Permanence, Almost Periodic Oscillations and Stability of Delayed Predator-Prey System with General Functional Response

Ting Yuan, Liyan Pang and Tianwei Zhang

Abstract—**By using some new analytical techniques and Mawhin's continuous theorem of coincidence degree theory, some new sufficient conditions for the existence of positive almost periodic solutions to a class of predator-prey system with general functional response and time delays are established. Secondly, by using the comparison theorem, we give a permanence result for the model. By using the Lyapunov method of differential equations, sufficient conditions which guarantee uniform asymptotical stability of the model are obtained. Finally, two examples and simulations are given to illustrate the main result of this paper.**

Index Terms—**Almost periodic oscillation; Coincidence degree; Predator-prey; Functional response.**

I. Introduction

It is well-known that the theoretical study of predator-prey systems in mathematical ecology has a long history starting with the pioneering work of Lotka and Volterra [1-2]. The principles of Lotka-Volterra model, conservation of mass and decomposition of the rates of change in birth and death processes, have remained valid until today and many theoretical ecologists adhere to these principles. This general approach has been applied to many biological systems in particular with functional response. In population dynamics, a functional response of the predator to the prey density refers to the change in the density of prey attached per unit time per predator as the prey density changes. During the last ten years, there has been extensively investigation on the dynamics of predator-prey models with the different functional responses in the literature, (see [1-15] and references therein). In particular, the existence of positive periodic solutions of the predator-prey system with some monotone or non-monotone functional responses has been studied extensively in the literature.

In [7], Wang and Li considered the following system with Holling III type functional response:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[b_1(t) - a_1(t)N_1(t-\mu_1(t)) \\ \qquad - \frac{\alpha_1(t)N_1(t)}{1+mN_1^2(t)}N_2(t-\nu(t))\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-b_2(t) - a_2(t)N_2(t) \\ \qquad + \frac{\alpha_2(t)N_1^2(t-\mu_3(t))}{1+mN_1^2(t-\mu_3(t))}\Big]. \end{cases} \tag{1.1}$$

Xu et al. [9] studied the following system with Holling II type functional response:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[b_1(t) - a_1(t)N_1(t-\mu_1(t)) \\ \qquad - \frac{\alpha_1(t)N_1(t)}{1+mN_1(t)}N_2(t)\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-b_2(t) - a_2(t)N_2(t-\mu_2(t)) \\ \qquad + \frac{\alpha_2(t)N_1(t-\mu_3(t))}{1+mN_1(t-\mu_3(t))}\Big]. \end{cases} \tag{1.2}$$

By using Mawhin's continuation theorem of coincidence degree theory, the authors [7, 9] obtained sufficient conditions which guarantee the existence of positive periodic solution of systems (1.1)-(1.2).

However, in real world phenomenon, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1) periods, then one has to consider the environment to be almost periodic [10] since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity.

Example 1. Let us consider the following simple population model:

$$\dot{N}(t) = N(t)\left[|\sin(\sqrt{2}t)| - |\sin(\sqrt{3}t)|N(t)\right]. \tag{1.3}$$

In Eq. (1.3), $|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that Eq. (1.3) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of Eq. (1.3). Thus, it is significant to study the existence of positive almost periodic solutions of Eq. (1.3).

In view of this, now we consider the following almost periodic predator-prey system with general functional response

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and time delays:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[b_1(t) - a_1(t)N_1^{q_1}(t-\mu_1(t)) \\ \qquad -\frac{\alpha_1(t)N_1^{p-1}(t)}{1+mN_1^p(t)}N_2(t-\nu(t))\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-b_2(t) - a_2(t)N_2^{q_2}(t-\mu_2(t)) \\ \qquad +\frac{\alpha_2(t)N_1^p(t-\mu_3(t))}{1+mN_1^p(t-\mu_3(t))}\Big], \end{cases} \tag{1.4}$$

where N_1 and N_2 represent the densities of the prey population and predator population, respectively, p is a positive constant and $p \geq 1$, m is a nonnegative constant and q_1, q_2 are positive constants. From systems (1.1)-(1.2), we know that the prey population and predator population obey the logistic growth. But many authors have suspected the reasonableness of the logistic equations [16,17]. Hence, some of them proposed single growth population models in succession, such as Gilpin model [16], Smith model [17] etc. Therefore, we consider system (1.4), which are more general and reasonable. Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively. Related to a continuous function f, we use the following notations:

$$f^l = \inf_{s\in\mathbb{R}} f(s), \quad f^M = \sup_{s\in\mathbb{R}} f(s),$$

$$|f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

Throughout this paper, we always make the following assumption for system (1.4):

(F_1) All the coefficients of system (1.4) are nonnegative almost periodic functions with $\bar{a}_i > 0$ and $\bar{b}_1 > 0$, $i = 1, 2$.

The initial conditions of system (1.4) are of the form

$$N_i(s) = \psi_i(s), \quad s \in [-\mu, 0], \quad \psi_i(0) > 0,$$

$$\psi_i \in C([-\mu, 0], [0, +\infty)), \quad i = 1, 2,$$

where $\mu := \max_{i=1,2,3}\{\mu_i^M, \nu^M\}$.

It is well known that Mawhin's continuation theorem of coincidence degree theory is an important method to investigate the existence of positive periodic solutions of some kinds of non-linear ecosystems (see [7-14]). However, it is difficult to be used to investigate the existence of positive almost periodic solutions of non-linear ecosystems. Therefore, to the best of the author's knowledge, so far, there are scarcely any papers concerning with the existence of positive almost periodic solutions of system (1.4). Motivated by the above reason, the main purpose of this paper is to establish some new sufficient conditions on the existence of positive almost periodic solutions of system (1.4) by using Mawhin's continuous theorem of coincidence degree theory.

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, we obtain some new sufficient conditions for the existence of at least one positive almost periodic solution of system (1.4) by way of Mawhin's continuous theorem of coincidence degree theory. Two illustrative examples and simulations are given in Section 4.

II. Preliminaries

Definition 1. ([18,19]) $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x,\epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x,\epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R}, \mathbb{R})$.

Lemma 1. *([18,19]) If $x \in AP(\mathbb{R})$, then x is bounded and uniformly continuous on $\mathbb{R}$.*

Lemma 2. *([18,19]) If $x \in AP(\mathbb{R})$, then $\int_0^t x(s)\,\mathrm{d}s \in AP(\mathbb{R})$ if and only if $\int_0^t x(s)\,\mathrm{d}s$ is bounded on $\mathbb{R}$.*

Lemma 3. *([21]) Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$. For arbitrary interval $[a,b]$ with $b - a = \omega > 0$, let $\xi, \eta \in [a,b]$ and*

$$I = \big\{s \in [\xi, b] : \dot{x}(s) \geq 0\big\}, \quad J = \big\{s \in [a, \eta] : \dot{x}(s) \geq 0\big\},$$

then ones have

$$x(t) \leq x(\xi) + \int_I \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b],$$

$$x(t) \geq x(\eta) - \int_J \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [a, \eta].$$

Lemma 4. *([21]) If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a,b]$ with $b - a = \omega > 0$, there exist $\xi \in [a,b]$, $\underline{\xi} \in (-\infty, a]$ and $\bar{\xi} \in [b, +\infty)$ such that*

$$x(\underline{\xi}) = x(\bar{\xi}) \quad \text{and} \quad x(\xi) \leq x(s), \ \forall s \in [\underline{\xi}, \bar{\xi}].$$

Lemma 5. *([21]) If $x \in AP(\mathbb{R})$, then for arbitrary interval $[a,b]$ with $I = b - a = \omega > 0$, there exist $\eta \in [a,b]$, $\underline{\eta} \in (-\infty, a]$ and $\bar{\eta} \in [b, +\infty)$ such that*

$$x(\underline{\eta}) = x(\bar{\eta}) \quad \text{and} \quad x(\eta) \geq x(s), \ \forall s \in [\underline{\eta}, \bar{\eta}].$$

Lemma 6. *([21]) If $x \in AP(\mathbb{R})$, then for $\forall n \in \mathbb{N}^+$, there exist $\alpha_n, \beta_n \in \mathbb{R}$ such that $x(\alpha_n) \in \big[x^* - \frac{1}{n}, x^*\big]$ and $x(\beta_n) \in \big[x_*, x_* + \frac{1}{n}\big]$, where $x^* = \sup_{s\in\mathbb{R}} x(s)$ and $x_* = \inf_{s\in\mathbb{R}} x(s)$.*

For $x \in AP(\mathbb{R})$, we denote by

$$\bar{x} = m(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s,$$

$$a(x, \varpi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)e^{-\mathrm{i}\varpi s}\,\mathrm{d}s,$$

$$\Lambda(x) = \left\{\varpi \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\}$$

the mean value and the set of Fourier exponents of x, respectively.

Lemma 7. *([21]) Assume that $x \in AP(\mathbb{R})$ and $\bar{x} > 0$, then for $\forall t_0 \in \mathbb{R}$, there exists a positive constant T_0 independent* *of* t_0 *such that*

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(s)\,\mathrm{d}s \in \left[\frac{\bar{x}}{2}, \frac{3\bar{x}}{2}\right], \quad \forall T \geq T_0.$$

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin ([20]).

Mawhin's Continuous Theorem. (*[20]) Let* $\Omega \subseteq \mathbb{X}$ *be an open bounded set,* L *be a Fredholm mapping of index zero and* N *be* L*-compact on* $\bar{\Omega}$*. If all the following conditions hold*:

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0,1)$;
(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;
(c) $\deg\{JQN, \Omega\cap\mathrm{Ker}L, 0\} \neq 0$, *where* $J: \mathrm{Im}Q \to \mathrm{Ker}L$ *is an isomorphism.*

Then $Lx = Nx$ *has a solution on* $\bar{\Omega} \cap \mathrm{Dom}L$.

Under the invariant transformation $(N_1, N_2)^T = (e^u, e^v)^T$, system (1.4) reduces to

$$\begin{cases} \dot{u}(t) = b_1(t) - a_1(t)e^{q_1 u(t-\mu_1(t))} \\ \qquad - \frac{\alpha_1(t)e^{(p-1)u(t)}}{1+me^{pu(t)}}e^{v(t-\nu(t))}, \\ \dot{v}(t) = -b_2(t) - a_2(t)e^{q_2 v(t-\mu_2(t))} \\ \qquad + \frac{\alpha_2(t)e^{pu(t-\mu_3(t))}}{1+me^{pu(t-\mu_3(t))}}. \end{cases} \tag{2.1}$$

Set $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \Big\{ z = (u,v)^T \in AP(\mathbb{R}, \mathbb{R}^2) : \forall \varpi \in \Lambda(u) \cup \Lambda(v), |\varpi| \geq \gamma_0 \Big\},$$

$$\mathbb{V}_2 = \big\{ z = (u,v)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbb{R} \big\},$$

where γ_0 is a given positive constant. Define the norm

$$\|z\|_{\mathbb{X}} = \max\left\{ \sup_{s\in\mathbb{R}} |u(s)|, \sup_{s\in\mathbb{R}} |v(s)| \right\}, \quad \forall z \in \mathbb{X} = \mathbb{Y}.$$

Lemma 8. (*[21]) Let* $x \in AP(\mathbb{R})$*. For* $\forall \varpi \in \Lambda(x)$ *with* $|\varpi| \geq \gamma_0 > 0$*, then* $\int_0^t x(s)\,\mathrm{d}s \in AP(\mathbb{R})$.

Lemma 9. (*[21]) Let* $f \in AP(\mathbb{R})$ *such that* $f(t) \sim \sum a(f, \lambda_n)e^{\mathrm{i}\lambda_n t}$ *with* $|\lambda_n| \geq \gamma_0 > 0$*. If* g *is the integral of* f *with* $a(g, 0) = 0$*, then there exists a constant* D *independent of* f*,* g *and* γ_0 *such that* $|g|_\infty \leq D|f|_\infty$.

Lemma 10. (*[21])* $\mathbb{X}$ *and* $\mathbb{Y}$ *are Banach spaces endowed with* $\|\cdot\|_{\mathbb{X}}$.

Lemma 11. (*[21]) Let* $L: \mathbb{X} \to \mathbb{Y}$*,* $Lz = L(u,v)^T = (\dot{u}, \dot{v})^T$*, then* L *is a Fredholm mapping of index zero.*

Lemma 12. (*[21]) Define* $N: \mathbb{X} \to \mathbb{Y}$*,* $P: \mathbb{X} \to \mathbb{X}$ *and* $Q: \mathbb{Y} \to \mathbb{Y}$ *by*

$$Nz = N\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1(t) - a_1(t)e^{q_1 u(t-\mu_1(t))} \\ -\frac{\alpha_1(t)e^{(p-1)u(t)}}{1+me^{pu(t)}}e^{v(t-\nu(t))} \\ -b_2(t) - a_2(t)e^{q_2 v(t-\mu_2(t))} \\ +\frac{\alpha_2(t)e^{pu(t-\mu_3(t))}}{1+me^{pu(t-\mu_3(t))}} \end{pmatrix},$$

$$Pz = P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m(u) \\ m(v) \end{pmatrix} = Qz, \quad \forall z \in \mathbb{X} = \mathbb{Y}.$$

Then N *is* L*-compact on* $\bar{\Omega}$*(*Ω *is an open and bounded subset of* $\mathbb{X}$*).*

III. Main results

Now we are in the position to present and prove our result on the existence of at least one positive almost periodic solution for system (1.4).

Take

$$l_0 := \max\{\sup_{s\in\mathbb{R}} \mu_1(s), \sup_{s\in\mathbb{R}} \mu_2(s)\}.$$

From (F_1) and Lemma 7, for $\forall k \in \mathbb{R}$, there exists a constant $\omega_0 \in (2l_0, +\infty)$ independent of k such that

$$\begin{aligned} &\frac{1}{T}\int_k^{k+T} a_i(s)\,\mathrm{d}s \in \left[\frac{\bar{a}_i}{2}, \frac{3\bar{a}_i}{2}\right], \\ &\frac{1}{T}\int_k^{k+T} b_1(s)\,\mathrm{d}s \in \left[\frac{\bar{b}_1}{2}, \frac{3\bar{b}_1}{2}\right], \end{aligned} \tag{3.1}$$

where $\forall T \geq \frac{\omega_0}{2}$, $i = 1, 2$.

Let

$$\rho_1 := \ln\left[\frac{6\bar{b}_1}{\bar{a}_1}\right]^{\frac{1}{q_1}} + b_1^M \omega_0,$$

$$\rho_2 := \ln\left[\frac{4\alpha_2^M e^{p\rho_1}}{(1+me^{p\rho_1})\bar{a}_2}\right]^{\frac{1}{q_2}} + \frac{\alpha_2^M e^{p\rho_1}\omega_0}{1+me^{p\rho_1}}.$$

Therefore, we may introduce a assumption as follows:

(F_2) $\bar{\Phi}_1 := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left[b_1(s) - e^{(p-1)\rho_1+\rho_2}\alpha_1(s)\right]\mathrm{d}s > 0$.

Similar to (3.1), for $\forall k \in \mathbb{R}$, there exists a constant $\tau_0 \in (\omega_0, +\infty)$ independent of k such that

$$\frac{1}{T}\int_k^{k+T} \Phi_1(s)\,\mathrm{d}s \in \left[\frac{\bar{\Phi}_1}{2}, \frac{3\bar{\Phi}_1}{2}\right], \quad \forall T \geq \tau_0. \tag{3.2}$$

Theorem 1. *Assume that* (F_1)*,* (F_2) *and the following condition hold:*

(F_3) $\bar{\Phi}_2 := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left[\frac{e^{p\rho_3}}{1+me^{p\rho_3}}\alpha_2(s) - b_2(s)\right]\mathrm{d}s > 0$*, where*

$$\rho_3 := \ln\left[\frac{\bar{\Phi}_1}{4a_1^M}\right]^{\frac{1}{q_1}} - b_1^M \pi_0,$$

$$\pi_0 := \max\left\{\tau_0, \frac{4a_1^M e^{q_1\rho_1} l_0}{\bar{\Phi}_1}\right\}.$$

Then system (1.4) *admits at least one positive almost periodic solution.*

Proof: It is easy to see that if system (2.1) has one almost periodic solution $(\bar{u}, \bar{v})^T$, then $(\bar{N}_1, \bar{N}_2)^T$=$(e^{\bar{u}}, e^{\bar{v}})^T$ is a positive almost periodic solution of system (1.4). Therefore, to complete the proof it suffices to show that system (2.1) has one almost periodic solution.

In order to use the Mawhin's continuous theorem, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 10 and L, N, P, Q the same as those defined in Lemmas 11 and 12, respectively. It remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Corresponding to the operator equation $Lz = \lambda z$, $\lambda \in$

$(0,1)$, we have

$$\begin{cases} \dot{u}(t) = \lambda\Big[b_1(t) - a_1(t)e^{q_1 u(t-\mu_1(t))} \\ \qquad -\frac{\alpha_1(t)e^{(p-1)u(t)}}{1+me^{pu(t)}}e^{v(t-\nu(t))}\Big], \\ \dot{v}(t) = \lambda\Big[-b_2(t) - a_2(t)e^{q_2 v(t-\mu_2(t))} \\ \qquad +\frac{\alpha_2(t)e^{pu(t-\mu_3(t))}}{1+me^{pu(t-\mu_3(t))}}\Big]. \end{cases} \tag{3.3}$$

Suppose that $(u,v)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.3) for some $\lambda \in (0,1)$, where $\mathrm{Dom}L = \{z = (u,v)^T \in \mathbb{X} : u, v \in C^1(\mathbb{R}), \dot{u}, \dot{v} \in C(\mathbb{R})\}$. By Lemma 6, there exist two sequences $\{T_n : n \in \mathbb{N}^+\}$ and $\{P_n : n \in \mathbb{N}^+\}$ such that

$$u(T_n) \in \left[u^* - \frac{1}{n}, u^*\right], \quad u^* = \sup_{s\in\mathbb{R}} u(s), \quad n \in \mathbb{N}^+, \tag{3.4}$$

$$v(P_n) \in \left[v^* - \frac{1}{n}, v^*\right], \quad v^* = \sup_{s\in\mathbb{R}} v(s), \quad n \in \mathbb{N}^+. \tag{3.5}$$

For $\forall n_0 \in \mathbb{N}^+$, we consider $[T_{n_0} - \omega_0, T_{n_0}]$ and $[P_{n_0} - \omega_0, P_{n_0}]$, where ω_0 is defined as that in (3.1). By Lemma 4, there exist $\xi \in [T_{n_0} - \omega_0, T_{n_0}]$, $\underline{\xi} \in (-\infty, T_{n_0} - \omega_0]$ and $\bar{\xi} \in [T_{n_0}, +\infty)$ such that

$$u(\underline{\xi}) = u(\bar{\xi}) \quad \text{and} \quad u(\xi) \le u(s), \quad \forall s \in [\bar{\xi}, \underline{\xi}]. \tag{3.6}$$

Integrating the first equation of system (3.3) from $\underline{\xi}$ to $\bar{\xi}$ leads to

$$\int_{\underline{\xi}}^{\bar{\xi}} \Big(b_1(s) - a_1(s)e^{q_1 u(s-\mu_1(s))} - \frac{\alpha_1(s)e^{(p-1)u(s)}}{1+me^{pu(s)}}e^{v(s-\nu(s))}\Big)\,\mathrm{d}s = 0,$$

which yields that

$$\int_{\underline{\xi}+l_0}^{\bar{\xi}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \le \int_{\underline{\xi}}^{\bar{\xi}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \le \int_{\underline{\xi}}^{\bar{\xi}} b_1(s)\,\mathrm{d}s.$$

By the integral mean value theorem and (3.1), there exists $s_0 \in [\underline{\xi} + l_0, \bar{\xi}]$ $(s_0 - \mu_1(s_0) \in [\underline{\xi}, \bar{\xi}])$ such that

$$\begin{aligned} &\frac{\bar{a}_1}{4}e^{q_1 u(s_0-\mu_1(s_0))} \\ &\le \frac{\bar{\xi} - \underline{\xi} - l_0}{\bar{\xi} - \underline{\xi}}\frac{\bar{a}_1}{2}e^{q_1 u(s_0-\mu_1(s_0))} \\ &\le \frac{\bar{\xi} - \underline{\xi} - l_0}{\bar{\xi} - \underline{\xi}}e^{q_1 u(s_0-\mu_1(s_0))}\frac{1}{\bar{\xi} - \underline{\xi} - l_0}\int_{\underline{\xi}+l_0}^{\bar{\xi}} a_1(s)\,\mathrm{d}s \\ &= \frac{1}{\bar{\xi} - \underline{\xi}}\int_{\underline{\xi}+l_0}^{\bar{\xi}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \\ &\le \frac{1}{\bar{\xi} - \underline{\xi}}\int_{\underline{\xi}}^{\bar{\xi}} b_1(s)\,\mathrm{d}s \\ &\le \frac{3\bar{b}_1}{2}, \end{aligned}$$

which implies from (3.6) that

$$u(\xi) \le \ln\left[\frac{6\bar{b}_1}{\bar{a}_1}\right]^{\frac{1}{q_1}}. \tag{3.7}$$

Let $I_1 = \{s \in [\xi, T_{n_0}] : \dot{u}(s) \ge 0\}$. It follows from system (3.3) that

$$\begin{aligned} \int_{I_1} \dot{u}(s)\,\mathrm{d}s &= \int_{I_1} \lambda\Big[b_1(s) - a_1(s)e^{q_1 u(s-\mu_1(s))} \\ &\qquad -\frac{\alpha_1(s)e^{(p-1)u(s)}}{1+me^{pu(s)}}e^{v(s-\nu(s))}\Big]\,\mathrm{d}s \\ &\le \int_{I_1} \lambda b_1(s)\,\mathrm{d}s \le \int_{T_{n_0}-\omega_0}^{T_{n_0}} b_1(s)\,\mathrm{d}s \\ &\le b_1^M\omega_0. \end{aligned} \tag{3.8}$$

By Lemma 3, it follows from (3.7)-(3.8) that

$$u(t) \le u(\xi) + \int_{I_1} \dot{u}(s)\,\mathrm{d}s \le \ln\left[\frac{6\bar{b}_1}{\bar{a}_1}\right]^{\frac{1}{q_1}} + b_1^M\omega_0 := \rho_1,$$

$\forall t \in [\xi, T_{n_0}]$, which implies that

$$u(T_{n_0}) \le \rho_1.$$

In view of (3.4), letting $n_0 \to +\infty$ in the above inequality leads to

$$u^* = \lim_{n_0\to+\infty} u(T_{n_0}) \le \rho_1. \tag{3.9}$$

Also, by Lemma 4, there exist $\zeta \in [P_{n_0} - \omega_0, P_{n_0}]$, $\underline{\zeta} \in (-\infty, P_{n_0} - \omega_0]$ and $\bar{\zeta} \in [P_{n_0}, +\infty)$ such that

$$v(\underline{\zeta}) = v(\bar{\zeta}) \quad \text{and} \quad v(\zeta) \le v(s), \ \forall s \in [\underline{\zeta}, \bar{\zeta}]. \tag{3.10}$$

Integrating the second equation of system (3.3) from $\underline{\zeta}$ to $\bar{\zeta}$ leads to

$$\int_{\underline{\zeta}}^{\bar{\zeta}} \Big(-b_2(s) - a_2(s)e^{q_2 v(s-\mu_2(s))} + \frac{\alpha_2(s)e^{pu(s-\mu_3(s))}}{1+me^{pu(s-\mu_3(s))}}\Big)\,\mathrm{d}s = 0, \tag{3.11}$$

which yields that

$$\int_{\underline{\zeta}+l_0}^{\bar{\zeta}} a_2(s)e^{q_2 v(s-\mu_2(s))}\,\mathrm{d}s \le \int_{\underline{\zeta}}^{\bar{\zeta}} a_2(s)e^{q_2 v(s-\mu_2(s))}\,\mathrm{d}s \le \int_{\underline{\zeta}}^{\bar{\zeta}} \frac{\alpha_2(s)e^{pu(s-\mu_3(s))}}{1+me^{pu(s-\mu_3(s))}}\,\mathrm{d}s.$$

By a similar argument as that in (3.7), there exists $c_0 \in [\underline{\zeta} + l_0, \bar{\zeta}]$ $(c_0 - \mu_2(c_0) \in [\underline{\zeta}, \bar{\zeta}])$ such that

$$\begin{aligned} \frac{\bar{a}_2}{4}e^{q_2 v(c_0-\mu_2(c_0))} &\le \frac{1}{\bar{\zeta} - \underline{\zeta}}\int_{\underline{\zeta}+l_0}^{\bar{\zeta}} a_2(s)e^{q_2 v(s-\mu_2(s))}\,\mathrm{d}s \\ &\le \frac{1}{\bar{\zeta} - \underline{\zeta}}\int_{\underline{\zeta}}^{\bar{\zeta}} \frac{\alpha_2(s)e^{pu(s-\mu_3(s))}}{1+me^{pu(s-\mu_3(s))}}\,\mathrm{d}s \\ &\le \frac{\alpha_2^M e^{p\rho_1}}{1+me^{p\rho_1}}, \end{aligned}$$

which implies from (3.10) that

$$v(\zeta) \le \ln\left[\frac{4\alpha_2^M e^{p\rho_1}}{(1+me^{p\rho_1})\bar{a}_2}\right]^{\frac{1}{q_2}}. \tag{3.12}$$

Let $I_2 = \{s \in [\zeta, P_{n_0}] : \dot{v}(s) \geq 0\}$. It follows from system (3.3) that

$$\begin{aligned}\int_{I_2} \dot{v}(s)\,\mathrm{d}s &= \int_{I_2} \lambda\Big[-b_2(s) - a_2(s)e^{q_2 v(s-\mu_2(s))} \\ &\quad + \frac{\alpha_2(s)e^{pu(s-\mu_3(s))}}{1+me^{pu(s-\mu_3(s))}}\Big]\,\mathrm{d}s \\ &\leq \int_{I_2} \lambda \frac{\alpha_2(s)e^{pu(s-\mu_3(s))}}{1+me^{pu(s-\mu_3(s))}}\,\mathrm{d}s \\ &\leq \int_{P_{n_0}-\omega_0}^{P_{n_0}} \frac{\alpha_2(s)e^{pu(s-\mu_3(s))}}{1+me^{pu(s-\mu_3(s))}}\,\mathrm{d}s \\ &\leq \frac{\alpha_2^M e^{p\rho_1}\omega_0}{1+me^{p\rho_1}}. \end{aligned} \tag{3.13}$$

By Lemma 3, it follows from (3.12)-(3.13) that

$$\begin{aligned} v(t) &\leq v(\zeta) + \int_{I_2} \dot{v}(s)\,\mathrm{d}s \\ &\leq \ln\left[\frac{4\alpha_2^M e^{p\rho_1}}{(1+me^{p\rho_1})\bar{a}_2}\right]^{\frac{1}{q_2}} + \frac{\alpha_2^M e^{p\rho_1}\omega_0}{1+me^{p\rho_1}} \\ &:= \rho_2, \quad \forall t \in [\zeta, P_{n_0}], \end{aligned}$$

which implies that

$$v(P_{n_0}) \leq \rho_2.$$

In view of (3.5), letting $n_0 \to +\infty$ in the above inequality leads to

$$v^* = \lim_{n_0 \to +\infty} v(P_{n_0}) \leq \rho_2. \tag{3.14}$$

On the other hand, by Lemma 6, there exists a sequence $\{H_n : n \in \mathbb{N}^+\}$ such that

$$u(H_n) \in \left[u_*, u_* + \frac{1}{n}\right],\ u_* = \inf_{s\in\mathbb{R}} u(s),\ n \in \mathbb{N}^+. \tag{3.15}$$

For $\forall n_0 \in \mathbb{N}^+$, we consider $[H_{n_0}, H_{n_0} + \pi_0]$. By Lemma 4, there exist $\eta \in [H_{n_0}, H_{n_0} + \pi_0]$, $\underline{\eta} \in (-\infty, H_{n_0}]$ and $\bar{\eta} \in [H_{n_0}, +\infty)$ such that

$$u(\underline{\eta}) = u(\bar{\eta}) \quad \text{and} \quad u(\eta) \geq u(s),\ \forall s \in [\underline{\eta}, \bar{\eta}]. \tag{3.16}$$

Integrating the first equation of system (3.3) from $\underline{\eta}$ to $\bar{\eta}$ leads to

$$\begin{aligned}&\int_{\underline{\eta}}^{\bar{\eta}} \Big(b_1(s) - a_1(s)e^{q_1 u(s-\mu_1(s))} \\ &\quad - \frac{\alpha_1(s)e^{(p-1)u(s)}}{1+me^{pu(s)}}e^{v(s-\nu(s))}\Big)\,\mathrm{d}s = 0, \end{aligned}\tag{3.17}$$

which yields from (3.9) and (3.14) that

$$\begin{aligned}&\int_{\underline{\eta}}^{\bar{\eta}} \left[b_1(s) - \alpha_1(s)e^{(p-1)\rho_1}e^{\rho_2}\right]\mathrm{d}s \\ &\leq \int_{\underline{\eta}}^{\bar{\eta}} \left[b_1(s) - \frac{\alpha_1(s)e^{(p-1)u(s)}}{1+me^{pu(s)}}e^{v(s-\nu(s))}\right]\mathrm{d}s \\ &= \int_{\underline{\eta}}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \\ &= \int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \\ &\quad + \int_{\underline{\eta}}^{\underline{\eta}+l_0} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \\ &\leq \int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s + a_1^M e^{q_1\rho_1} l_0, \end{aligned}$$

which implies from (3.2) that

$$\begin{aligned}\frac{\bar{\Phi}_1}{2} &\leq \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} \left[b_1(s) - \alpha_1(s)e^{(p-1)\rho_1}e^{\rho_2}\right]\mathrm{d}s \\ &\leq \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s + \frac{a_1^M e^{q_1\rho_1} l_0}{\bar{\eta}-\underline{\eta}} \\ &\leq \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s + \frac{a_1^M e^{q_1\rho_1} l_0}{\pi_0} \\ &\leq \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s + \frac{\bar{\Phi}_1}{4}. \end{aligned}\tag{3.18}$$

In view of (3.18), by the integral mean value theorem and (3.16), there exists $s_1 \in [\underline{\eta}+l_0, \bar{\eta}]\,(s_1 - \mu_1(s_1) \in [\underline{\eta}, \bar{\eta}])$ such that

$$\begin{aligned}\frac{\bar{\Phi}_1}{4} &\leq \frac{e^{q_1 u(s_1-\mu_1(s_1))}}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)\,\mathrm{d}s \\ &\leq a_1^M e^{q_1 u(\eta)}\frac{\bar{\eta}-\underline{\eta}-l_0}{\bar{\eta}-\underline{\eta}} \leq a_1^M e^{q_1 u(\eta)}, \end{aligned}\tag{3.19}$$

which implies that

$$u(\eta) \geq \ln\left[\frac{\bar{\Phi}_1}{4a_1^M}\right]^{\frac{1}{q_1}}. \tag{3.20}$$

Let $J = \{s \in [H_{n_0}, \eta] : \dot{u}(s) \geq 0\}$. It follows from system (3.3) that

$$\begin{aligned}\int_J \dot{u}(s)\,\mathrm{d}s &= \int_J \lambda\Big[b_1(s) - a_1(s)e^{q_1 u(s-\mu_1(s))} \\ &\quad - \frac{\alpha_1(s)e^{(p-1)u(s)}}{1+me^{pu(s)}}e^{v(s-\nu(s))}\Big]\,\mathrm{d}s \\ &\leq \int_J \lambda b_1(s)\,\mathrm{d}s \leq \int_{H_{n_0}}^{H_{n_0}+\pi_0} b_1(s)\,\mathrm{d}s \\ &\leq b_1^M \pi_0. \end{aligned}\tag{3.21}$$

By Lemma 3, it follows from (3.19)-(3.20) that

$$\begin{aligned} u(t) &\geq u(\eta) - \int_J \dot{u}(s)\,\mathrm{d}s \geq \ln\left[\frac{\bar{\Phi}_1}{4a_1^M}\right]^{\frac{1}{q_1}} - b_1^M\pi_0 \\ &:= \rho_3, \quad \forall t \in [H_{n_0}, \eta], \end{aligned}\tag{3.22}$$

which implies that

$$u(H_{n_0}) \geq \rho_3.$$

In view of (3.16), letting $n_0 \to +\infty$ in the above inequality leads to

$$u_* = \lim_{n_0 \to +\infty} u(H_{n_0}) \geq \rho_3. \tag{3.23}$$

Take

$$\sigma_1 := \max\left\{\pi_0, \frac{4a_2^M e^{q_2\rho_2} l_0}{\bar{\Phi}_2}\right\}.$$

From (F_3) and Lemma 7, for $\forall k \in \mathbb{R}$, there exists a constant $\sigma_0 \in (\sigma_1, +\infty)$ independent of k such that

$$\frac{1}{T}\int_k^{k+T} \Phi_2(s)\,\mathrm{d}s \in \left[\frac{\bar{\Phi}_2}{2}, \frac{3\bar{\Phi}_2}{2}\right], \quad \forall T \geq \sigma_0. \tag{3.24}$$

Also, there exist $\varsigma \in [n_0\sigma_0, n_0\sigma_0 + \sigma_0] (\forall n_0 \in \mathbb{Z})$, $\underline{\varsigma} \in$

$(-\infty, n_0\sigma_0]$ and $\bar{\varsigma} \in [n_0\sigma_0 + \sigma_0, +\infty)$ such that

$$v(\underline{\varsigma}) = v(\bar{\varsigma}) \quad \text{and} \quad v(\varsigma) \geq v(s), \quad \forall s \in [\underline{\varsigma}, \bar{\varsigma}]. \quad (3.25)$$

Integrating the second equation of system (3.3) from $\underline{\varsigma}$ to $\bar{\varsigma}$ leads to

$$\int_{\underline{\varsigma}}^{\bar{\varsigma}} \bigg(- b_2(s) - a_2(s) e^{q_2 v(s-\mu_2(s))} + \frac{\alpha_2(s) e^{pu(s-\mu_3(s))}}{1 + m e^{pu(s-\mu_3(s))}} \bigg) \, \mathrm{d}s = 0,$$

which yields that

$$\begin{aligned} &\int_{\underline{\varsigma}}^{\bar{\varsigma}} \bigg[\frac{e^{p\rho_3}}{1 + m e^{p\rho_3}} \alpha_2(s) - b_2(s) \bigg] \mathrm{d}s \\ \leq &\int_{\underline{\varsigma}}^{\bar{\varsigma}} \bigg[- b_2(s) + \frac{\alpha_2(s) e^{pu(s-\mu_3(s))}}{1 + m e^{pu(s-\mu_3(s))}} \bigg] \mathrm{d}s \\ = &\int_{\underline{\varsigma}}^{\bar{\varsigma}} a_2(s) e^{q_2 v(s-\mu_2(s))} \, \mathrm{d}s \\ = &\int_{\underline{\varsigma}+l_0}^{\bar{\varsigma}} a_2(s) e^{q_2 v(s-\mu_2(s))} \, \mathrm{d}s + \int_{\underline{\varsigma}}^{\underline{\varsigma}+l_0} a_2(s) e^{q_2 v(s-\mu_2(s))} \, \mathrm{d}s \\ \leq &\int_{\underline{\varsigma}+l_0}^{\bar{\varsigma}} a_2(s) e^{q_2 v(s-\mu_2(s))} \, \mathrm{d}s + a_2^M e^{q_2\rho_2} l_0. \end{aligned}$$

Similar to the argument as that in (3.18), we obtain that

$$\frac{\bar{\Phi}_2}{2} \leq \frac{1}{\bar{\varsigma} - \underline{\varsigma}} \int_{\underline{\varsigma}+l_0}^{\bar{\varsigma}} a_2(s) e^{q_2 v(s-\mu_2(s))} \, \mathrm{d}s + \frac{\bar{\Phi}_2}{4}. \quad (3.26)$$

In view of (3.24), by the integral mean value theorem and (3.23), there exists $c_1 \in [\underline{\varsigma} + l_0, \bar{\varsigma}]$ ($c_1 - \mu_2(c_1) \in [\underline{\varsigma}, \bar{\varsigma}]$) such that

$$\begin{aligned} \frac{\bar{\Phi}_2}{4} &\leq \frac{e^{q_2 v(c_1-\mu_2(c_1))}}{\bar{\varsigma} - \underline{\varsigma}} \int_{\underline{\varsigma}+l_0}^{\bar{\varsigma}} a_2(s) \, \mathrm{d}s \\ &\leq a_2^M e^{q_2 v(\varsigma)} \frac{\bar{\varsigma} - \underline{\varsigma} - l_0}{\bar{\varsigma} - \underline{\varsigma}} \\ &\leq a_2^M e^{q_2 v(\varsigma)}, \end{aligned}$$

which implies that

$$v(\varsigma) \geq \ln \bigg[\frac{\bar{\Phi}_2}{4 a_2^M} \bigg]^{\frac{1}{q_2}}. \quad (3.27)$$

Further, we obtain from system (3.3) that

$$\begin{aligned} &\int_{n_0\sigma_0}^{n_0\sigma_0+\sigma_0} |\dot{v}(s)| \, \mathrm{d}s \\ = &\int_{n_0\sigma_0}^{n_0\sigma_0+\sigma_0} \lambda \bigg| - b_2(s) - a_2(s) e^{q_2 v(s-\mu_2(s))} + \frac{\alpha_2(s) e^{pu(s-\mu_3(s))}}{1 + m e^{pu(s-\mu_3(s))}} \bigg| \mathrm{d}s \\ \leq &[b_2^M + a_2^M e^{q_2\rho_2} + \alpha_2^M e^{p\rho_1}]\sigma_0 := \Theta_2. \quad (3.28) \end{aligned}$$

It follows from (3.25)-(3.26) that

$$\begin{aligned} v(t) &\geq v(\varsigma) - \int_{n_0\sigma_0}^{n_0\sigma_0+\sigma_0} |\dot{v}(s)| \, \mathrm{d}s \\ &\geq \ln \bigg[\frac{\bar{\Phi}_2}{4 a_2^M} \bigg]^{\frac{1}{q_2}} - \Theta_2 \\ &:= \rho_4, \quad \forall t \in [n_0\sigma_0, n_0\sigma_0 + \sigma_0]. \quad (3.29) \end{aligned}$$

Obviously, ρ_4 is a constant independent of n_0. So it follows from (3.27) that

$$\begin{aligned} v_* = \inf_{s\in\mathbb{R}} v(s) &= \inf_{n_0\in\mathbb{Z}} \Big\{ \min_{s\in[n_0\sigma_0, n_0\sigma_0+\sigma_0]} v(s) \Big\} \\ &\geq \inf_{n_0\in\mathbb{Z}} \{\rho_4\} = \rho_4. \quad (3.30) \end{aligned}$$

Set $K = |\rho_1| + |\rho_2| + |\rho_3| + |\rho_4| + 1$, then $\|z\|_{\mathbb{X}} = \|(u, v)^T\| < K$. Clearly, K is independent of $\lambda \in (0, 1)$. Consider the algebraic equations $QNz_0 = 0$ for $z_0 = (u^0, v^0)^T \in \mathbb{R}^2$ as follows:

$$\begin{cases} 0 = \bar{b}_1 - \bar{a}_1 e^{q_1 u^0} - \frac{\bar{\alpha}_1 e^{(p-1)u^0}}{1+me^{pu^0}} e^{v^0}, \\ 0 = -\bar{b}_2 - \bar{a}_2 e^{q_2 v^0} + \frac{\bar{\alpha}_2 e^{pu^0}}{1+me^{pu^0}}. \end{cases}$$

Similar to the arguments as that in (3.9), (3.14), (3.21) and (3.28), we can easily obtain that

$$\rho_3 \leq u^0 \leq \rho_1, \quad \rho_4 \leq v^0 \leq \rho_2.$$

Then $\|z_0\|_{\mathbb{X}} = |u^0| + |v^0| < K$. Let $\Omega = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < K\}$, then Ω satisfies conditions (a) and (b) of Mawhin's continuous theorem.

Finally, we will show that condition (c) of Mawhin's continuous theorem is satisfied. Let us consider the homotopy

$$H(\iota, z) = \iota QNz + (1 - \iota)Fz, \quad (\iota, z) \in [0, 1] \times \mathbb{R}^2,$$

where

$$Fz = F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \bar{b}_1 - \bar{a}_1 e^{q_1 u} \\ -\bar{b}_2 - \bar{a}_2 e^{q_2 v} + \frac{\bar{\alpha}_2 e^{pu}}{1+me^{pu}} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, z) \neq 0$ on $\partial\Omega \cap \mathrm{Ker}L$. By the invariance property of homotopy, we have

$$\begin{aligned} \deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) &= \deg\big(QN, \Omega \cap \mathrm{Ker}L, 0\big) \\ &= \deg\big(F, \Omega \cap \mathrm{Ker}L, 0\big), \end{aligned}$$

where $\deg(\cdot, \cdot, \cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$.

Note that the equations of the following system

$$\begin{cases} \bar{b}_1 - \bar{a}_1 e^{q_1 u} = 0, \\ -\bar{b}_2 - \bar{a}_2 e^{q_2 v} + \frac{\bar{\alpha}_2 e^{pu}}{1+me^{pu}} = 0 \end{cases}$$

has a solution:

$$(u^*, v^*) = \left(\ln \bigg[\frac{\bar{b}_1}{\bar{a}_1} \bigg]^{\frac{1}{q_1}}, \ln \bigg[\frac{\frac{\bar{\alpha}_2 e^{pu^*}}{1+me^{pu^*}} - \bar{b}_2}{\bar{a}_2} \bigg]^{\frac{1}{q_2}} \right) \in \Omega.$$

It follows that

$$\begin{aligned} &\deg\big(JQN, \Omega \cap \mathrm{Ker}L, 0\big) \\ = &\deg\big(F, \Omega \cap \mathrm{Ker}L, 0\big) \\ = &\,\mathrm{sign} \begin{vmatrix} -q_1 \bar{a}_1 e^{q_1 u} & 0 \\ \frac{\mathrm{d}}{\mathrm{d}u} \Big[\frac{\bar{\alpha}_2 e^{pu}}{1 + m e^{pu}} \Big] & -q_2 \bar{a}_2 e^{q_2 v} \end{vmatrix}_{(u,v)=(u^*,v^*)} \\ = &\,\mathrm{sign}\big(e^{q_1 u^*} e^{q_2 v^*}\big) \\ = &\,1. \end{aligned}$$

Obviously, all the conditions of Mawhin's continuous theorem are satisfied. Therefore, system (2.1) has one almost periodic solution, that is, system (1.4) has at least one positive almost periodic solution. This completes the proof. ■

Theorem 2. *Assume that* (F_1) *and the following conditions hold:*

(F_4) $p > 1 + me^{p\rho_1}$.

(F_5) $\bar{\Psi}_1 := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left[b_1(s) - \frac{e^{(p-1)\rho_1+\rho_2}}{1+me^{p\rho_1}}\alpha_1(s)\right] \mathrm{d}s > 0$.

(F_6) $\bar{\Psi}_2 := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left[\frac{e^{p\tilde{\rho}_3}}{1+me^{p\tilde{\rho}_3}}\alpha_2(s) - b_2(s)\right] \mathrm{d}s > 0$, *where*

$$\tilde{\rho}_3 := \ln\left[\frac{\bar{\Psi}_1}{4a_1^M}\right]^{\frac{1}{q_1}} - b_1^M\tilde{\pi}_0,$$

$$\tilde{\pi}_0 := \max\left\{\tau_0, \frac{4a_1^M e^{q_1\rho_1} l_0}{\bar{\Psi}_1}\right\}.$$

Then system (1.4) *admits at least one positive almost periodic solution.*

Proof: Let

$$L(x) = \frac{x^{p-1}}{1+mx^p}, \quad \forall x \in (0, e^{\rho_1}].$$

By (F_4), we are easily obtain that

$$\dot{L}(x) = \frac{x^{p-2}(p-1-mx^p)}{(1+mx^p)^2} > 0, \quad \forall x \in (0, e^{\rho_1}],$$

which implies that

$$\max_{x\in(0,e^{\rho_1}]} L(x) = \frac{e^{(p-1)\rho_1}}{1+me^{p\rho_1}}. \tag{3.31}$$

By the same arguments as that in Theorem 1, we have (3.10), (3.15)-(3.18). In view of (3.18), it follows from (3.10), (3.15) and (3.30) that

$$\begin{aligned}\frac{\bar{\Psi}_1}{2} &\le \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} \left[b_1(s) - \alpha_1(s)\frac{e^{(p-1)\rho_1+\rho_2}}{1+me^{p\rho_1}}\right] \mathrm{d}s \\ &\le \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} \left[b_1(s) - \frac{\alpha_1(s)e^{(p-1)u(s)}}{1+me^{pu(s)}}e^{v(s-\nu(s))}\right] \mathrm{d}s \\ &= \int_{\underline{\eta}}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s \\ &\le \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+l_0}^{\bar{\eta}} a_1(s)e^{q_1 u(s-\mu_1(s))}\,\mathrm{d}s + \frac{\bar{\Psi}_1}{4}.\end{aligned}$$

Similar to the argument as that in (3.20), we have

$$u(\eta) \ge \ln\left[\frac{\bar{\Psi}_1}{4a_1^M}\right]^{\frac{1}{q_1}}.$$

The remaining proof is similar to Theorem 1, so we omit it. This completes the proof. ■

From the proves of Theorems 1-2, we can show that

Corollary 1. *Assume that* (F_1)-(F_3) *hold. Suppose further that* a_i, b_i, α_i, μ_j *and* ν *are continuous nonnegative periodic functions with different periods,* $i = 1, 2$, $j = 1, 2, 3$, *then system* (1.1) *admits at least one positive almost periodic solution.*

Remark 1. By Corollary 1, it is easy to obtain the existence of at least one positive almost periodic solution of Eq. (1.3) in Example 1, although there is no a priori reason to expect the existence of positive periodic solutions of Eq. (1.3).

Corollary 2. *Assume that* (F_1), (F_4)-(F_6) *hold. Suppose further that* a_i, b_i, α_i, μ_j *and* ν *are continuous nonnegative periodic functions with different periods,* $i = 1, 2$, $j = 1, 2, 3$, *then system* (1.1) *admits at least one positive almost periodic solution.*

Assume that all coefficients of system (1.4) are ω-periodic functions, let

$$\hat{\rho}_1 := \ln\left[\frac{\bar{b}_1}{\bar{a}_1}\right]^{\frac{1}{q_1}} + \bar{b}_1\omega,$$

$$\hat{\rho}_2 := \ln\left[\frac{\alpha_2^u e^{p\hat{\rho}_1}}{(1+me^{p\hat{\rho}_1})\bar{a}_2}\right]^{\frac{1}{q_2}} + \min\left\{(\bar{B}_2+\bar{b}_2)\omega, \frac{\bar{\alpha}_2 e^{p\hat{\rho}_1}\omega}{1+me^{p\hat{\rho}_1}}\right\}.$$

From the proves of Theorems 1-2, we can show that

Corollary 3. *Assume that* (F_1) *and the following conditions hold:*

(F_7) $\bar{b}_1 > e^{(p-1)\hat{\rho}_1+\hat{\rho}_2}\bar{\alpha}_1$,

(F_8) $e^{p\hat{\rho}_3}\bar{\alpha}_2 > (1+me^{p\hat{\rho}_3})\bar{b}_2$, *where*

$$\hat{\rho}_3 := \ln\left[\frac{\bar{b}_1 - e^{(p-1)\hat{\rho}_1+\hat{\rho}_2}\bar{\alpha}_1}{\bar{a}_1}\right]^{\frac{1}{q_1}} - \bar{b}_1\omega.$$

Then system (1.4) *admits at least one positive* ω*-periodic solution.*

Corollary 4. *Assume that* (F_1) *and the following conditions hold:*

(F_9) $p > 1 + me^{p\hat{\rho}_1}$,

(F_{10}) $(1+me^{p\hat{\rho}_1})\bar{b}_1 > e^{(p-1)\hat{\rho}_1+\hat{\rho}_2}\bar{\alpha}_1$,

(F_{11}) $e^{p\vec{\rho}_3}\bar{\alpha}_2 > (1+me^{p\vec{\rho}_3})\bar{b}_2$, *where*

$$\vec{\rho}_3 := \ln\left[\frac{\bar{b}_1 - \frac{e^{(p-1)\hat{\rho}_1+\hat{\rho}_2}}{1+me^{p\hat{\rho}_1}}\bar{\alpha}_1}{\bar{a}_1}\right]^{\frac{1}{q_1}} - \bar{b}_1\omega.$$

Then system (1.4) *admits at least one positive* ω*-periodic solution.*

IV. Uniform persistence

Our object in this section is to prove the uniform persistence of system (1.4).

Theorem 3. *Let* $p = q_1 = q_2 = 1$ *in system (1.4). Assume that*

(H_1) $b_1^- > \alpha_1^+ M_2$, $(1+mM_1)^{-1}\alpha_2^- N_1 > b_2^+$,

then for any positive solution $(N_1, N_2)^T$ *of system* (1.4) *satisfies*

$$N_i \le N_i(t) \le M_i, \quad i = 1, 2,$$

where N_i *and* M_i *are defined as those in* (4.1)-(4.4), $i = 1, 2, 3$. *That is, system* (1.4) *is uniformly persistent.*

Proof: We have from the first equation of system (1.4) that

$$\dot{N}_1(t) \le N_1(t)\left[b_1^+ - a_1^- N_1(t)\right].$$

By Lemmas 2.3 and 2.4 in [22], we have from (4.1) that

$$N_1(t) \le \frac{b_1^+}{a_1^-} := M_1. \tag{4.1}$$

We have from the second equation of system (1.4) that

$$\dot{N}_2(t) \leq N_2(t)\left[m^{-1}\alpha_2^+ - a_2^- N_2(t)\right].$$

By Lemmas 2.3 and 2.4 in [22], we have from (4.1) that

$$N_2(t) \leq \frac{m^{-1}\alpha_2^+}{a_2^-} := M_2. \tag{4.2}$$

In view of the first equation of system (1.4), it follows that

$$\dot{N}_1(t) \geq N_1(t)\Big[b_1^- - \alpha_1^+ M_2 - a_1^+ N_1(t)\Big],$$

which implies that

$$N_1(t) \geq \frac{b_1^- - \alpha_1^+ M_2}{a_1^+} := N_1. \tag{4.3}$$

Similar to the argument as that in (4.3), we obtain from the second equation of system (1.4) that

$$\begin{aligned} N_2(t) \;&\geq\; \frac{(1+mM_1)^{-1}\alpha_2^- N_1 - b_2^+}{a_2^+} \\ &:= N_2. \end{aligned} \tag{4.4}$$

The proof is completed. ■

V. Uniform asymptotical stability

The main result of this paper concerns the uniformly asymptotically stable of system (1.4).

Theorem 4. *Let $p = q_1 = q_2 = 1$ and $\mu_1 = \mu_2 = \mu_3 = \nu \equiv 0$ in system (1.4). Suppose (H_1) and the following condition hold:*

(H_2) *there exists a constant μ such that*

$$a_1^- - \alpha_1^+ m M_1 - \alpha_2^+ > \mu,$$
$$a_2^- - \alpha_2^+ > \mu,$$

where M_1 is defined as that in Theorem 3. Then system (1.4) is uniformly asymptotically stable.

Proof: Suppose that $Z(t) = (\ln N_1(t), \ln N_2(t))^T$ and $Z^*(t) = (\ln N_1^*(t), \ln N_2^*(t))^T$ are any two solutions of system (1.4). Let $V(t) = V_1(t) + V_2(t)$, where $V_1(t) = |\ln N_1(t) - \ln N_1^*(t)|$ and $V_2(t) = |\ln N_2(t) - \ln N_2^*(t)|$.

Calculating the upper right derivative of $V_1(t)$ along the solution of system (1.4), we have

$$\begin{aligned} D^+V_1(t) \;\leq\; & -a_1^-|N_1(t) - N_1^*(t)| \\ & +\alpha_1^+ m M_1 |N_1(t) - N_1^*(t)| \\ & +\alpha_1^+|N_2(t) - N_2^*(t)|, \end{aligned}$$

similarly,

$$\begin{aligned} D^+V_2(t) \;\leq\; & -a_2^-|N_2(t) - N_2^*(t)| \\ & +\alpha_2^+|N_1(t) - N_1^*(t)|. \end{aligned}$$

Then

$$\begin{aligned} & D^+V(t) \\ \leq\; & [-a_1^- + \alpha_1^+ m M_1 + \alpha_2^+]|N_1(t) - N_1^*(t)| \\ & +[-a_2^- + \alpha_2^+]|N_2(t) - N_2^*(t)| \\ \leq\; & -\mu V(t). \end{aligned} \tag{5.1}$$

Therefore, V is non-increasing. Integrating (5.1) from 0 to t leads to

$$V(t) + \mu\int_0^t V(s)\,\mathrm{d}s \leq V(0) < +\infty, \quad \forall t \geq 0,$$

that is,

$$\int_0^{+\infty} V(s)\,\mathrm{d}s < +\infty,$$

which implies that

$$\lim_{s\to+\infty}|N_1(t) - N_1^*(t)| = \lim_{s\to+\infty}|N_2(t) - N_2^*(t)| = 0.$$

Thus, system (1.4) is uniformly asymptotically stable. This completes the proof. ■

VI. Examples and simulations

Example 2. Consider the following delayed predator-prey system:

$$\begin{cases} \dot{N}_1(t) \;=\; N_1(t)\Big[1 - |\sin\sqrt{3}t|N_1^2(t-0.8) \\ \qquad\qquad -\frac{N_1(t)}{2e^{10}[1+N_1^2(t)]}N_2(t-0.8)\Big], \\ \dot{N}_2(t) \;=\; N_2(t)\Big[-\frac{e^{-18}}{2+2e^{-18}} - \cos^2(\sqrt{2}t)N_2^2(t-0.8) \\ \qquad\qquad +\frac{N_1^2(t-0.8)}{1+N_1^2(t-0.8)}\Big]. \end{cases} \tag{6.1}$$

Corresponding to system (1.4), we have $\bar{b}_1 = 1$, $\bar{b}_2 = \frac{e^{-18}}{2+2e^{-18}}$, $\bar{a}_1 = \frac{2}{\pi}$, $\bar{a}_2 = \frac{1}{2}$, $l_0 = e^{-10}$, $m = 1$, $q_1 = q_2 = p = 2$. Further, for $\forall k \in \mathbb{R}$, we can choose $\omega_0 = \frac{2\sqrt{3}\pi}{3}$ so that (3.1) holds, that is,

$$\frac{1}{T}\int_k^{k+T} a_1(s)\,\mathrm{d}s \in \left[\frac{1}{\pi}, \frac{3}{\pi}\right],$$
$$\frac{1}{T}\int_k^{k+T} a_2(s)\,\mathrm{d}s \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \forall T \geq \omega_0 = \frac{2\sqrt{3}\pi}{3}.$$

By a easy calculation, we obtain that

$$\rho_1 \approx 5.1253, \quad \rho_2 \approx 5.0695.$$

Hence $\Phi_1(t) \equiv \frac{1}{2}$, $\forall t \in \mathbb{R}$, which implies that (F_2) holds. Take $\tau_0 = 2\pi$. So $\pi_0 = 8$ and

$$\rho_3 \approx -9,$$

which yields that

$$\bar{\Phi}_2 := \lim_{T\to+\infty}\frac{1}{T}\int_0^T\left[\frac{e^{-18}}{1+e^{-18}} - \frac{e^{-18}}{2+2e^{-18}}\right]\mathrm{d}s > 0,$$

which implies that (F_3) holds. Therefore, all the conditions of Corollary 1 are satisfied. By Corollary 1, system (6.1) admits at least one positive almost periodic solution (see Figures 1-2). It is easy to verify that all the conditions of Theorems 2-3 are satisfied. By Theorems 2-3, system (6.1) is permanent and uniform asymptotical stability.

Remark 2. In system (6.1), corresponding to system (1.4) and Corollary 3.3, $\varphi_1 = \frac{\pi}{\sqrt{3}}$, $\varphi_2 = \frac{\pi}{\sqrt{2}}$, β_i, γ_i, σ_j and ψ are arbitrary constants, $i = 1, 2$, $j = 1, 2, 3$. So system (6.1) is with incommensurable periods. Through all the coefficients of system (6.1) are periodic functions, the positive periodic solutions of system (6.1) could not possibly exist. However, by Corollary 1, the positive almost periodic solutions of system (6.1) exactly exist.

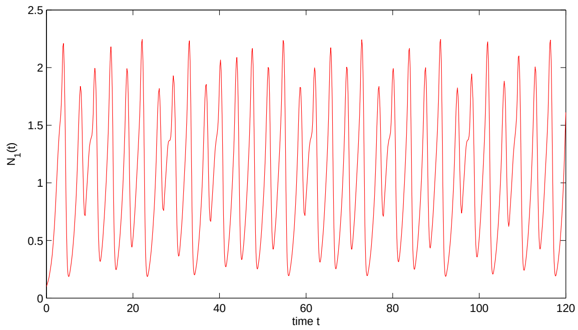

Fig. 1 State variable N_1 of system (6.1)

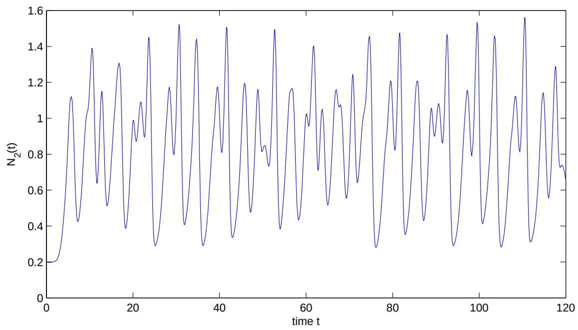

Fig. 2 State variable N_2 of system (6.1)

Example 3. Consider the following delayed almost periodic predator-prey system:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[1 - \frac{|\sin\sqrt{2}t| + |\sin\sqrt{3}t|}{2} N_1^2(t-0.8) \\ \qquad - \frac{N_1(t)}{2e^{10}[1+N_1^2(t)]} N_2(t-0.8)\Big], \\ \dot{N}_2(t) = N_2(t)\Big[- \frac{e^{-18}}{2+2e^{-18}} \\ \qquad - \frac{\cos^2(\sqrt{2}t)+\cos^2(\sqrt{3}t)}{2} N_2^2(t-0.8) \\ \qquad + \frac{N_1^2(t-0.8)}{1+N_1^2(t-0.8)}\Big]. \end{cases} \tag{6.2}$$

In system (6.2), $\frac{|\sin\sqrt{2}t| + |\sin\sqrt{3}t|}{2}$ and $\frac{\cos^2(\sqrt{2}t)+\cos^2(\sqrt{3}t)}{2}$ are almost periodic functions, which are not periodic functions. Similar to the argument as that in Example 2, by Theorem 1, it is easy to obtain that system (6.2) admits at least one positive almost periodic solution (see Figures 3-4).

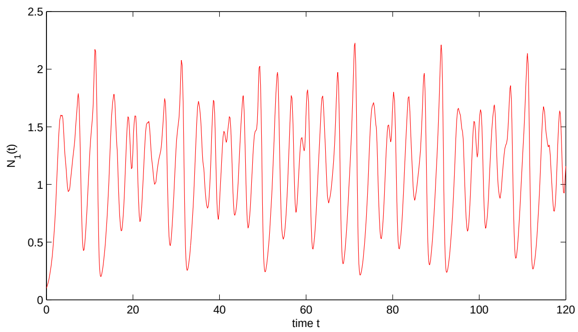

Fig. 3 State variable N_1 of system (6.2)

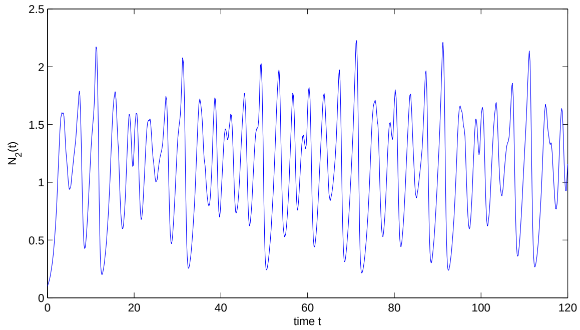

Fig. 4 State variable N_2 of system (6.2)

VII. Conclusion

In this paper we have obtained the uniform permanence and existence of a positive almost periodic solution for a delayed predator-prey system with general functional response. The approach is based on the continuation theorem of coincidence degree theory and the comparison theorem. And Lemma 2 in Section 2 and Lemmas 2.3-2.4 in [22] are critical to study the permanence of the biological model. It is important to notice that the approach used in this paper can be extended to other types of biological model such as epidemic models, Lotka-Volterra systems and other similar models of first order.

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