C^n-\tilde{C}^n, & \phi_{C,r}^n & := & w_{C,r}^n-\tilde{C}^n,\\
\theta_{C,r}^n & := & C_r^n-\tilde{C}^n, & E_{C,r}^n & := & C^n-w_{C,r}^n\\
e_{C,r}^n & = & C^n-C_r^n, & & &
\end{array}$$

where $(\tilde{\mathbf{U}}^{\mathbf{n}},\tilde{T}^n,\tilde{C}^n)$ are L^2 projections of $(\mathbf{u}^n,T^n,C^n)$ in $(\mathbf{X}_r,W_r,\Psi_r)$ at time t^n.

Letting $S_r=\phi_{T,r}^{n+1}$ in (50), and reorganizing it, we get

$$\begin{aligned}
&\left(\frac{3E_{T,r}^{n+1}-4e_{T,r}^n+e_{T,r}^{n-1}}{2\Delta t},\phi_{T,r}^{n+1}\right)\\
&+\gamma(\nabla E_{T,r}^{n+1},\nabla\phi_{T,r}^{n+1})+b_2(\mathbf{u}^{n+1},T^{n+1},\phi_{T,r}^{n+1})\\
&-b_2(\mathbf{w}_{u,r}^{n+1},w_{T,r}^{n+1},\phi_{T,r}^{n+1})\\
&+\left(T_t^{n+1}-\frac{3T^{n+1}-4T^n+T^{n-1}}{2\Delta t},\phi_{T,r}^{n+1}\right)=0.
\end{aligned} \tag{52}$$

Utilizing $E_{T,r}^{n+1}=\eta_T^{n+1}-\phi_{T,r}^{n+1}$, $e_{T,r}^n=\eta_T^n-\theta_{T,r}^n$,

(16) and (33), we get

$$\frac{1}{4\Delta t}\|\phi_{T,r}^{n+1}\|^2-\frac{1}{4\Delta t}\|\theta_{T,r}^{n}\|^2 + \frac{1}{4\Delta t}(\|2\phi_{T,r}^{n+1}-\theta_{T,r}^{n}\|^2-\|2\theta_{T,r}^{n}-\theta_{T,r}^{n-1}\|^2) + \frac{1}{4\Delta t}\|\phi_{T,r}^{n+1}-2\theta_{T,r}^{n}+\theta_{T,r}^{n-1}\|^2+\gamma\|\nabla\phi_{T,r}^{n+1}\|^2 = \gamma(\nabla\eta_T^{n+1},\nabla\phi_{T,r}^{n+1}) + b_2(\mathbf{w}_{u,r}^{n+1}, w_{T,r}^{n+1},\phi_{T,r}^{n+1}) - b_2(\mathbf{u}^{n+1},T^{n+1},\phi_{T,r}^{n+1}) - \left(T_t^{n+1}-\frac{3T^{n+1}-4T^n+T^{n-1}}{2\Delta t},\phi_{T,r}^{n+1}\right). \tag{53}$$

Adding and subtracting $\tilde{T}$ in (27) on both sides to get

$$\left(\frac{\phi_{T,r}^{n+1}-\theta_{T,r}^{n+1}}{\Delta t},S_r\right) = \left(\alpha_2\widetilde{P}_{T,R}\nabla(\frac{\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1}+2\tilde{T}}{2}),\widetilde{P}_{T,R}\nabla S_r\right). \tag{54}$$

Setting $S_r=\frac{\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1}}{2}$ in (54) produces

$$\|\phi_{T,r}^{n+1}\|^2= \|\theta_{T,r}^{n+1}\|^2 + \frac{\alpha_2\Delta t}{2}\|\widetilde{P}_{T,R}\nabla(\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1})\|^2 + \Delta t(\alpha_2\widetilde{P}_{T,R}\nabla\tilde{T}^{n+1},\widetilde{P}_{T,R}\nabla(\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1})). \tag{55}$$

In a similar setting, for concentration we have

$$\|\phi_{C,r}^{n+1}\|^2= \|\theta_{C,r}^{n+1}\|^2 + \frac{\alpha_3\Delta t}{2}\|\widetilde{P}_{C,R}\nabla(\phi_{C,r}^{n+1}+\theta_{C,r}^{n+1})\|^2 + \Delta t(\alpha_3\widetilde{P}_{C,R}\nabla\tilde{C}^{n+1},\widetilde{P}_{C,R}\nabla(\phi_{C,r}^{n+1}+\theta_{C,r}^{n+1})). \tag{56}$$

Rewriting the nonlinear terms, we have

$$b_2(\mathbf{w}_{u,r}^{n+1},w_{T,r}^{n+1},\phi_{T,r}^{n+1})-b_2(\mathbf{u}^{n+1},T^{n+1},\phi_{T,r}^{n+1}) = -b_2(\boldsymbol{\eta}_{\mathbf{u}}^{n+1},T^{n+1},\phi_{T,r}^{n+1})+b_2(\boldsymbol{\phi}_{\mathbf{u},r}^{n+1},T^{n+1},\phi_{T,r}^{n+1}) - b_2(\mathbf{w}_{\mathbf{u},r}^{n+1},\eta_T^{n+1},\phi_{T,r}^{n+1}). \tag{57}$$

After substituting (57) in (53) and multiplying $4\Delta t$, we bound the right hand side of (53) as follows

$$|\gamma(\nabla\eta_T^{n+1},\nabla\phi_{T,r}^{n+1})| \leq C\gamma\|\nabla\eta_T^{n+1}\|^2+\frac{\gamma}{10}\|\nabla\phi_{T,r}^{n+1}\|^2,$$

$$|b_2(\boldsymbol{\eta}_{\mathbf{u}}^{n+1},T^{n+1},\phi_{T,r}^{n+1})| \leq \frac{C}{\gamma}\|\nabla\boldsymbol{\eta}_{\mathbf{u}}^{n+1}\|^2\|\nabla T^{n+1}\|^2+\frac{\gamma}{10}\|\nabla\phi_{T,r}^{n+1}\|^2,$$

$$|b_2(\boldsymbol{\phi}_{\mathbf{u},r}^{n+1},T^{n+1},\phi_{T,r}^{n+1})| \leq \frac{C}{\nu\gamma^2}\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2\|\nabla T^{n+1}\|^4+\frac{\gamma}{10}\|\nabla\phi_{T,r}^{n+1}\|^2+\nu\|\mathbb{D}\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2,$$

$$|b_2(\mathbf{w}_{\mathbf{u},r}^{n+1},\eta_T^{n+1},\phi_{T,r}^{n+1})| \leq \frac{C}{\gamma}\|\nabla\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\|\nabla\eta_T^{n+1}\|^2+\frac{\gamma}{10}\|\nabla\phi_{T,r}^{n+1}\|^2,$$

$$|(\alpha_2\widetilde{P}_{T,R}\nabla\tilde{T}^{n+1},\widetilde{P}_{T,R}\nabla(\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1}))| \leq \alpha_2\|\widetilde{P}_{T,R}\nabla\tilde{T}^{n+1}\|^2+\frac{\alpha_2}{4}\|\widetilde{P}_{T,R}\nabla(\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1})\|^2. \tag{58}$$

For the last term in the right hand side of (53), using Taylor series expansion with the remainder in integral form with Cauchy Schwarz and the triangle inequality, we obtain

$$|(T_t^{n+1}-\frac{3T^{n+1}-4T^n+T^{n-1}}{2\Delta t},\phi_{T,r}^{n+1})| \leq C\gamma^{-1}\Delta t\|T_{tt}\|^2_{L^2(0,\tau;H^1(\Omega))}+\frac{\gamma}{10}\|\nabla\phi_{T,r}^{n+1}\|^2. \tag{59}$$

Inserting all bounds in (53), using $\tilde{T}^{n+1}=T^{n+1}-\eta_T^{n+1}$ and summing over the time steps produces

$$\|\theta_{T,r}^{M+1}\|^2+\|2\theta_{T,r}^{M+1}-\theta_{T,r}^{M}\|^2 + \frac{\alpha_2\Delta t}{4}\sum_{n=1}^{M}\|\widetilde{P}_{T,R}\nabla(\phi_{T,r}^{n+1}+\theta_{T,r}^{n+1})\|^2 + 2\gamma\Delta t\sum_{n=1}^{M}\|\nabla\phi_{T,r}^{n+1}\|^2 \leq \|\theta_{T,r}^{0}\|^2+\|2\theta_{T,r}^{1}-\theta_{T,r}^{0}\|^2 + K\Delta t\Big(\gamma\sum_{n=1}^{M}\|\nabla\eta_T{}^{n+1}\|^2 + \alpha_2\|\widetilde{P}_{T,R}\nabla(T^{n+1}-\eta_T^{n+1})\|^2 + \gamma^{-1}\sum_{n=1}^{M}\|\nabla\boldsymbol{\eta}_u{}^{n+1}\|^2\|\nabla T^{n+1}\|^2 + \nu^{-1}\gamma^{-2}\sum_{n=1}^{M}\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2\|\nabla T^{n+1}\|^4 + \gamma^{-1}\sum_{n=1}^{M}\|\nabla\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\|\nabla\eta_T{}^{n+1}\|^2 + \nu\sum_{n=1}^{M}\|\mathbb{D}\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2 + \gamma^{-1}\Delta t\|T_{tt}\|^2_{L^2(0,\tau;H^1(\Omega))}\Big). \tag{60}$$

By using (30), (47) and Lemma II.1 in (60) results in

$$\|\theta_{T,r}^{M+1}\|^2+2\gamma\Delta t\sum_{n=1}^{M}\|\nabla\phi_{T,r}^{n+1}\|^2\leq \|\theta_{T,r}^{0}\|^2 +K\Big(h^{2m}+(\|S_{u,r}\|_2+\|S_{T,r}\|_2 +\|S_{T,R}\|_2)h^{2m+2}+\varepsilon_{\mathbf{u},r}^2+\varepsilon_{T,r}^2+\varepsilon_{T,R}^2 +\nu^{-1}\gamma^{-2}|||\nabla T|||_{\infty,0}^4\Delta t\sum_{n=1}^{M}\|\boldsymbol{\phi}_{u,r}^{n+1}\|^2 +\nu\Delta t\sum_{n=1}^{M}\|\mathbb{D}\boldsymbol{\phi}_{u,r}^{n+1}\|^2+(\Delta t)^2\Big), \quad (61)$$

where

$$\varepsilon_{T,r}=(\sum_{i=r_2+1}^{d}\|\phi_i\|_1^2\mu_i)^{\frac{1}{2}},\ \varepsilon_{T,R}=(\sum_{i=R_2+1}^{d}\|\phi_i\|_1^2\mu_i)^{\frac{1}{2}},$$

$$\varepsilon_{\mathbf{u},r}=(\sum_{i=r_1+1}^{d}\|\boldsymbol{\psi}_i\|_1^2\lambda_i)^{\frac{1}{2}}.$$

Similarly, the error estimation for the concentration is given by

$$\|\theta_{C,r}^{M+1}\|^2+2D_c\Delta t\sum_{n=1}^{M}\|\nabla\phi_{C,r}^{n+1}\|^2\leq \|\theta_{C,r}^{0}\|^2 +K\Big(h^{2m}+(\|S_{u,r}\|_2+\|S_{C,r}\|_2 +\|S_{C,R}\|_2)h^{2m+2}+\varepsilon_{\mathbf{u},r}^2+\varepsilon_{C,r}^2+\varepsilon_{C,R}^2 +\nu^{-1}{D_c}^{-2}|||\nabla C|||_{\infty,0}^4\Delta t\sum_{n=1}^{M}\|\boldsymbol{\phi}_{u,r}^{n+1}\|^2 +\nu\Delta t\sum_{n=1}^{M}\|\mathbb{D}\boldsymbol{\phi}_{u,r}^{n+1}\|^2+(\Delta t)^2\Big), \quad (62)$$

where

$$\varepsilon_{C,r}=(\sum_{i=r_3+1}^{d}\|\eta_i\|_1^2\xi_i)^{\frac{1}{2}},\ \varepsilon_{C,R}=(\sum_{i=R_3+1}^{d}\|\eta_i\|_1^2\xi_i)^{\frac{1}{2}}.$$

In a similar manner, setting $\mathbf{v}_r=\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}$ in the (49), we have

$$\Big(\frac{3\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}-4\boldsymbol{\theta}_{\mathbf{u},r}^{n}+\boldsymbol{\theta}_{\mathbf{u},r}^{n-1}}{2\Delta t},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\Big) +2\nu\|\mathbb{D}\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2+Da^{-1}\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2 = \Big(\frac{3\boldsymbol{\eta}_{\mathbf{u}}^{n+1}-4\boldsymbol{\eta}_{\mathbf{u}}^{n}+\boldsymbol{\eta}_{\mathbf{u}}^{n-1}}{2\Delta t},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\Big) +2\nu(\mathbb{D}\boldsymbol{\eta}_{\mathbf{u}}^{n+1},\mathbb{D}\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}) +b(\mathbf{u}^{n+1},\mathbf{u}^{n+1},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}) -b(\mathbf{w}_{\mathbf{u},r}^{n+1},\mathbf{w}_{\mathbf{u},r}^{n+1},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}) +(Da^{-1}\boldsymbol{\eta}_{\mathbf{u}}^{n+1},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1})-(p^{n+1},\nabla\cdot\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}) +\Big(\mathbf{u}_t^{n+1}-\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\Delta t},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\Big) +\beta_T(\mathbf{g}(T^{n+1}-w_{T,r}^{n+1}),\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}) +\beta_C(\mathbf{g}(C^{n+1}-w_{C,r}^{n+1}),\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}). \quad (63)$$

Note that from (16), we get

$$(\boldsymbol{\eta}_{\mathbf{u}}^{n-1},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1})=(\boldsymbol{\eta}_{\mathbf{u}}^{n},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1})=(\boldsymbol{\eta}_{\mathbf{u}}^{n+1},\boldsymbol{\phi}_{\mathbf{u},r}^{n+1})=0.$$

Using (33) and inserting all bounds for the right hand side terms of (63) and multiplying both sides by $4\Delta t$ gives

$$\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2-\|\boldsymbol{\theta}_{\mathbf{u},r}^{n}\|^2+\|2\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}-\boldsymbol{\theta}_{\mathbf{u},r}^{n}\|^2 -\|2\boldsymbol{\theta}_{\mathbf{u},r}^{n}-\boldsymbol{\theta}_{\mathbf{u},r}^{n-1}\|^2+\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}-2\boldsymbol{\theta}_{\mathbf{u},r}^{n}+\boldsymbol{\theta}_{\mathbf{u},r}^{n-1}\|^2 +4\nu\Delta t\|\mathbb{D}\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2+2Da^{-1}\Delta t\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2 \leq K\Big(\nu\Delta t\|\nabla\boldsymbol{\eta}_{\mathbf{u}}^{n+1}\|^2+Da^{-1}\Delta t\|\boldsymbol{\eta}_{\mathbf{u}}^{n+1}\|^2 +\nu^{-1}\Delta t\|\nabla\boldsymbol{\eta}_{\mathbf{u}}^{n+1}\|^2\|\nabla\mathbf{u}^{n+1}\|^2 +\frac{\Delta t}{\nu^3}\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2\|\nabla\mathbf{u}^{n+1}\|^4 +\nu^{-1}\Delta t\|p^{n+1}-q_h\|^2 +\nu^{-1}\Delta t\|\nabla\mathbf{w}_{\mathbf{u},r}^{n+1}\|^2\|\nabla\boldsymbol{\eta}_{\mathbf{u}}^{n+1}\|^2 +\nu^{-1}\Delta t\|\mathbf{u}_t^{n+1}-\frac{3\mathbf{u}^{n+1}-4\mathbf{u}^{n}+\mathbf{u}^{n-1}}{2\Delta t}\|^2 +\nu^{-1}\Delta t\beta_T^2\|\mathbf{g}\|_\infty^2(\|\eta_T^{n+1}\|^2+\|\phi_{T,r}^{n+1}\|^2) +\nu^{-1}\Delta t\beta_C^2\|\mathbf{g}\|_\infty^2(\|\eta_C^{n+1}\|^2+\|\phi_{C,r}^{n+1}\|^2)\Big). \quad (64)$$

To get a bound for $\|\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}\|^2$, write (26) by adding and subtracting the true solution projection $\widetilde{\mathbf{U}}^{n+1}$ on both sides, then one obtains

$$(\frac{\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}-\boldsymbol{\theta}_{\mathbf{u},r}^{n+1}}{\Delta t},\boldsymbol{\psi})= (\alpha_1\widetilde{P}_{\mathbf{u},R}\nabla\frac{(\boldsymbol{\phi}_{\mathbf{u},r}^{n+1}+\boldsymbol{\theta}_{\mathbf{u},r}^{n+1}+2\mathcal{U}^{n+1})}{2},\widetilde{P}_{\mathbf{u},R}\nabla\boldsymbol{\psi}). \quad (65)$$

Choosing $\psi = \dfrac{(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})}{2}$ in (65), we get

$$\begin{aligned}\|\phi^{n+1}_{\mathbf{u},r}\|^2 &= \|\theta^{n+1}_{\mathbf{u},r}\|^2 \\ &+ \frac{1}{2}\Delta t\alpha_1 \|\widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})\|^2 \\ &+ \Delta t(\alpha_1 \widetilde{P}_{\mathbf{u},R}\nabla\mathcal{U}^{n+1}, \widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})).\end{aligned} \quad (66)$$

Noting $\mathcal{U}^{n+1} = \mathbf{u}^{n+1} - \eta^{n+1}_{\mathbf{u}}$ and inserting (66) into (64) results into

$$\begin{aligned}&\|\theta^{n+1}_{\mathbf{u},r}\|^2 - \|\theta^{n}_{\mathbf{u},r}\|^2 + \frac{1}{2}\Delta t\alpha_1\|\widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})\|^2 \\ &\quad + 4\nu\Delta t\|\mathbb{D}\phi^{n+1}_{\mathbf{u},r}\|^2 + 2Da^{-1}\Delta t\|\phi^{n+1}_{\mathbf{u},r}\|^2 \\ \le\ & K\Big(\nu\Delta t\|\nabla\eta^{n+1}_{\mathbf{u}}\|^2 + \frac{\Delta t}{\nu}\|\nabla\eta^{n+1}_{\mathbf{u}}\|^2\|\nabla\mathbf{u}^{n+1}\|^2 \\ &\quad + \frac{\Delta t}{\nu^3}\|\nabla\mathbf{u}^{n+1}\|^4\Big[\|\theta^{n+1}_{\mathbf{u},r}\|^2 \\ &\quad + \frac{1}{2}\Delta t\alpha_1\|\widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})\|^2 \\ &\quad + \Delta t(\alpha_1\widetilde{P}_{\mathbf{u},R}\nabla(\mathbf{u}^{n+1} - \eta^{n+1}_{\mathbf{u}}), \widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r}))\Big] \\ &\quad + \Delta t(\alpha_1\widetilde{P}_{\mathbf{u},R}\nabla(\eta^{n+1}_{\mathbf{u}} - \mathbf{u}^{n+1}), \widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})) \\ &\quad + \frac{\Delta t}{\nu}\|\nabla\mathbf{w}^{n+1}_{\mathbf{u},r}\|^2\|\nabla\eta^{n+1}_{\mathbf{u}}\|^2 + \frac{\Delta t}{\nu}\|p^{n+1} - q_h\|^2 \\ &\quad + \frac{\Delta t}{\nu}\|\mathbf{u}^{n+1}_t - \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t}\|^2 \\ &\quad + \nu^{-1}\Delta t\beta_T^2\|\mathbf{g}\|^2_\infty(\|\eta^{n+1}_T\|^2 + \|\phi^{n+1}_{T,r}\|^2) \\ &\quad + \nu^{-1}\Delta t\beta_C^2\|\mathbf{g}\|^2_\infty(\|\eta^{n+1}_C\|^2 + \|\phi^{n+1}_{C,r}\|^2)\Big).\end{aligned} \quad (67)$$

Applying Cauchy-Schwarz and Young's inequalities for fifth and sixth term of the right hand side of (67), inserting (55) and (56) in (67) and using $\Delta t \le \dfrac{\nu^3}{8C\|\nabla\mathbf{u}\|^4}$, we obtain

$$\begin{aligned}&\|\theta^{n+1}_{\mathbf{u},r}\|^2 + \frac{1}{8}\alpha_1\Delta t\|\widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})\|^2 \\ &\quad + 4\nu\Delta t\|\mathbb{D}\phi^{n+1}_{\mathbf{u},r}\|^2 + 2Da^{-1}\|\phi^{n+1}_{\mathbf{u},r}\|^2 \\ \le\ & \|\theta^{n}_{\mathbf{u},r}\|^2 + K\Big[\nu\Delta t\|\nabla\eta^{n+1}_{\mathbf{u}}\|^2 \\ &\quad + \frac{\Delta t}{\nu}\|\nabla\mathbf{u}^{n+1}\|^2\|\nabla\eta^{n+1}_{\mathbf{u}}\|^2 \\ &\quad + \Delta t\alpha_1\|\widetilde{P}_{\mathbf{u},R}\nabla(\mathbf{u}^{n+1} - \eta^{n+1}_{\mathbf{u}})\|^2 \\ &\quad + \frac{\Delta t}{\nu}\|\nabla\mathbf{w}^{n+1}_{\mathbf{u},r}\|^2\|\nabla\eta^{n+1}_{\mathbf{u}}\|^2 \\ &\quad + \frac{\Delta t}{\nu}\|p^{n+1} - q_h\|^2 \\ &\quad + \frac{\Delta t}{\nu}\|\mathbf{u}^{n+1}_t - \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t}\|^2 \\ &\quad + \nu^{-1}\Delta t\beta_T^2\|\mathbf{g}\|^2_\infty\Big(\|\eta^{n+1}_T\|^2 + \|\theta^{n+1}_{T,r}\|^2 \\ &\quad + \frac{\alpha_2\Delta t}{2}\|\widetilde{P}_{T,R}\nabla(\phi^{n+1}_{T,r} + \theta^{n+1}_{T,r})\|^2 \\ &\quad + \alpha_2\Delta t(\widetilde{P}_{T,R}\nabla\tilde{T}^{n+1}, \widetilde{P}_{T,R}\nabla(\phi^{n+1}_{T,r} + \theta^{n+1}_{T,r}))\Big) \\ &\quad + \nu^{-1}\beta_C^2\|\mathbf{g}\|^2_\infty\Big(\|\eta^{n+1}_C\|^2 + \|\theta^{n+1}_{C,r}\|^2 \\ &\quad + \frac{\alpha_3\Delta t}{2}\|\widetilde{P}_{C,R}\nabla(\phi^{n+1}_{C,r} + \theta^{n+1}_{C,r})\|^2 \\ &\quad + \alpha_3\Delta t(\widetilde{P}_{C,R}\nabla\tilde{C}^{n+1}, \widetilde{P}_{C,R}\nabla(\phi^{n+1}_{C,r} + \theta^{n+1}_{C,r}))\Big)\Big] \\ &\quad + \frac{K\Delta t}{\nu^3}\sum_{n=0}^{M-1}\|\nabla\mathbf{u}^{n+1}\|^4\|\theta^{n+1}_{\mathbf{u},r}\|^2\end{aligned} \quad (68)$$

Summing from $n = 0$ to $M-1$, using (29), Lemma II.1 in (68), and applying the regularity assumptions (47) leads to

$$\begin{aligned}&\|\theta^{M}_{\mathbf{u},r}\|^2 + \sum_{n=0}^{M-1}\Big[\frac{1}{8}\Delta t\alpha_1\|\widetilde{P}_{\mathbf{u},R}\nabla(\phi^{n+1}_{\mathbf{u},r} + \theta^{n+1}_{\mathbf{u},r})\|^2 \\ &\quad + 4\nu\Delta t\|\mathbb{D}\phi^{n+1}_{\mathbf{u},r}\|^2 + 2Da^{-1}\|\phi^{n+1}_{\mathbf{u},r}\|^2\Big] \\ &\le \theta^0_{\mathbf{u},r} + K\Big(h^{2m} + (\Delta t)^2 + h^{2m+2}(\|S_{u,r}\|_2 \\ &\quad + \|S_{u,R}\|_2) + \varepsilon_{\mathbf{u},r} + \varepsilon_{\mathbf{u},R} + \sum_{i=r_2+1}^{d}\mu_i + \sum_{i=r_3+1}^{d}\xi_i\Big) \\ &\quad + K\nu^{-1}\beta_T^2\|\mathbf{g}\|^2_\infty\Delta t\sum_{n=0}^{M-1}\|\theta^{n+1}_{T,r}\|^2 \\ &\quad + K\nu^{-1}\beta_C^2\|\mathbf{g}\|^2_\infty\Delta t\sum_{n=0}^{M-1}\|\theta^{n+1}_{C,r}\|^2 \\ &\quad + \frac{K\Delta t}{\nu^3}\sum_{n=0}^{M-1}|||\nabla\mathbf{u}|||^4_{\infty,0}\|\theta^{n+1}_{\mathbf{u},r}\|^2.\end{aligned} \quad (69)$$

Adding (61) and (62) to (69), one gets

$$\begin{aligned}&\|\theta^{M}_{u,r}\|^2 + \|\theta^{M}_{T,r}\|^2 + \|\theta^{M}_{C,r}\|^2 + 2\Delta t\sum_{n=0}^{M-1}\Big(\nu\|\mathbb{D}\phi^{n+1}_{u,r}\|^2 \\ &\quad + Da^{-1}\|\phi^{n+1}_{u,r}\|^2 + \gamma\|\nabla\phi^{n+1}_{T,r}\|^2 + D_c\|\nabla\phi^{n+1}_{C,r}\|^2\Big) \\ &\le \|\mathbf{u}^0_r - \tilde{\mathbf{u}}^0\|^2 + \|T^0_r - \tilde{T}^0\|^2 + \|C^0_r - \tilde{C}^0\|^2 \\ &\quad K\Big(h^{2m} + (\Delta t)^2 + h^{2m+2}(1 + \|S_{u,r}\|_2 + \|S_{T,r}\|_2 \\ &\quad + \|S_{C,r}\|_2 + \|S_{\mathbf{u},R}\|_2 + \|S_{T,R}\|_2 + \|S_{C,R}\|_2) \\ &\quad + \sum_{i=r_2+1}^{d}\mu_i + \sum_{i=r_3+1}^{d}\xi_i + \varepsilon^2_{\mathbf{u},r} \\ &\quad + \varepsilon^2_{T,r} + \varepsilon^2_{C,r} + \varepsilon^2_{\mathbf{u},R} + \varepsilon^2_{T,R} + \varepsilon^2_{C,R}\Big) \qquad (70) \\ &\quad + K\nu^{-1}\beta_T^2\|\mathbf{g}\|^2_\infty\Delta t\sum_{n=0}^{M-1}\|\theta^{n+1}_{T,r}\|^2 \\ &\quad + K\nu^{-1}\beta_C^2\|\mathbf{g}\|^2_\infty\Delta t\sum_{n=0}^{M-1}\|\theta^{n+1}_{C,r}\|^2 \\ &\quad + K\Big(\nu^{-3}|||\nabla\mathbf{u}|||^4_{\infty,0} + \nu^{-1}\gamma^{-2}|||\nabla T|||^4_{\infty,0} \\ &\quad + \nu^{-1}D_c^{-2}|||\nabla C|||^4_{\infty,0}\Big)\Delta t\sum_{n=0}^{M-1}\|\theta^{n+1}_{\mathbf{u},r}\|^2.\end{aligned} \quad (71)$$

Applying Gronwall inequality for sufficiently small time step,

$$\Delta t \leq \min\{(K\nu^{-1}(\nu^{-2}+\gamma^{-2}+D_c{}^{-2}))^{-1}, (K\nu^{-1}\beta_T^2)^{-1}, (K\nu^{-1}\beta_C^2)^{-1}\} \quad (72)$$

we have

$$\|\boldsymbol{\theta}_{u,r}^M\|^2+\|\theta_{T,r}^M\|^2+\|\theta_{C,r}^M\|^2+2\Delta t\sum_{n=0}^{M-1}\Big(\nu\|\mathbb{D}\boldsymbol{\phi}_{u,r}^{n+1}\|^2 +Da^{-1}\|\boldsymbol{\phi}_{u,r}^{n+1}\|^2+\gamma\|\nabla\phi_{T,r}^{n+1}\|^2+D_c\|\nabla\phi_{C,r}^{n+1}\|^2\Big)$$
$$\leq \|\mathbf{u}_r^0-\tilde{\mathbf{u}}^0\|^2+\|T_r^0-\tilde{T}^0\|^2+\|C_r^0-\tilde{C}^0\|^2$$
$$K\Big(h^{2m}+(\Delta t)^2+h^{2m+2}(1+\|S_{u,r}\|_2+\|S_{T,r}\|_2 +\|S_{C,r}\|_2+\|S_{\mathbf{u},R}\|_2+\|S_{T,R}\|_2+\|S_{C,R}\|_2)$$
$$+\sum_{i=r_2+1}^{d}\mu_i+\sum_{i=r_3+1}^{d}\xi_i+\varepsilon_{\mathbf{u},r}^2+\varepsilon_{T,r}^2+\varepsilon_{C,r}^2 +\varepsilon_{\mathbf{u},R}^2+\varepsilon_{T,R}^2+\varepsilon_{C,R}^2\Big),$$

where

$$\varepsilon_{\mathbf{u},R} = (\sum_{i=R_1+1}^{d}\|\boldsymbol{\psi}_i\|_1^2\lambda_i)^{1/2}.$$

Applying triangle inequality yields the stated result. ■

IV. Numerical Studies

In this section, we present results of numerical tests using the VMS-POD studied above. We consider a test problem from [13], [14]. The space domain is rectangular box $[0,1]\times[0,2]$, the time domain is from 0 to 1. The boundary conditions are given as

$$\mathbf{u}=\mathbf{0} \text{ on } \partial\Omega,\ T=C=0, \text{ for } x=0,$$
$$T=C=1 \text{ for } x=1.$$
$$\nabla T\cdot n=\nabla C\cdot n=0 \text{ for } y=0, y=2,$$

and initial conditions are taken as $\mathbf{u}_0=T_0=C_0=0$. We fix $Pr=1, Le=2, N=0.8, \nu=1$. No porosity case is considered, i.e., $Da=\infty$. The VMS cut off numbers are chosen $R_1=R_2=R_3=R$. The fine mesh solution was calculated by using BDF2 finite element scheme with Taylor-Hood velocity-pressure elements, and continuous quadratic elements as in [12].

A. Test 1: Convergence rates wrt R

In VMS-POD scheme, the basis truncations dominate the error sources. When the temporal and spatial error neglected, POD cut off r and VMS cut off R become dominant. Since our special interest is to measure the effect of VMS method, we test the method with scaling R. We fix $Ra=10^4$, $\Delta t=0.000015625$, $T=0.01$. Errors and convergence rates with respect to R are given in Table I. We observe that the rates approximate 1 or higher expected by analysis.

TABLE I: Convergence of the VMS-POD for varying R.

r	R	$\varepsilon_{\mathbf{u},R}$	$\|\mathbf{u}^M-\mathbf{u}_r^M\|$	rate
12	2	87.9396	2.79821	-
12	4	21.8237	0.44505	1.32
12	6	8.8818	0.22490	0.76
r	R	$\varepsilon_{T,R}$	$\|T^M-T_r^M\|$	rate
12	2	3.1932	0.11978	-
12	4	1.5694	0.01749	2.71
12	6	0.4319	0.00772	0.63
r	R	$\varepsilon_{C,R}$	$\|C^M-C_r^M\|$	rate
12	2	4.6149	0.13724	-
12	4	1.5529	0.03620	1.22
12	6	0.6768	0.00682	2.01

B. Test 2: Efficiency of VMS-POD

In this test, we compare the process time of full order system and process time of reduced order system.

TABLE II: Process times (in seconds) for DNS, POD, and efficiency for different Ra

Ra	DNS	VMS-POD	Efficiency
10^4	1186.973710	52.844179	22.46
10^5	1285.602149	53.234042	24.15
10^6	966.318515	199.2651460	4.85

For $Ra=10^4$ and $Ra=10^5$, the VMS-POD method is remarkably faster than DNS solution. The efficiency of POD is slightly reduced for $Ra=10^6$. However, POD reduces the processing time for each case.

C. Test 3: Accuracy of the method

In this test, we check the accuracy of the method for $Ra=10^6$. We choose $\Delta t=0.00025$, $R=15$, $\alpha_1=2$, $\alpha_2=\alpha_3=\frac{1}{4}$. The velocity, temperature and concentration solutions for the simulations using DNS, POD and VMS-POD using 40 modes at $t=0.8$ are shown in the Figure 1, Figure 2, Figure 3.

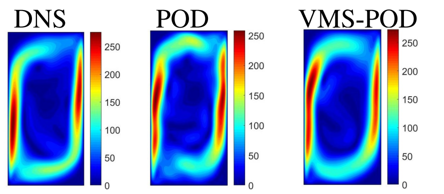

Fig. 1: Speed solution plots for the simulations using DNS, POD and VMS-POD.

We observe that POD causes numerical instability for high Ra. However, combining with the VMS method provides better results.

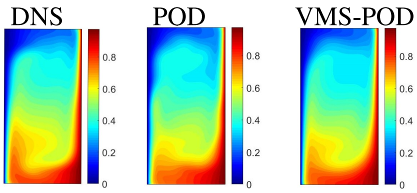

Fig. 2: Temperature solution plots for the simulations using DNS, POD and VMS-POD.

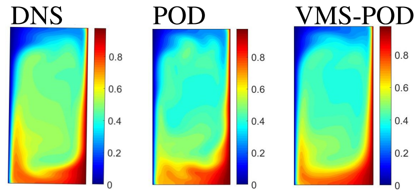

Fig. 3: Concentration solution plots for the simulations using DNS, POD and VMS-POD.

V. Conclusions

We proposed a modular regularization with the VMS-POD method for double diffusive system. In this approach, the stabilization is added for each fluid variables. We proved the stability and convergence results for the VMS-POD scheme, and gave results of several numerical tests. For higher Ra, our tests showed, POD did not perform well without stabilization, but adding VMS-type stabilization, gave good qualitative results.

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