Sufficient Conditions for The Existence of Fractional Factors in Different Settings

Wei Gao, *Member, IAENG,* and Ce Shi

***Abstract*—The fractional factor theory of graphs originated from the feasible flow problem in communication networks. With the development of computer networks, this problem has been highly valued by scholars from the field of computer science and mathematics. This paper studies the sufficient conditions for the existence of fractional factors in the different setting of network from a theoretical perspective. These theoretical results provide the basis for the initial network designing. We first study the relationship between vulnerable parameter in networks and the existence of fractional factor, and an isolated toughness condition for a graph to be fractional (g, f, n')-critical is determined. Then, we illustrate some neighborhood union conditions for independent-set-deletable deleted graphs, and point out that the conditions are best in some senses. Moreover, we present an independent set condition for a graph to be fractional ID-(g, f, m)-deleted. Also, the result is tight on independent set degree condition. Finally, we introduce the concept of all fractional ID-(g, f, m)-deleted graph, and several conclusions are obtained from the known results.**

***Index Terms*—graph, fractional factor, all fractional factor, fractional critical deleted graph, isolated toughness, neighborhood union condition.**

I. INTRODUCTION

ALL graphs considered in this paper are finite, loopless, and have no multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. For two vertex-disjoint subsets S and T of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* $d_G(x, y)$ between two vertices x and y is defined to be the length of a shortest path connecting them. The notation and terminology used but undefined in this paper can be found in [1].

In the whole context, we always assume that $n = |V(G)|$ and G is not complete. Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \le g(x) \le f(x)$ for all $x \in V(G)$. A *fractional* (g, f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \le d_G^h(x) \le f(x)$, where $d_G^h(x) = \sum_{e \in E(x)} h(e)$ is called the *fractional degree* of x in G. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional $[a, b]$-factor. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional f-factor. Moreover, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then a fractional (g, f)-factor is just a fractional k-factor.

A graph G is called a *fractional* (g, f, m)*-deleted graph* if for each edge subset $H \subseteq E(G)$ with $|H| = m$, there exists a fractional (g, f)-factor h such that $h(e) = 0$ for all $e \in H$. That is, after removing any m edges, the resulting graph still has a fractional (g, f)-factor. A graph G is called a *fractional* (g, f, n')*-critical graph* if after delated any n' vertices from G, the resulting graph still has a fractional (g, f)-factor.

The first author of this paper first introduced the concept of a fractional (g, f, n', m)-critical deleted graph [2]. A graph G is called a *fractional* (g, f, n', m)*-critical deleted graph* if after deleting any n' vertices from G, the resulting graph is still a fractional (g, f, m)-deleted graph. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are fractional (a, b, m)-deleted graph, fractional (a, b, n')-critical graph, and fractional (a, b, n', m)-critical deleted graph, respectively. If $g(x) = f(x)$ for all $x \in V(G)$, then fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are fractional (f, m)-deleted graph, fractional (f, n')-critical graph, and fractional (f, n', m)-critical deleted graph, respectively. Furthermore, if $g(x) = f(x) = k$ ($k \ge 1$ is an integer) for all $x \in V(G)$, then fractional (g, f, m)-deleted graph, fractional (g, f, n')-critical graph, and fractional (g, f, n', m)-critical deleted graph are just fractional (k, m)-deleted graph, fractional (k, n')-critical graph, and fractional (k, n', m)-critical deleted graph, respectively.

We say that G has all fractional (g, f)-factors if G has a fractional p-factor for each $p : V(G) \to \mathbb{N}$ satisfies $g(x) \le p(x) \le f(x)$ for any $x \in V(G)$. If $g(x) = a$, $f(x) = b$ for each vertex x and G has all fractional (g, f)-factors, then we say that G has all fractional $[a, b]$-factors.

Lu [3] presented the sufficient and necessary condition for a graph with all fractional (g, f)-factors. Zhou and Sun [4] introduced the concept of all fractional (a, b, n')-critical graph, i.e., a graph G is called an all fractional (a, b, n')-critical graph if after deleting any n' vertices of G the remaining graph of G has all fractional $[a, b]$-factors. Also, the necessary and sufficient condition for a graph to be all fractional (a, b, n')-critical is determined.

Gao et al. [5] combined two concepts all fractional (g, f, m)-deleted graph and all fractional (g, f, m)-critical graph together. A graph G is called an all fractional (g, f, n', m)-critical deleted graph if after deleting any n' vertices of G the remaining graph of G is still an all fractional

Manuscript received September 13, 2018; revised November 30, 2018. We thank the reviewers for their constructive comments in improving the quality of this paper. This work was supported in part by the National Natural Science Foundation of China (11761083).

W. Gao is with the School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China. E-mail: gaowei@ynnu.edu.cn.

C. Shi is with the School of Mathematics and Information Science, Shanghai Lixin University of Commerce, Shanghai 201620, China. Email: shice060@lixin.edu.cn

(g, f, m)-deleted graph. If $g(x) = a$, $f(x) = b$ for each $x \in V(G)$, then all fractional (g, f, n', m)-critical deleted graph becomes all fractional (a, b, n', m)-critical deleted graph, it means, after deleting any n' vertices of G the remaining graph of G still an all fractional (a, b, m)-deleted graph.

Chang et al. [6] introduced the concept of *fractional ID-k-factor-critical graph* (if $G - I$ has a fractional k-factor for any independent set I of G) and derived that G is a fractional ID-k-factor-critical graph if $\delta(G) \geq \frac{2n}{3}$ and $n \geq 6k-8$. Later, this concept was extended to the *fractional ID-$[a,b]$-factor-critical graph* by Zhou et al. in [7], i. e., a graph G is fractional ID-$[a, b]$-factor-critical if $G - I$ admits a fractional $[a, b]$-factor for every independent set I of G. Zhou et al. [7] determined that a graph G to be a fractional ID-$[a, b]$-factor-critical graph if $\delta(G) \geq \frac{(a+b)n}{a+2b}$ and $n \geq \frac{(a+2b)(a+b-2)+1}{b}$. Theoretical results on ID-k-factor-critical graphs can be refered to Zhou [8] and [9], and Zhou et al. [10]. Furthermore, conclusions on fractional ID-$[a, b]$-factor-critical graphs can be refered to Zhou et al. [11].

In Zhou [12] and Zhou et al. [13], the setting was different from the previous situations in which there is a difference Δ between $g(x)$ and $f(x)$ for each vertex $x \in V(G)$, i.e., $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. We observe that if $\Delta = 0$ ($a \leq g(x) \leq f(x) \leq b$), then binding number (minimum $\frac{|N(X)|}{|X|}$ where $X \neq \emptyset$, $X \subset V(G)$) condition for ID-(g, f)-factor-critical graph is

$$bind(G) > \frac{(2a+b-1)(n-1)}{an-(a+b-2)}.$$

After adding the variable Δ (i.e., $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$), by the conclusion obtained by Zhou et al. [13], the binding number condition becomes

$$bind(G) > \frac{(2a+b+\Delta-1)(n-1)}{(a+\Delta)n-(a+b-2)}.$$

This fact reveals that if the setting changes, the lower bound of binding number for ID-(g, f)-factor-critical graph is changed as well, and the new binding number heavily depend on Δ.

More sufficient conditions for graphs to have fractional factors can be found in Gao and Gao [14], Gao et al. [15], [16], [17] and [18], and Gao and Wang [19] and [20].

Gao et al. [21] extended the concept of fractional ID-$[a, b]$-factor-critical graph. A graph is called *fractional independent-set-deletable (g, f, m)-deleted graph* (in short, *fractional ID-(g, f, m)-deleted graph*) if $G-I$ is a fractional (g, f, m)-deleted graph for every independent set I of G. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional ID-(g, f, m)-deleted graph is a fractional ID-(f, m)-deleted graph.

In Gao et al. [22], it determines the following result on the neighborhood union condition for fractional (k, m)-deleted graphs.

Theorem 1. (Gao et al. [22]) Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph with $n \geq 8k + 4m - 7$ and $\delta(G) \geq k + m$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

Furthermore, the the neighborhood union condition for fractional (g, f, m)-deleted graphs is presented in Gong et al. [23].

Theorem 2. (Gong et al. [23]) Let G be a graph of order n. Let a, b, Δ, m be four integers with $1 \leq a \leq b - \Delta$ and $m \geq 0$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{b^2}{a} + 2m$, $n > \frac{(a+b)(2(a+b)+2m-1)}{a}$, and

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{bn}{a+b}$$

for any non-adjacent vertices $x_1, x_2 \in V(G)$, then G is a fractional (g, f, m)-deleted graph.

In Gao et al. [21], they stated the following results on ID-(g, f, m)-deleted graph.

Theorem 3. (Gao et al. [21]) Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \leq a \leq b$ and $n > \frac{(2a+b)(a+b+2m-2)}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies $\delta(G) \geq \frac{(a+b)n}{2a+b}$, then G is a fractional ID-(g, f, m)-deleted graph.

Theorem 4. (Gao et al. [21]) Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \leq a \leq b$, $n > \frac{(2a+b)(a+b+2m-1)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2}{a} + m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is a fractional ID-(g, f, m)-deleted graph.

Theorem 5. (Gao et al. [21]) Let G be a graph of order n, and let a, b, and m be non-negative integers such that $2 \leq a \leq b$, $n > \frac{(2a+b)(a+b+2m-2)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2}{a} + m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$, and $\sigma_2(G) = \min\{d_G(u)+d_G(v)\}$ for each pair of non-adjacent vertices u and v of G. If G satisfies $\sigma_2(G) \geq \frac{2(a+b)n}{2a+b}$, then G is a fractional ID-(g, f, m)-deleted graph.

The notion of *isolated toughness* was first introduced by Yang et al. [24] which was stated as follows: if G is a complete graph, $I(G) = \infty$. If G is not complete,

$$I(G) = \min\{\frac{|S|}{i(G-S)}\Big|S \subset V(G), i(G-S) \geq 2\}$$

and where $i(G - S)$ is the number of isolated vertices of $G - S$.

The proof of our main result in the first part is based on the following lemma which is the necessary and sufficient condition of all fractional (g, f, n', m)-critical deleted graphs.

Lemma 1. (Gao et al. [5]) Let a, b, m and n' be nonnegative integers with $1 \leq a \leq b$, and let G be a graph of order n with $n \geq b + n' + m + 1$. Let $g, f : V(G) \to \mathbb{Z}^+$ be two valued functions with $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$, and H be a subgraph of G with m edges. Then G is all fractional (g, f, n', m)-critical deleted if and only if for any

$S \subseteq V(G)$ with $|S| \geq n'$,

$$\begin{aligned} & g(S) - f(T) + \sum_{x\in T} d_{G-S}(x) \\ \geq & \max_{U\subseteq S,|U|=n',H\subseteq E(G-U),|H|=m} \{g(U) \\ & + \sum_{x\in T} d_H(x) - e_H(S,T)\}, \end{aligned}$$

where

$$T = \{x : x \in V(G)\backslash S, d_{G-S}(x) - d_H(x) + e_H(x,S) < f(x)\}.$$

The equal version of Lemma 1 is stated as follows.

Lemma 2. (Gao et al. [5]) Let a, b, m and n' be nonnegative integers with $1 \leq a \leq b$, and let G be a graph of order n with $n \geq b + n' + m + 1$. Let $g, f : V(G) \to \mathbb{Z}^+$ be two valued functions with $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$, and H be a subgraph of G with m edges. Then G is all fractional (g, f, n', m)-critical deleted if and only if

$$\begin{aligned} & g(S) - f(T) + \sum_{x\in T} d_{G-S}(x) \quad (1) \\ \geq & \max_{U\subseteq S,|U|=n',H\subseteq E(G-U),|H|=m} \{g(U) \\ & + \sum_{x\in T} d_H(x) - e_H(S,T)\}, \end{aligned}$$

for any non-disjoint subsets $S, T \subseteq V(G)$ with $|S| \geq n'$.

The contributions of our paper are four-fold: first we present the isolated toughness condition of all fractional (g, f, n')-critical graphs; second, we determine several the neighborhood union conditions for independent-set-deletable deleted graphs; third, an independent set degree condition for fractional ID-(g, f, m)-deleted graphs is proposed; at last, we introduce all fractional ID-(g, f, m)-deleted graph and present some immediate conclusions from the known results.

II. Isolated toughness condition for a graph to be all fractional (g, f, n')-critical

In this section, we research the relations between isolated toughness $I(G)$ and the all fractional (g, f, n')-critical graphs. Our main result in this part can be formulated as follows.

Theorem 6. Let G be a graph and g, f be two non-negative integer valued functions defined on vertex set of G which satisfy $a \leq g(x) \leq f(x) \leq b$ for any vertex x (here $1 \leq a \leq b$ and $b \geq 2$). Set $\Delta = b - a$. If $\delta(G) \geq n' + \frac{(b+2)^2}{4a} + b - 1$ and $I(G) \geq \frac{b^2-\Delta}{a} + n'$, then G is an all fractional (g, f, n')-critical graph.

One thing we must emphasize here is that $\Delta = b - a$ only hold in this section. From next part, we go to the other setting, and Δ is denoted to be a non-negative integer number.

The following two lemmas are presented by Liu and Zhang [25] which will be used in our proof.

Lemma 3. (Liu and Zhang [25]) Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k - 1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k - 2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \ldots, k-1$.

Clearly, Lemma 3 is also correct for $\delta(H) \geq 0$. By the proving process of Lemma 2.2 in [25], we obtain the following important Lemma.

Lemma 4. (Liu and Zhang [25]) Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and the covering set $C = V(H) - I$ of H satisfying

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

where $I^{(i)} = \{x \in I, d_H(x) = k - i\}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k}|I^{(i)}| = |I|$.

Proof of Theorem 6. If G is complete, the result is obtained by means of $\delta(G) \geq n' + \frac{(b+2)^2}{4a} + b - 1$. In what follows, we assume that G is not complete. Suppose that G satisfies the conditions of Theorem 6, but is not an all fractional (g, f, n')-critical graph. By Lemma 2 (consider $m = 0$), there are exist disjoint subsets S ($|S| \geq n'$) and T of $V(G)$ satisfying

$$\begin{aligned} & a|S| - b|T| + \sum_{x\in T} d_{G-S}(x) - an' \\ \leq & a(|S| - n') + \sum_{x\in T}(d_{G-S}(x) - b) \\ \leq & g(S-U) + \sum_{x\in T}(d_{G-S}(x) - f(x)) \leq -1. \quad (2) \end{aligned}$$

We select S and T such that $|T|$ is minimum. Clearly, $T \neq \emptyset$. If there exists $x \in T$ satisfying $d_{G-S}(x) \geq f(x)$, then the subsets S and $T \setminus \{x\}$ satisfy (2) as well. This contradicts the selection rule of S and T. It implies that $d_{G-S}(x) \leq f(x) - 1 \leq b - 1$ for any $x \in T$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H')|d_{G-S} = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_b. Let S' be a set of vertices that contains exactly $b-1$ vertices in each component of K_b in H'.

If $|V(H)| = 0$, then from (2) we obtain $|S| < \frac{b(|T_0|+l)+an'}{a}$. Clearly, $i(G - S - S') \geq |T_0| + l \geq 1$. If $|T_0| + l = 1$, then $d_{G-S}(x) + |S| \geq d_G(x) \geq \delta(G) \geq n' + \frac{(b+1)^2}{4a} + b - 1$ and $d_{G-S}(x) \geq n' + \frac{(b+1)^2}{4a} + b - 1 - |S| > n' + \frac{b}{a} + b - 1 - \frac{an'+b}{a}$ which contradicts $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for any $x \in T$. Hence $i(G - S \cup S') \geq |T_0| + l \geq 2$ and $I(G) \leq \frac{|S\cup S'|}{i(G-S\cup S')} \leq \frac{b(|T_0|+l)+an'+al(b-1)}{a(|T_0|+l)} \leq \frac{b}{a} + \frac{n'}{2} + b - 1$. This contradicts $I(G) > \frac{b^2-\Delta}{a} + n'$ and $b \geq 2$. Therefore, we have $|V(H)| > 0$.

Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = b - 1$ for every vertex $x \in$

$V(H_1)$ and $H_2 = H - H_1$. By Lemma 4, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq \sum_{i=1}^{b}(b-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \quad (3)$$

and

$$|C_1| \leq \sum_{i=1}^{b}(b-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \quad (4)$$

where $I^{(i)} = \{x \in I_1, d_{H_1}(x) = b - i\}$ for $1 \leq i \leq b$ and $\sum_{i=1}^{b} |I^{(i)}| = |I_1|$. On the other hand, let $T_j = \{x \in V(H_2)|d_{G-S}(x) = j\}$ for $1 \leq j \leq b-1$. By the definition of H and H_2 we can also see that each component of H_2 has a vertex of degree at most $b-2$ in $G-S$. According to Lemma 3, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \quad (5)$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, b-1$. Set $W = V(G) - S - T$ and $U = S \cup S' \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. We have

$$\begin{aligned} & |C_2| + |N_G(I_2) \cap W| \\ = \; & |V(H_2)| - |I_2| + |N_{G-S-T}(I_2)| \\ = \; & |V(H_2)| - |I_2| + |N_{G-S}(I_2)| - |N_T(I_2)| \\ = \; & (|V(H_2)| - |I_2| - |N_T(I_2)|) + |N_{G-S}(I_2)| \\ \leq \; & (|V(H_2)| - |I_2| - |N_{H_2}(I_2)|) + |N_{G-S}(I_2)| \\ \leq \; & 0 + \sum_{j=1}^{b-1} j i_j = \sum_{j=1}^{b-1} j i_j. \end{aligned}$$

Furthermore, we get

$$|U| \leq |S| + l(b-1) + |C_1| + \sum_{j=1}^{b-1} j i_j + \sum_{i=1}^{b}(i-1)|I^{(i)}| \quad (6)$$

and

$$i(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \quad (7)$$

where $t_0 = |T_0|$. When $i(G-U) > 1$, we have

$$|U| \geq I(G) i(G-U). \quad (8)$$

If $i(G-U) = 1$, then $G[T]$ is a clique with less than b vertices. By (2), we infer

$$\begin{aligned} |S| &< \frac{an' + b|T| - d_{G-S}(T)}{a} \\ &\leq \frac{an' + b|T| - |T|(|T|-1)}{a} \\ &\leq \frac{an' + b\frac{b+1}{2} - \frac{b+1}{2}(\frac{b+1}{2} - 1)}{a} \\ &= n' + \frac{(b+1)^2}{4a} \end{aligned}$$

and

$$\begin{aligned} & b-1 \geq d_{G-S}(x) \geq n' + \frac{(b+1)^2}{4a} + b - 1 - |S| \\ > \; & n' + \frac{(b+1)^2}{4a} + b - 1 - (n' + \frac{(b+1)^2}{4a}) \end{aligned}$$

for any $x \in T$. A contradiction.

By (6)-(8), we yield

$$\begin{aligned} |S| + |C_1| \geq \; & \sum_{j=1}^{b-1}(I(G) - j)i_j + I(G)(t_0 + l) \quad (9) \\ & + I(G)|I_1| - \sum_{i=1}^{b}(i-1)|I^{(i)}| - l(b-1). \end{aligned}$$

Thus, from $b|T| - d_{G-S}(T) > a|S| - an'$ we have

$$bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1}(b-j)i_j + \sum_{j=1}^{b-1}(b-j)c_j > a|S| - an'.$$

Combining with (9), we have

$$\begin{aligned} & |V(H_1)| + \sum_{j=1}^{b-1}(b-j)c_j + a|C_1| \quad (10) \\ > \; & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + (aI(G) - b)(t_0 + l) \\ & + aI(G)|I_1| - a\sum_{i=1}^{b}(i-1)|I^{(i)}| - an' - la(b-1). \end{aligned}$$

By (3) and (4), we get

$$\begin{aligned} |V(H_1)| + a|C_1| \leq \; & \sum_{i=1}^{b}(ab - ai + b - (i-1))|I^{(i)}| \\ & - \frac{(a+1)|I^{(1)}|}{2}. \quad (11) \end{aligned}$$

Using (5),(10) and (11), we have

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b \\ & -(i-1))|I^{(i)}| \\ > \; & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + (aI(G) - b)(t_0 + l) \\ & + aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ & -an' - la(b-1). \quad (12) \end{aligned}$$

Now, we discuss the following cases according to the value of $t_0 + l$.

Case 1. $t_0 + l \geq 1$**.** In this case, by $b \geq 2$ and $aI(G) \geq b^2 + an' - \Delta$, we have (12) and $(aI(G) - b)(t_0 + l) - an' - la(b-1) \geq 0$. Thus (12) becomes

$$\begin{aligned} & \sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b \\ & -(i-1))|I^{(i)}| \\ > \; & \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| \\ & + \frac{(a+1)|I^{(1)}|}{2} - a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \quad (13) \end{aligned}$$

Thus, at least one of the following two subcases must happen.

Subcase 1. There is at least one j such that

$$(b-2)(b-j) > aI(G) - aj - b + j.$$

Hence,

$$\begin{aligned} aI(G) &< b(b-2)+(a-b+1)j+b \\ &\le b(b-1)+(a-b+1)+b \\ &= (b^2-1)+(a-b)+(2-b) \\ &\le b^2-\Delta-1, \end{aligned}$$

which contradicts $I(G) \ge \frac{b^2-\Delta}{a}+n'$.

Subcase 2.

$$\begin{aligned} &\sum_{i=1}^{b}(ab-ai+b-(i-1))|I^{(i)}| \\ > \;& aI(G)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ \ge \;& (b^2+an'-\Delta)|I_1|+\frac{(a+1)|I^{(1)}|}{2} \\ & -a\sum_{i=1}^{b}(i-1)|I^{(i)}| \\ \ge \;& (b^2-\Delta)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned}$$

That is to say,

$$\begin{aligned} & |I^{(1)}|(ab+b-\frac{3}{2}a-b^2-\frac{1}{2}+\Delta) \\ + & \sum_{i=2}^{b}(ab+b-a-i+1-b^2+\Delta)|I^{(i)}|>0. \end{aligned}$$

Let

$$\Omega_1(b)=-b^2+(a+1)b-\frac{3}{2}a-\frac{1}{2}+\Delta.$$

In light of $\Delta \ge 0$, we infer

$$\begin{aligned} & \max\{\Omega_1(b)\}=\Omega_1(a+\Delta) \\ = \;& -\Delta^2-(a-2)\Delta-\frac{a+1}{2}<0. \end{aligned}$$

On the other hand, $ab+b-a-i+1-b^2+\Delta \le -b^2+(a+2)b-2a-1$ due to $i \ge 2$. Let

$$\Omega_2(b)=-b^2+(a+2)b-2a-1.$$

We deduce

$$\max \Omega_2(b)=\Omega_2(a)<0$$

due to $b \ge a$. This is a contradiction.

Case 2. $t_0+l=0$**.** In this case, by (12) we have,

$$\begin{aligned} &\sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b \\ & -(i-1))|I^{(i)}| \\ > \;& \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j+aI(G)|I_1| \\ & +\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-an'. \quad (14) \end{aligned}$$

From what we have discussed in Subcase 1, we deduce

$$\sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j \ge \sum_{j=1}^{b-1}(b-2)(b-j)i_j.$$

If $|I_1|>0$, we obtain

$$\begin{aligned} &\sum_{i=1}^{b}(ab-ai+b-(i-1))|I^{(i)}| \\ > \;& aI(G)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-an' \\ \ge \;& (b^2+an'-\Delta)|I_1|+\frac{(a+1)|I^{(1)}|}{2} \\ & -a\sum_{i=1}^{b}(i-1)|I^{(i)}|-an' \\ \ge \;& (b^2-\Delta)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|. \end{aligned}$$

A contradiction follows from what we discussed in Subcase 2 above.

The last situation is $|I_1|=0$ and

$$\sum_{j=1}^{b-1}(b-2)(b-j)i_j > \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j-an'.$$

Let $\Omega_3=(b-2)(b-j)-(aI(G)-aj-b+j)i_j+an'$. We have

$$\begin{aligned} \Omega_3 &= b(b-2)+(a-b+1)j+b-aI(G)+an' \\ &\le b(b-2)+(a-b+1)+b \\ & \quad -(b^2+an'-b+a)+an' \\ &= 1-b<0, \end{aligned}$$

a contradiction.

Therefore, we complete the proof of the desired result. □

III. The neighborhood union condition for a graph to be independent-set-deletable deleted

We assume that from the second part, the non-negative integer number Δ is not equal to $b-a$.

In this part, we discuss the relationship between neighborhood union condition and independent-set-deletable deleted graphs. The main result in this part can be formulated as follows.

Theorem 7. Let $k \ge 2$ and $m \ge 0$ be two integers, and let G be a graph with $n \ge 12k+6m-11$ and $\delta(G) \ge \frac{n}{3}+k+m$. If

$$|N_G(x) \cup N_G(y)| \ge \frac{2n}{3}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional ID-(k,m)-deleted graphs.

We will show that the bounds for neighborhood union condition, the order and the minimum degree of G are all sharp. And, we need the following Lemma.

Lemma 5. (Gao [2]) Let $k \ge 1$ and $m \ge 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if and only if

$$k|S|+\sum_{x\in T}d_{G-S}(x)-k|T| \ge \sum_{x\in T}d_H(x)-e_H(S,T),$$

for all disjoint subsets S and T of $V(G)$.

The second main contribution of this part is to manifest a neighborhood union condition for fractional fractional ID-(g,f,m)-deleted graph, and the main conclusion can be stated as follows.

Theorem 8. Let G be a graph of order n, and let a, b, m, Δ be nonnegative integers with $1 \leq a \leq b-\Delta$, $n > \frac{(2a+b)(2(a+b)+2m-1)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x)-\Delta \leq b-\Delta$ for each $x \in V(G)$. If G satisfies

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional ID-(g, f, m)-deleted graph.

Set $\Delta = 0$ in Theorem 8, we get the following result for a graph to be fractional ID-(g, f, m)-deleted.

Corollary 1. Let G be a graph of order n, and let a, b, m be nonnegative integers with $1 \leq a \leq b$, $n > \frac{(2a+b)(2(a+b)+2m-1)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional ID-(g, f, m)-deleted graph.

Set $m = 0$ in Theorem 8, then we deduce the following corollary on the neighborhood union condition of fractional ID-(g, f)-factor-critical graph.

Corollary 2. Let G be a graph of order n, and let a, b, Δ be nonnegative integers with $1 \leq a \leq b-\Delta$, $n > \frac{(2a+b)(2(a+b)-1)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x)-\Delta \leq b-\Delta$ for each $x \in V(G)$. If G satisfies

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional ID-(g, f)-factor-critical graph.

Set $f(x) = g(x)$ for any $x \in V(G)$ in Corollary 1, we yield the following corollary on the neighborhood union condition of a fractional ID-(f, m)-deleted graph.

Corollary 3. Let G be a graph of order n, and let a, b, m be nonnegative integers with $1 \leq a \leq b$, $n > \frac{(2a+b)(2(a+b)+2m-1)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2}{a} + 2m$. Let f be a integer-valued functions defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$|N_G(x_1) \cup N_G(x_2)| \geq \frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional ID-(f, m)-deleted graph.

Our presentation of sharpness of Theorem 8 is depends heavily on the following Lemma which manifests the necessary and sufficient condition of fractional (g, f, m)-deleted graph.

Lemma 6. (Gao [2]) Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let m be two non-negative integers. Then G is a fractional (g, f, m)-deleted graph if and only if

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) \\ \geq & \max_{H \subseteq E(G), |H|=m} \{\sum_{x \in T} d_H(x) - e_H(T, S)\} \end{aligned} \quad (15)$$

for all disjoint subsets S, T of $V(G)$.

The proof of Theorem 7 can follow the trick in Gao et al. [21] and Theorem 1, and the proof of Theorem 8 can follow the trick in Gao et al. [21] and Theorem 2. Hence, we skip the main proof of Theorem 7 and Theorem 8. Next, we mainly present the sharpness of bound in Theorem 7 and Theorem 8.

We construct some graphs to show that the bounds in the Theorem 7 are best possible.

First, $\delta(G) \geq \frac{n}{3} + k + m$ can't be replaced by $\delta(G) \geq \frac{n}{3} + k + m - 1$. Otherwise, select an appropriate graph which can be expressed as $\frac{n}{3}K_1 \vee H$, choose a vertex v such that $d_H(v) = k + m - 1$. Delete m edges incident to v. The resulting graph has minimal degree $k - 1$, which has no fractional k-factor by the definition.

Let $G = (4k+2m-4)K_1 \vee (K_{4k+2m-4} \vee (4k+2m-3)K_1)$. Then $n = 12k+6m-11$, $\delta(G) = 8k+4m-8 > \frac{n}{3} + k+m$, but $|N_G(x_1) \cup N_G(x_2)| = 8k+4m-8 < \frac{2n}{3}$ for each non-adjacent vertex x_1 and x_2 in $(4k+2m-3)K_1$. Let $G' = K_{4k+2m-4} \vee (4k+2m-3)K_1$, $S = K_{4k+2m-4}$ and $T = (4k+2m-3)K_1$. Then $d_{G'-S}(T) = 0$ and $\sum_{x \in T} d_H(x) - e_H(S, T) = 0$. We have $k|S| + \sum_{x \in T} d_{G'-S}(x) - k|T| - (\sum_{x \in T} d_H(x) - e_H(S, T)) = -k < 0$. Thus, G' is not a fractional (k, m)-deleted graph by Lemma 5, and G is not a fractional ID-(k, m)-deleted graph. The condition $|N_G(x_1) \cup N_G(x_2)| \geq \frac{2n}{3}$ is sharp.

Let $G = (4k+2m-4)K_1 \vee (K_{4k+2m-6} \vee (2k+m-1)K_2)$. Then $n \geq 12k+6m-12$, $\delta(G) = 8k+4m-9 \geq \frac{n}{3}+k+m$ and $|N_G(x_1) \cup N_G(x_2)| = 8k+4m-8 = \frac{2n}{3}$ for any non-adjacent vertices x_1 and x_2 in G. Let $G' = K_{4k+2m-6} \vee (2k+m-1)K_2$, $S = K_{4k+2m-6}$ and $T = (2k+m-1)K_2$. Let H be the set of m edges such that $H \subseteq (2k+m-1)K_2$, then $\sum_{x \in T} d_H(x) - e_H(S, T) = 2m$ and $\sum_{x \in T} d_{G'-S}(x) = 4k+2m-2$. We have, $k|S| + \sum_{x \in T} d_{G'-S}(x) - k|T| - (\sum_{x \in T} d_H(x) - e_H(S, T)) = k(4k+2m-6) - k(4k+2m-2) + (4k+2m-2) - 2m = -2 < 0$. Thus, G' is not a fractional (k, m)-deleted graph by Lemma 5 and G is not a fractional ID-(k, m)-deleted graph. Therefore, the condition $n \geq 12k+6m-11$ is sharp.

Theorem 8 is best possible, to some extent, under the condition. Actually, we can construct some graphs such that the neighborhood union condition in Theorem 8 can't be replaced by $|N_G(x_1) \cup N_G(x_2)| \geq \frac{(a+b)n}{2a+b} - 1$.

Considering a graph $G = (at+1)K_1 \vee K_{bt} \vee (at+1)K_1$, where t is a sufficiently large positive integer. Clearly, $n = (2a+b)t+2$. Let $a = g(x) = f(x) = b$ for all $x \in V(G)$. We have

$$\begin{aligned} & \frac{(a+b)n}{2a+b} > |N_G(x_1) \cup N_G(x_2)| \\ = & \quad (a+b)t+1 > \frac{(a+b)n}{2a+b} - 1 \end{aligned}$$

for any x_1, x_2 satisfy $x_1x_2 \neq E(G)$.

Let $I = (at+1)K_1$. For $G' = K_{bt} \vee (at+1)K_1$, let $S = K_{bt}$ and $T = (at+1)K_1$. Then we have $\sum_{x \in T} d_H(x) - e_H(T, S) = 0$ for any subset H of $E(G')$ with m edges. Therefore,

$$\begin{aligned} & f(S) - g(T) + d_{G'-S}(T) - (\sum_{x \in T} d_H(x) - e_H(T, S)) \\ = & \quad a(bt) - b(at+1) \\ = & \quad -b. \end{aligned}$$

Thus, G' is not a fractional (g, f, m)-deleted graph by Lemma 6. In conclusion, G is not a fractional ID-(g, f, m)-deleted graph.

IV. An independent set degree condition for a graph to be fractional ID-(g, f, m)-deleted

In this section, we consider the relationship between independent set degree condition and fractional ID-(g, f, m)-deleted graphs. The main contribution of this part is to determine an independent set degree condition for fractional fractional ID-(g, f, m)-deleted graph, and it can be stated as follows.

Theorem 9. Let G be a graph of order n, and let a, b, m, Δ, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b-\Delta$, $n > \frac{(2a+b)(ib+2m-2)}{a}$ and $\delta(G) \geq \frac{an}{2a+b}+\frac{b^2(i-1)}{a}+2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x)-\Delta \leq b-\Delta$ for each $x \in V(G)$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{(a+b)n}{2a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional ID-(g, f, m)-deleted graph.

Set $\Delta = 0$ in Theorem 9, we get the following result for a graph to be fractional ID-(g, f, m)-deleted.

Corollary 4. Let G be a graph of order n, and let a, b, m, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b$, $n > \frac{(2a+b)(ib+2m-2)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2(i-1)}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{(a+b)n}{2a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional ID-(g, f, m)-deleted graph.

Set $a = b = k$ in Theorem 9, then $f(x) = g(x) = k$ for any $x \in V(G)$, and $\Delta = 0$. The independent set degree condition for fractional ID-(k, m)-deleted graph can be described as follows.

Corollary 5. Let G be a graph of order n, and let k, m, and i be non-negative integers such that $i \geq 2$, $k \geq 1$, $n > 3(ik + 2m - 2)$ and $\delta(G) \geq \frac{n}{3} + k(i-1) + 2m$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{2n}{3}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional ID-(k, m)-deleted graph.

Set $m = 0$ in Theorem 9, then we deduce the following corollary on the independent set degree condition of fractional ID-(g, f)-factor-critical graph.

Corollary 6. Let G be a graph of order n, and let a, b, Δ, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b - \Delta$, $n > \frac{(2a+b)(ib-2)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2(i-1)}{a}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{(a+b)n}{2a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional ID-(g, f)-factor-critical graph.

Set $f(x) = g(x)$ for any $x \in V(G)$ in Corollary 4, we yield the following corollary on the independent set degree condition of a fractional ID-(f, m)-deleted graph.

Corollary 7. Let G be a graph of order n, and let a, b, m, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b$, $n > \frac{(2a+b)(ib+2m-2)}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{b^2(i-1)}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{(a+b)n}{2a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional ID-(f, m)-deleted graph.

If setting $m = 0$ in Corollary 5, then we deduce the following independent set degree condition of fractional ID-k-factor-critical graph.

Corollary 8. Let G be a graph of order n, and let k and i be non-negative integers such that $i \geq 2$, $k \geq 1$, $n > 3(ik-2)$ and $\delta(G) \geq \frac{n}{3} + k(i-1)$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{2n}{3}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional ID-k-deleted graph.

Our presentation of sharpness of Theorem 9 is depended heavily on the Lemma 6 which manifests the necessary and sufficient condition of fractional (g, f, m)-deleted graph. Furthermore, the proof of Theorem 9 relied on the following Lemma which is a corollary of the main result in Gao et al. [26].

Lemma 7. (Gao et al. [26]) Let G be a graph of order n, and let a, b, m, Δ, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b - \Delta$, $n > \frac{(a+b)(ib+2m-2)}{a}$ and $\delta(G) \geq \frac{b^2(i-1)}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x)-\Delta \leq b - \Delta$ for each $x \in V(G)$. If G satisfies

$$\max\{d_1, d_2, \cdots, d_i\} \geq \frac{bn}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, m)-deleted graph.

Proof of Theorem 9. For every independent set I, let $G' = G - I$.

If $|I| = 1$, then $|V(G')| > \frac{(2a+b)(ib+2m-2)-a}{a} > \frac{(a+b)(ib+2m-2)}{a}$. It is easy to verify that $\delta(G') \geq \frac{b^2(i-1)}{a}+2m$ and $\max\{d_{G'}(x_1), d_{G'}(x_2), \cdots, d_{G'}(x_i)\} \geq \frac{b|V(G')|}{a+b} = \frac{b(n-1)}{a+b}$ for any independent subset $\{x_1, x_2, \cdots, x_i\}$ of G'. Thus, the result holds from Lemma 7.

We now consider $|I| \geq 2$ and G' is not complete. By degree condition, we obtain $|V(G')| \geq \frac{(a+b)n}{2a+b} > \frac{(a+b)(ib+2m-2)}{a}$. If $\max\{d_{G'}(u), d_{G'}(v)\} < \frac{b|V(G')|}{a+b}$ for some non-adjacent vertices u, v in G', then $\frac{(a+b)(|V(G')|+|I|)}{2a+b} \leq \max\{d_G(u), d_G(v)\} < \frac{b|V(G')|}{a+b} + |I|$, i.e., $|V(G')| < \frac{a+b}{a}|I| \leq \frac{a+b}{a}\frac{an}{2a+b} = \frac{(a+b)n}{2a+b}$. This contradicts $\max\{d_1, d_2, \cdots, d_i\} \geq \frac{(a+b)n}{2a+b}$ for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$ and $|I| \geq 2$. Therefore, $\max\{d_{G'}(x_1), d_{G'}(x_2), \cdots, d_{G'}(x_i)\} \geq \frac{b|V(G')|}{a+b}$ for all independent subset $\{x_1, x_2, \ldots, x_i\}$ in G'. Furthermore, we obtain $\delta(G') \geq \frac{b^2(i-1)}{a} + 2m$ by $|I| \leq \frac{an}{2a+b}$ and $\delta(G) \geq$

$\frac{an}{2a+b}+\frac{b^2(i-1)}{a}+2m$. Then, the result follows from Lemma 7.

Thus, we complete the proof of Theorem 9. □

Theorem 9 is best possible, to some extent, under the condition. Actually, we can construct some graphs such that the independent set degree condition in Theorem 9 can't be replaced by $\max\{d_1,d_2,\cdots,d_i\}\geq\frac{(a+b)n}{2a+b}-1$.

Considering a graph $G=(at+1)K_1\vee K_{bt}\vee(at+1)K_1$, where t is a sufficiently large positive integer. Clearly, $n=(2a+b)t+2$. Let $a=g(x)=f(x)=b$ for all $x\in V(G)$. We have

$$\begin{aligned}\frac{(a+b)n}{2a+b} &> \max\{d_1,d_2,\cdots,d_i\}\\ &= (a+b)t+1>\frac{(a+b)n}{2a+b}-1\end{aligned}$$

for any independent subset $\{x_1,x_2,\ldots,x_i\}$ of $V(G)$.

Let $I=(at+1)K_1$. For $G'=K_{bt}\vee(at+1)K_1$, let $S=K_{bt}$ and $T=(at+1)K_1$. Then we have $\sum_{x\in T}d_H(x)-e_H(T,S)=0$ for any subset H of $E(G')$ with m edges. Therefore,

$$\begin{aligned}& f(S)-g(T)+d_{G'-S}(T)-(\sum_{x\in T}d_H(x)-e_H(T,S))\\ =\ & a(bt)-b(at+1)\\ =\ & -b.\end{aligned}$$

Thus, G' is not a fractional (g,f,m)-deleted graph by Lemma 6. In conclusion, G is not a fractional ID-(g,f,m)-deleted graph.

V. All fractional ID-(g,f,m)-deleted graph

As the last part of our paper, this section aims to introduce the concept of all fractional ID-(g,f,m)-deleted graph and yield several related conclusions.

A graph is called *all fractional independent-set-deletable (g,f,m)-deleted graph* (in short, *all fractional ID-(g,f,m)-deleted graph*) if $G-I$ is an all fractional (g,f,m)-deleted graph for any independent set I of G. If $g(x)=f(x)$ for all $x\in V(G)$, then an all fractional ID-(g,f,m)-deleted graph is an all fractional ID-(f,m)-deleted graph. If $g(x)=a$ and $f(x)=b$ for all $x\in V(G)$, then an all fractional ID-(g,f,m)-deleted graph is an all fractional ID-(a,b,m)-deleted graph. If $m=0$, then an all fractional ID-(g,f,m)-deleted graph is just an all fractional ID-(g,f)-factor-critical graph.

Since Gao et al. [5] presented the necessary and sufficient condition for a graph to be an all fractional (g,f,n',m)-critical deleted graph, we can directly derive the following lemma on the necessary and sufficient condition for a graph to be an all fractional (g,f,m)-deleted graph which is also regarded as a corollary of Lemma 2.

Lemma 8. Let a, b and m be nonnegative integers with $1\leq a\leq b$, and let G be a graph of order n with $n\geq b+m+1$. Let $g,f:V(G)\to\mathbb{Z}^+$ be two valued functions with $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$, and H be a subgraph of G with m edges. Then G is all fractional (g,f,m)-deleted if and only if

$$\begin{aligned}& g(S)-f(T)+\sum_{x\in T}d_{G-S}(x) \qquad (16)\\ \geq\ & \sum_{x\in T}d_H(x)-e_H(S,T),\end{aligned}$$

for any non-disjoint subsets $S,T\subseteq V(G)$.

Using this lemma, we will check the sharpness of results given in this section.

Very recently, Wu et al. [27] presented several sharp degree conditions for a graph to be an all fractional (g,f,n',m)-critical deleted graph which can be used in resource dispatching of NFV (Network Functions Virtualization) networks. By setting $n'=0$ in the main conclusions in [27], we obtain the following conclusions on all fractional (g,f,m)-deleted graph.

Theorem 10. (Wu et al. [27]) Let G be a graph of order n, and let a,b, and m be non-negative integers such that $2\leq a\leq b$ and $n>\frac{(a+b)(a+b+2m-2)}{a}$. Let g,f be two integer-valued functions defined on $V(G)$ such that $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$. If G satisfies $\delta(G)\geq\frac{bn}{a+b}$, then G is an all fractional (g,f,m)-deleted graph.

Theorem 11. (Wu et al. [27]) Let G be a graph of order n, and let a,b, and m be non-negative integers such that $2\leq a\leq b$, $n>\frac{(a+b)(a+b+2m-1)}{a}$ and $\delta(G)\geq\frac{b^2}{a}+m$. Let g,f be two integer-valued functions defined on $V(G)$ such that $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$. If G satisfies

$$\max\{d_G(x),d_G(y)\}\geq\frac{bn}{a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is an all fractional (g,f,m)-deleted graph.

Theorem 12. (Wu et al. [27]) Let G be a graph of order n, and let a,b, and m be non-negative integers such that $2\leq a\leq b$, $n>\frac{(a+b)(a+b+2m-2)}{a}$ and $\delta(G)\geq\frac{b^2}{a}+m$. Let g,f be two integer-valued functions defined on $V(G)$ such that $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$. If G satisfies $\sigma_2(G)\geq\frac{2bn}{a+b}$, then G is an all fractional (g,f,m)-deleted graph.

In light of Theorem 10, Theorem 11 and Theorem 12, we get the following results on an all fractional ID-(g,f,m)-deleted graph. The proofs are similar to Gao et al. [21] and Theorem 9 in this paper, so we skip them here.

Theorem 13. Let G be a graph of order n, and let a,b, and m be non-negative integers such that $2\leq a\leq b$ and $n>\frac{(2a+b)(a+b+2m-2)}{a}$. Let g,f be two integer-valued functions defined on $V(G)$ such that $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$. If G satisfies $\delta(G)\geq\frac{(a+b)n}{2a+b}$, then G is an all fractional ID-(g,f,m)-deleted graph.

Theorem 14. Let G be a graph of order n, and let a,b, and m be non-negative integers such that $2\leq a\leq b$, $n>\frac{(2a+b)(a+b+2m-1)}{a}$ and $\delta(G)\geq\frac{an}{2a+b}+\frac{b^2}{a}+m$. Let g,f be two integer-valued functions defined on $V(G)$ such that $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$. If G satisfies

$$\max\{d_G(x),d_G(y)\}\geq\frac{(a+b)n}{2a+b}$$

for each pair of non-adjacent vertices x and y of G, then G is an all fractional ID-(g,f,m)-deleted graph.

Theorem 15. Let G be a graph of order n, and let a,b, and m be non-negative integers such that $2\leq a\leq b$, $n>\frac{(2a+b)(a+b+2m-2)}{a}$ and $\delta(G)\geq\frac{an}{2a+b}+\frac{b^2}{a}+m$. Let g,f be two integer-valued functions defined on $V(G)$ such that $a\leq g(x)\leq f(x)\leq b$ for each $x\in V(G)$. If G satisfies $\sigma_2(G)\geq\frac{2(a+b)n}{2a+b}$, then G is an all fractional ID-(g,f,m)-deleted graph.

The next example shows the main degree conditions in Theorem 13, Theorem 14 and Theorem 15 are sharp.

Consider a graph $G = (at+1)K_1 \vee K_{bt} \vee (at+1)K_1$, where t is a sufficiently large positive integer. Clearly, $n = (2a+b)t+2$. We have

$$\frac{(a+b)n}{2a+b} > \delta(G) = (a+b)t+1 > \frac{(a+b)n}{2a+b} - 1,$$

$$\begin{aligned}\frac{(a+b)n}{2a+b} &> \max\{d_G(u), d_G(v)\} = (a+b)t+1 \\ &> \frac{(a+b)n}{2a+b} - 1,\end{aligned}$$

$$\frac{2(a+b)n}{2a+b} > \sigma_2(G) = 2(a+b)t+2 > \frac{2(a+b)n}{2a+b} - 1.$$

Let $I = (at+1)K_1$. For $G' = K_{bt} \vee (at+1)K_1$, let $S = K_{bt}$ and $T = (at+1)K_1$. Let $a = g(x)$ and $f(x) = b$ for all $x \in V(G)$. Then we have $\sum_{x\in T} d_H(x) - e_H(T,S) = 0$ for any subset H of $E(G')$ with m edges. Therefore,

$$\begin{aligned}& g(S) - f(T) + d_{G'-S}(T) - (\sum_{x\in T} d_H(x) - e_H(T,S)) \\ =\ & a(bt) - b(at+1) \\ =\ & -b.\end{aligned}$$

Thus, G' is not a fractional (g,f,m)-deleted graph by Lemma 8. In conclusion, G is not a fractional ID-(g,f,m)-deleted graph.

Wu et al. [28] considered the resource scheduling problem in NFV networks by using graph theory, and an independent set degree condition and an independent set neighborhood union condition for all fractional (g,f,n',m)-critical deleted graphs are determined. Furthermore, they show that the results are tight on independent set conditions. By setting $n' = 0$ in Wu's contribution [28], we get the following two conclusions on independent set degree condition and independent set neighborhood union condition for all fractional (g,f,m)-deleted graph.

Theorem 16. (Wu et al. [28]) Let G be a graph of order n, and let a, b, m, and i be non-negative integers such that $i \ge 2$, $1 \le a \le b$, $n > \frac{(a+b)(2m+ib-2)}{a}$ and $\delta(G) \ge \frac{b^2(i-1)}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \ge \frac{bn}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is an all fractional (g,f,m)-deleted graph.

Theorem 17. (Wu et al. [28]) Let G be a graph of order n. Let a, b, i be four integers with $i \ge 2$, $1 \le a \le b$ and $m \ge 0$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $\delta(G) \ge \frac{b^2(i-1)}{a} + 2m$, $n > \frac{(a+b)(i(a+b)+2m-2)}{a}$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{bn}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is an all fractional (g,f,m)-deleted graph.

Using Theorem 16 and Theorem 17, we can deduce the following two results on an all fractional ID-(g,f,m)-deleted graph. The proof of Theorem 18 and Theorem 19 are similar to the proof of Theorem 9, and we skip them here.

Theorem 18. Let G be a graph of order n, and let a, b, m, and i be non-negative integers such that $i \ge 2$, $1 \le a \le b$, $n > \frac{(2a+b)(2m+ib-2)}{a}$ and $\delta(G) \ge \frac{an}{2a+b} + \frac{b^2(i-1)}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If G satisfies

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \ge \frac{(a+b)n}{2a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is an all fractional ID-(g,f,m)-deleted graph.

Theorem 19. Let G be a graph of order n. Let a, b, i be integers with $i \ge 2$, $1 \le a \le b$ and $m \ge 0$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $\delta(G) \ge \frac{an}{2a+b} + \frac{b^2(i-1)}{a} + 2m$, $n > \frac{(2a+b)(i(a+b)+2m-2)}{a}$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{(a+b)n}{2a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is an all fractional ID-(g,f,m)-deleted graph.

Theorem 18 and Theorem 19 are best possible, to some extent, under the condition. Actually, we can construct some graphs such that the independent set degree condition in Theorem 18 can't be replaced by $\max\{d_1, d_2, \cdots, d_i\} \ge \frac{(a+b)n}{2a+b} - 1$, and the independent set neighborhood union condition in Theorem 19 can't be replaced by

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \ge \frac{(a+b)n}{2a+b} - 1.$$

Consider a graph $G = (at+1)K_1 \vee K_{bt} \vee (at+1)K_1$, where t is a sufficiently large positive integer. Clearly, $n = (2a+b)t+2$. We have

$$\begin{aligned}& \frac{(a+b)n}{2a+b} > \max\{d_1, d_2, \cdots, d_i\} \\ =\ & (a+b)t+1 > \frac{(a+b)n}{2a+b} - 1\end{aligned}$$

and

$$\begin{aligned}& \frac{(a+b)n}{2a+b} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \\ =\ & (a+b)t+1 > \frac{(a+b)n}{2a+b} - 1\end{aligned}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$.

Let $I = (at+1)K_1$. For $G' = K_{bt} \vee (at+1)K_1$, let $S = K_{bt}$ and $T = (at+1)K_1$. Let $a = g(x)$ and $f(x) = b$ for all $x \in V(G)$. Then we have $\sum_{x\in T} d_H(x) - e_H(T,S) = 0$ for any subset H of $E(G')$ with m edges. Therefore,

$$\begin{aligned}& g(S) - f(T) + d_{G'-S}(T) - (\sum_{x\in T} d_H(x) - e_H(T,S)) \\ =\ & a(bt) - b(at+1) \\ =\ & -b.\end{aligned}$$

Thus, G' is not an all fractional (g,f,m)-deleted graph by Lemma 8. In conclusion, G is not an all fractional ID-(g,f,m)-deleted graph.

VI. Conclusion

In a data transmission network, the feasibility of data transmission at certain time can be expressed by the existence of a fractional factor. The existence of fractional factors under different settings corresponds to the feasibility of data transmission in networks at different conditions. This paper starts with the network parameters, finds the relationship between these parameters and the existence of fractional factors under different frameworks, and thus obtains the corresponding bounds. Since these graph parameters are of great significance in data transmission networks which are key parameters to be considered in network design, the results obtained in this paper have important theoretical guiding significance for the analysis of network design and data transmission network, and have certain reference value for engineers and related practitioners.

VII. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Springer, Berlin, 2008.
[2] W. Gao, *Some results on fractional deleted graphs*, Doctoral disdertation of Soochow university, 2012.
[3] H. L. Lu, “Simplified existence theorems on all fractional $[a, b]$-factors,” *Discrete Applied Mathematics*, vol. 161, pp. 2075-2078, 2013.
[4] S. Z. Zhou and Z. R. Sun, “On all fractional (a, b, k)-critical graphs,” *Acta Mathematica Sinica*, vol. 30, no. 4, pp. 696-702, 2014.
[5] W. Gao, Y. Q. Zhang, and Y. J. Chen, “Neighborhood condition for all fractional (g, f, n', m)-critical deleted graphs,” *Open Physics*, vol. 16, pp. 544-553, 2018.
[6] R. Chang, G. Liu, and Y. Zhu, “Degree conditions of fractional ID-k-factor-critical graphs,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 33, no. 3, pp. 355-360, 2010.
[7] S. Z. Zhou, Z. R. Sun, and H. Liu, “A minimum degree condition for fractional ID-$[a, b]$-factor-critical graphs,” *Bulletin of the Australian Mathematical Society*, vol. 86, no. 2, pp. 177-183, 2012.
[8] S. Z. Zhou, “Independence number and minimum degree for fractional ID-k-factor-critical graphs,” *Aequationes Mathematicae*, vol. 84, no. 1-2, pp. 71-76, 2012.
[9] S. Z. Zhou, “Binding numbers for fractional ID-k-factor-critical graphs,” *Acta Mathematica Sinica*, vol. 30, no. 1, pp. 181-186, 2014.
[10] S. Z. Zhou, Q. X. Bian, and J. C. Wu, “A result on fractional ID-k -factor-critical graphs,” *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 87, pp. 229-236, 2013.
[11] S. Z. Zhou, J. Wu, and Q. R. Pan, “A result on fractional ID-$[a, b]$ -factor-critical graphs,” *Australasian Journal of Combinatorics*, vol. 58, pp. 172-177, 2014.
[12] S. Z. Zhou and Q. X. Bian, “An existence theorem on fractional deleted graphs,” *Periodica Mathematica Hungarica*, vol. 71, pp. 125-133, 2015.
[13] S. Z. Zhou, Z. R. Sun, and Y. Xu, “A theorem on fractional ID-(g, f)-factor-critical-graphs,” *Contributions to Discrete Mathematics*, vol. 10, no. 2, pp. 31-38, 2015.
[14] W. Gao and Y. Gao, “Toughness condition for a graph to be a fractional (g, f, n)-critical deleted graph,” *The Scientific World Journal*, Vol. 2014, Article ID 369798, 7 pages, http://dx.doi.org/10.1155/2014/369798.
[15] W. Gao, L. Liang, T. W. Xu, and J. X. Zhou, “Tight toughness condition for fractional (g, f, n)-critical graphs,” *Journal of the Korean Mathematical Society*, vol. 51, pp. 55-65, 2014.
[16] W. Gao and L. Shi, “Szeged related indices of unilateral polyomino chain and unilateral hexagonal chain,” *IAENG International Journal of Applied Mathematics*, vol. 45, no. 2, 138-150, 2015.
[17] W. Gao, L. Shi, and M. R. Farahani, “Distance-based indices for some families of dendrimer nanostars,” *IAENG International Journal of Applied Mathematics*, vol. 46, no. 2, 168-186, 2016.
[18] W. Gao, M. K. Jamil, W. Nazeer, and M. Amin, “Degree-based multiplicative atom-bond connectivity index of nanostructures,” *IAENG International Journal of Applied Mathematics*, vol. 47, no. 4, 388-397, 2017.
[19] W. Gao and W. F. Wang, “Degree conditions for fractional (k, m)-deleted graphs,” *Ars Combinatoria*, vol. 113A, pp. 273-285, 2014.
[20] W. Gao and W. F. Wang, “Toughness and fractional critical deleted graph,” *Utilitas Mathematica*, vol. 98, pp. 295-310, 2015.
[21] W. Gao, L. Liang, T. W. Xu, and J. X. Zhou, “Degree conditions for fractional (g, f, n', m)-critical deleted graphs and fractional ID-(g, f, m)-deleted graphs,” *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 39, pp. 315-330, 2016.
[22] Y. Gao, M. R. Farahani, and W. Gao, “A neighborhood union condition for fractional (k, n', m)-critical deleted graphs,” *Transactions on Combinatorics*, vol. 6, no. 1, pp. 13-19, 2017.
[23] S. Gong, L. Q. Zhang, and W. Gao, “Neighborhood union condition on fractional (g, f, n', m)-critical deleted graphs in new setting,” *Journal of Southwest China Normal University (Natural Science Edition)*, vol. 42, no. 8, pp. 13-17, 2017.
[24] J. B. Yang, Y. H. Ma, and G. Z. Liu, “Fractional (g, f)-factors in graphs,” *Applied Mathematics-A Journal of Chinese Universities Series A*, vol. 16, pp. 385-390, 2001.
[25] G. Z. Liu and L. J. Zhang, “Toughness and the existence of fractional k-factors of graphs,” *Discrete Mathematics*, vol. 308, pp. 1741-1748, 2008.
[26] W. Gao, J. L. G. Guirao, M. Abdel-Aty, and W. F. Xi, “An independent set degree condition for fractional critical deleted graphs,” *Discrete and Continuous Dynamical Systems, Series S*, vol. 12, no. 4-5, 877-886, 2019.
[27] J. Z. Wu, J. B. Yuana, and D. Dimitrov, “Graph-based feasibility analysis of resource dispatching in NFV networks,” *Journal of Intelligent & Fuzzy Systems*, vol. 35, no. 4, 4487-4494, 2018.
[28] J. Z. Wu, J. B. Yuana, and M. K. Siddiqui, “Independent set conditions for all fractional (g, f, n', m)-critical deleted NFV networks,” *Journal of Intelligent & Fuzzy Systems*, vol. 35, no. 4, 4495-4502, 2018.

Wei Gao obtained PhD degree in mathematical department at Soochow University, China in 2012. After that, he worked in School of Information Science and Technology, Yunnan Normal University. His research interests are Graph Theory, Theoretical Chemistry, Statistical Learning Theory, Computation Topics on Energy and Environmental Science, and Artificial Intelligence etc. He is a committee member of China Society of Industrial and Applied Mathematics (C-SIAM) Graph Theory and Combinatorics with Applications Committee, International Association of Engineers (IAENG), Asia Society of Applied Mathematics and Engineering (Asia-SAME), and act as academic adviser of Center for Energy Research (Iran). He is an editor of several journals, and also the chair of ICED 2017 and ISGTCTC 2018. Among his more than 100 publications in SCI index journals, 8 of them are included in Essential Science Indicators as Highly Cited Papers.

Ce Shi was born in Bengbu, Anhui, China in 1983. He received a BSc in Mathematics and Applied Mathematics from Anhui Polytechnic University (2006), M.S. (2009) and Ph.D. (2012) in Applied Mathematics from Soochow University. He has worked in the School of Statistics and Mathematics at Shanghai Lixin University of Accounting and Finance since 2012. He became a reviewer of Mathematical Reviews in 2013. His research interests include combinatorics, software testing and coding theory.