Existence of Solutions for the Four-point Fractional Boundary Value Problems Involving the P-Laplacian Operator

Gaihong Feng and Yitao Yang

***Abstract*—This paper is concerned with four-point fractional boundary value problems involving the p-Laplacian operator. By employing the Banach contraction mapping principle, we establish the existence of solutions for the four-point fractional boundary value problems involving the p-Laplacian operator. The interesting point is fractional differential equation with the p-Laplacian operator.**

***Index Terms*—Four-point fractional boundary value problems, P-Laplacian operator, Caputo's fractional derivative, Banach contraction mapping principle.**

I. Introduction

THE aim of this paper is to consider the existence and uniqueness of solutions for the four-point fractional boundary value problems with the p-Laplacian operator as follows

$$(\phi_p(^C D^\alpha u(t)))' = f(t, u(t)), \quad 0 < t < 1, \quad 0 < \alpha \leq 1, \tag{1}$$

$$\phi_p(^C D^\alpha u(0)) = a\phi_p(^C D^\alpha u(\xi)), \quad u(1) = bu(\eta), \tag{2}$$

where ϕ_p is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2} \cdot s,\ p > 1,\ (\phi_p)^{-1} = \phi_q,\ \ \frac{1}{p} + \frac{1}{q} = 1,\ {}^C D^\alpha$ is the standard Caputo derivative, $0 < \xi < \eta < 1,\ 0 < a < 1,\ b \neq 1,\ f \in C([0,1] \times R, R)$ is a given nonlinear function.

Fractional differential equations arise in many mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium or polymer rheology. In the literature [1-3], the derivatives of fractional order are involved. Because the fractional order models are more accurate than integer order models, many scholars study fractional differential equations. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properies of various materials and processes. This memory term insures the history and its impact to the present and future, see [4]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [5-12] and the reference therein. Recent results on fractional differential equations can be seen in the literature [13-19].

Recently, there have appeared a very large number of papers which study the existence of solutions of boundary value problems and initial value problems for the fractional differential equations. In paper [21], the authors discussed the existence of solutions for the following fractional differential equation with multi-point boundary condition

$$^c D^q_{0+} u(t) + f(t, u(t), (Ku)(t), (Hu)(t)) = 0, \quad t \in (0,1),$$

$$a_1 u(0) - b_1 u'(0) = d_1 u(\xi_1), \quad a_2 u(1) + b_2 u'(1) = d_2 u(\xi_2),$$

where $1 < q \leq 2$ is a real number.

In [20], by applying some standard fixed point theorems, the authors proved the existence and uniqueness of solutions for a four-point nonlocal boundary value problem of nonlinear integro-differential equations of fractional order

$$^c D^q x(t) = f(t, x(t), (\phi x)(t), (\psi x)(t)), \quad 0 < t < 1,\ 1 < q \leq 2,$$

$$x'(0) + a x(\eta_1) = 0,\ \ b x'(1) + x(\eta_2) = 0, \quad 0 < \eta_1 \leq \eta_2 < 1,$$

where $^c D^q$ is the Caputo's fractional derivative.

On the other hand, the p-Laplacian operator is widely used in analyzing mathematical models of physical phenomena, mechanics, nonlinear dynamics and many other related fields. In consequence, the subject of boundary value problems with p-Laplacian operator is gaining much importance and attention. For details, see [22-26] and the references therein.

In paper [22], the authors studied the existence of multiple positive solutions for Sturm-Liouville-like four-point boundary value problem with p-Laplacian

$$(\phi_p(u'))'(t) + f(t, u(t)) = 0, \quad t \in (0,1),$$

$$u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0,$$

by using a fixed-point theorem of operators on a cone.

However, as far as we know, very few papers have combined the fractional differential equation with the equation involving the p-Laplacian operator. As we all know, when the p passing from $p = 2$ to $p \neq 2$, difficulties appeared immediately. For the first case, for $p = 2$, we can change the differential equation into a equivalent integral equation easily and therefore, a Green's function exists, however, for $p \neq 2$, it is impossible since the differential operator $(\phi_p(^C D^\alpha u(t)))'$ is nonlinear. Inspired by the above mentioned works, in this paper, we study the existence and uniqueness of solutions for the four-point fractional boundary value problems with the p-Laplacian operator. To the best knowledge of the authors, no work has been done to obtain the positive solution of the problem (1), (2). It is interesting to note that the fractional differential equation with the p-Laplacian operator.

The organization of this paper is as follows. In section 2, we present some necessary definitions and preliminary

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results that will be used to prove our main results. The proofs of our main results are given in section 3. In section 4, we will give an example to ensure our main result.

II. The preliminary lemmas

For the convenience of readers, we provide some background material in this section.

Definition 2.1 [20] The Riemann-Liouville fractional integral of order α for function y is defined as

$$I^\alpha y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,\quad \alpha>0.$$

Definition 2.2 [20] The Caputo's derivative for function y is defined as

$$^C D^\alpha y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{y^{(n)}(s)ds}{(t-s)^{\alpha+1-n}},\quad n=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of real number α.

Lemma 2.1 Let $\alpha>0$, then the fractional differential equation

$$^C D^\alpha u(t)=0$$

has solutions

$$u(t)=c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1},\ c_i\in R,\ \ i=1,2,\cdots,n,n=[\alpha]+1.$$

Lemma 2.2 [20] Let $\alpha>0$, then

$$I^\alpha\ {}^C D^\alpha u(t)=u(t)+c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1}$$

for some $c_i\in R,\quad i=1,2,\cdots,n,n=[\alpha]+1$.

Lemma 2.3 If ϕ_p is a p-Laplacian operator, then it has the following properties.

(1) If $1<p<2,\ xy>0$, and $|x|,|y|\geq m>0$, then

$$|\phi_p(x)-\phi_p(y)|\leq(p-1)m^{p-2}|x-y|. \quad (3)$$

(2) If $p>2,\ |x|,|y|\leq M$, then

$$|\phi_p(x)-\phi_p(y)|\leq(p-1)M^{p-2}|x-y|. \quad (4)$$

Lemma 2.4 Suppose that $y\in C[0,1],\ a,b\neq 1$. Then the following four-point fractional boundary value problem

$$(\phi_p(^C D^\alpha u(t)))'=y(t),\qquad 0<t<1,\quad 0<\alpha\leq 1, \quad (5)$$

$$\phi_p(^C D^\alpha u(0))=a\phi_p(^C D^\alpha u(\xi)),\quad u(1)=bu(\eta), \quad (6)$$

is equivalent to the following integral equation:

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\phi_q\Big(\int_0^\tau h(s)ds\\&+\frac{a}{1-a}\int_0^\xi h(s)ds\Big)d\tau+\\&+\frac{b}{(1-b)\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\phi_q\Big(\int_0^\tau h(s)ds\\&+\frac{a}{1-a}\int_0^\xi h(s)ds\Big)d\tau\\&-\frac{1}{(1-b)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\phi_q\Big(\int_0^\tau h(s)ds\\&+\frac{a}{1-a}\int_0^\xi h(s)ds\Big)d\tau.\end{aligned} \quad (7)$$

proof By the equation $(\phi_p(^C D^\alpha u(t)))'=y(t),\qquad 0<t<1$, we have

$$\phi_p(^C D^\alpha u(t))=\phi_p(^C D^\alpha u(0))+\int_0^t y(s)ds. \quad (8)$$

Since $\phi_p(^C D^\alpha u(0))=a\phi_p(^C D^\alpha u(\xi))$, we deduce that

$$\phi_p(^C D^\alpha u(t))=a\phi_p(^C D^\alpha u(\xi))+\int_0^t y(s)ds. \quad (9)$$

We choose $t=\xi$ in (9), we have

$$\phi_p(^C D^\alpha u(\xi))=a\phi_p(^C D^\alpha u(\xi))+\int_0^\xi y(s)ds,$$

so,

$$\phi_p(^C D^\alpha u(\xi))=\frac{1}{1-a}\int_0^\xi y(s)ds. \quad (10)$$

Hence, from (9), (10), we have the following form

$$\phi_p(^C D^\alpha u(t))=\frac{a}{1-a}\int_0^\xi y(s)ds+\int_0^t y(s)ds,$$

and then

$$^C D^\alpha u(t)=\phi_q\left[\frac{a}{1-a}\int_0^\xi y(s)ds+\int_0^t y(s)ds\right]. \quad (11)$$

Using Lemma 2.2 to Eq. (11), we can write

$$\begin{aligned}u(t)&=I^q\left\{\phi_q\left[\frac{a}{1-a}\int_0^\xi y(s)ds+\int_0^t y(s)ds\right]\right\}+x(0)\\&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds\\&\quad+\textstyle\int_0^\tau y(s)ds\Big]d\tau+x(0).\end{aligned} \quad (12)$$

Then

$$\begin{aligned}u(1)=&\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds\\&+\textstyle\int_0^\tau y(s)ds\Big]d\tau+x(0),\end{aligned}$$

$$\begin{aligned}u(\eta)=&\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds\\&+\textstyle\int_0^\tau y(s)ds\Big]d\tau+x(0).\end{aligned}$$

By the boundary condition $u(1)=bu(\eta)$, we can have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds+\\&\textstyle\int_0^\tau y(s)ds\Big]d\tau+x(0)\\&-\frac{b}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds+\\&\textstyle\int_0^\tau y(s)ds\Big]d\tau-bx(0)=0.\end{aligned}$$

So, we get

$$\begin{aligned}x(0)=&\frac{b}{(1-b)\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds+\\&\textstyle\int_0^\tau y(s)ds\Big]d\tau\\&-\frac{1}{(1-b)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\phi_q\Big[\frac{a}{1-a}\int_0^\xi y(s)ds\\&+\textstyle\int_0^\tau y(s)ds\Big]d\tau.\end{aligned} \quad (13)$$

Substituting (13) into (12), it is easy to get that

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\phi_q\Bigg[\frac{a}{1-a}\int_0^\xi y(s)ds + \textstyle\int_0^\tau y(s)ds\Bigg]d\tau + \frac{b}{(1-b)\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\phi_q\Bigg[\frac{a}{1-a}\int_0^\xi y(s)ds + \textstyle\int_0^\tau y(s)ds\Bigg]d\tau - \frac{1}{(1-b)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\phi_q\Bigg[\frac{a}{1-a}\int_0^\xi y(s)ds + \textstyle\int_0^\tau y(s)ds\Bigg]d\tau.$$

The proof is completed.

III. Main result

This section is devoted to give an existence and uniqueness of solutions for the four-point fractional boundary value problem (1), (2).
To this end, we define the operator $T_0:\ C[0,1]\to C[0,1]$ by

$$T_0u(t)=\phi_q\left[\frac{a}{1-a}\int_0^\xi f(s,u(s))ds+\int_0^t f(s,u(s))ds\right],$$

and $T_1:\ C[0,1]\to C[0,1]$ by

$$T_1u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)d\tau + \frac{b}{(1-b)\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}u(\tau)d\tau - \frac{1}{(1-b)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}u(\tau)d\tau.$$

Let $T=\ T_1\circ T_0$, then $T:\ C[0,1]\to C[0,1]$ is a completely continuous operator. It is clear that u is a solution of the fractional boundary value problem (1), (2) if and only if u is a fixed point of T.
Let $E=C[0,1]$ be the Banach space endowed with the usual supremum norm $\|\cdot\|$.

Theorem 3.1 Suppose $p>2,\ 0<a<1,\ b\neq 1$,let

$$\Upsilon=\left(1+|\tfrac{b}{1-b}|+|\tfrac{1}{1-b}|\right)\tfrac{\Gamma(\lambda(q-2)+2)}{\Gamma(\lambda(q-2)+2+\alpha)} + \left(\tfrac{a\xi}{1-a}+|\tfrac{ab\xi}{(1-a)(1-b)}|+|\tfrac{a\xi}{(1-a)(1-b)}|\right)\tfrac{\Gamma(\lambda(q-2)+1)}{\Gamma(\lambda(q-2)+1+\alpha)}(q-1)\beta^{q-2}$$

and the following condition holds:
(H_1) There exist constants $\beta>0,\ \alpha+\lambda(q-2)>0,$

$$0<N<\frac{1}{\Upsilon}.$$

such that

$$\beta\lambda t^{\lambda-1}\le f(t,u),\ \text{ for any } (t,u)\in[0,1]\times R, \qquad (14)$$

and

$$|f(t,u)-f(t,v)|\le N|u-v|,\ \text{ for } t\in[0,1],\ \text{ and } (u,v)\in R. \qquad (15)$$

Then the fractional boundary value problem (1), (2) has a unique solution.

proof By (14), we can get

$$\beta t^\lambda\le\int_0^t f(s,u)ds,\ \text{ for any } (t,u)\in[0,1]\times R.$$

By Lemma 2.3 and the definition of operator T_0, for any $u,v\in E$, we have

$$\begin{aligned}
&|T_0u(t)-T_0v(t)|\\
&=\Bigg|\phi_q\left[\frac{a}{1-a}\int_0^\xi f(s,u(s))ds+\int_0^t f(s,u(s))ds\right] -\phi_q\left[\frac{a}{1-a}\int_0^\xi f(s,v(s))ds+\int_0^t f(s,v(s))ds\right]\Bigg|\\
&\le(q-1)(\beta t^\lambda)^{q-2}\Bigg|\frac{a}{1-a}\int_0^\xi f(s,u(s))ds + \textstyle\int_0^t f(s,u(s))ds-\displaystyle\frac{a}{1-a}\int_0^\xi f(s,v(s))ds - \textstyle\int_0^t f(s,v(s))ds\Bigg|\\
&\le(q-1)(\beta t^\lambda)^{q-2}\Bigg(\Bigg|\frac{a}{1-a}\int_0^\xi f(s,u(s))ds -\frac{a}{1-a}\int_0^\xi f(s,v(s))ds\Bigg| + \Big|\textstyle\int_0^t f(s,u(s))ds-\int_0^t f(s,v(s))ds\Big|\Bigg)\\
&\le(q-1)(\beta t^\lambda)^{q-2}\Bigg(\frac{a}{1-a}\int_0^\xi|f(s,u(s))-f(s,v(s))|ds + \textstyle\int_0^t\Big|f(s,u(s))ds-\int_0^t f(s,v(s))\Big|ds\Bigg)\\
&\le(q-1)(\beta t^\lambda)^{q-2}\left(\frac{a\xi}{1-a}N\|u-v\|+tN\|u-v\|\right)\\
&=N(q-1)(\beta t^\lambda)^{q-2}\left(\frac{a\xi}{1-a}+t\right)\|u-v\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|Tu(t)-Tv(t)|=|T_1(T_0u)(t)-T_1(T_0v)(t)|\\
&=\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left((T_0u)(\tau)-(T_0v)(\tau)\right)d\tau +\frac{b}{(1-b)\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\left((T_0u)(\tau)-(T_0v)(\tau)\right)d\tau -\frac{1}{(1-b)\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\left((T_0u)(\tau)-(T_0v)(\tau)\right)d\tau\Bigg|\\
&\le\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\lambda(q-2)}\left(\frac{a\xi}{1-a}+\tau\right)d\tau + \left|\frac{b}{(1-b)\Gamma(\alpha)}\right|\textstyle\int_0^\eta(\eta-\tau)^{\alpha-1}\tau^{\lambda(q-2)}\left(\frac{a\xi}{1-a}+\tau\right)d\tau + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right|\textstyle\int_0^1(1-\tau)^{\alpha-1}\tau^{\lambda(q-2)}\left(\frac{a\xi}{1-a}+\tau\right)d\tau\Bigg)\\
&\quad N(q-1)\beta^{q-2}\|u-v\|\\
&=\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-\tau)^{\alpha-1}t^{\lambda(q-2)+1}\tau^{\lambda(q-2)+1}td\tau + \tfrac{a\xi}{(1-a)\Gamma(\alpha)}\textstyle\int_0^1 t^{\alpha-1}(1-\tau)^{\alpha-1}t^{\lambda(q-2)}\tau^{\lambda(q-2)}td\tau\\
&\quad+\left|\tfrac{b}{(1-b)\Gamma(\alpha)}\right|\textstyle\int_0^1\eta^{\alpha-1}(1-\tau)^{\alpha-1}\eta^{\lambda(q-2)+1}\tau^{\lambda(q-2)+1}\eta d\tau + \left|\tfrac{ab\xi}{(1-a)(1-b)\Gamma(\alpha)}\right|\textstyle\int_0^1\eta^{\alpha-1}(1-\tau)^{\alpha-1}\eta^{\lambda(q-2)}\tau^{\lambda(q-2)}\eta d\tau\\
&\quad+\left|\tfrac{1}{(1-b)\Gamma(\alpha)}\right|\textstyle\int_0^1(1-\tau)^{\alpha-1}\tau^{\lambda(q-2)+1}d\tau + \left|\tfrac{a\xi}{(1-a)(1-b)\Gamma(\alpha)}\right|\textstyle\int_0^1(1-\tau)^{\alpha-1}\tau^{\lambda(q-2)}d\tau\Bigg)\\
&\quad N(q-1)\beta^{q-2}\|u-v\|
\end{aligned}$$

$$= \Big(\frac{1}{\Gamma(\alpha)} t^{\alpha+1+\lambda(q-2)} B(\lambda(q-2)+2,\alpha) + \tfrac{a\xi}{(1-a)\Gamma(\alpha)} t^{\alpha+\lambda(q-2)} B(\lambda(q-2)+1,\alpha) + \left|\tfrac{b}{(1-b)\Gamma(\alpha)}\right| \eta^{\alpha+1+\lambda(q-2)} B(\lambda(q-2)+2,\alpha) + \left|\tfrac{ab\xi}{(1-a)(1-b)\Gamma(\alpha)}\right| \eta^{\alpha+\lambda(q-2)} B(\lambda(q-2)+1,\alpha) + \left|\tfrac{1}{(1-b)\Gamma(\alpha)}\right| B(\lambda(q-2)+2,\alpha) + \left|\tfrac{a\xi}{(1-a)(1-b)\Gamma(\alpha)}\right| B(\lambda(q-2)+1,\alpha) \Big) N(q-1)\beta^{q-2}\|u-v\|$$

$$< \Big(\Big(\frac{1}{\Gamma(\alpha)} + \left|\frac{b}{(1-b)\Gamma(\alpha)}\right| + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right| \Big) B(\lambda(q-2)+2,\alpha) + \Big(\tfrac{a\xi}{(1-a)\Gamma(\alpha)} + \left|\tfrac{ab\xi}{(1-a)(1-b)\Gamma(\alpha)}\right| + \left|\tfrac{a\xi}{(1-a)(1-b)\Gamma(\alpha)}\right| \Big) B(\lambda(q-2)+1,\alpha) \Big) N(q-1)\beta^{q-2}\|u-v\|$$

$$= \Big(\Big(1 + \left|\tfrac{b}{(1-b)}\right| + \left|\tfrac{1}{(1-b)}\right|\Big) \tfrac{\Gamma(\lambda(q-2)+2)}{\Gamma(\lambda(q-2)+2+\alpha)} + \Big(\tfrac{a\xi}{(1-a)} + \left|\tfrac{ab\xi}{(1-a)(1-b)}\right| + \left|\tfrac{a\xi}{(1-a)(1-b)}\right| \Big) \tfrac{\Gamma(\lambda(q-2)+1)}{\Gamma(\lambda(q-2)+1+\alpha)} \Big) N(q-1)\beta^{q-2}\|u-v\|$$

$$= L\|u-v\|,$$

where

$$L = \Big(1 + \left|\frac{b}{1-b}\right| + \left|\frac{1}{1-b}\right|\Big) \frac{\Gamma(\lambda(q-2)+2)}{\Gamma(\lambda(q-2)+2+\alpha)} + \Big(\frac{a\xi}{1-a} + \left|\frac{ab\xi}{(1-a)(1-b)}\right| + \left|\frac{a\xi}{(1-a)(1-b)}\right| \Big) \frac{\Gamma(\lambda(q-2)+1)}{\Gamma(\lambda(q-2)+1+\alpha)} N(q-1)\beta^{q-2}.$$

By the condition (H_1) of this theorem, we can get that $0 < L < 1$, then

$$\|Tu - Tv\| \le L\|u-v\|.$$

This implies that $T: E \to E$ is a contraction mapping. In view of the Banach contraction mapping principle, we get that T has a unique fixed point in E, that is the boundary value problem (1), (2) has a unique solution. The proof is completed.

Theorem 3.2 Suppose $1 < p < 2,\ 0 < a < 1,\ b \neq 1$, let

$$\Theta = (q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big(\frac{1}{\Gamma(\alpha)} + \left|\frac{b}{(1-b)\Gamma(\alpha)}\right| + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right| \Big) \Big(\tfrac{1}{\alpha(\alpha+1)} + \tfrac{a\xi}{1-a}\tfrac{1}{\alpha} \Big)$$

and the following condition holds:
(H_2) There exists a nonnegative function $h \in L[0,1]$ and $H := \int_0^1 h(t)dt > 0$ such that

$$|f(t,u)| \le h(t),\ \text{for any } (t,u) \in [0,1]\times R, \qquad (16)$$

and there exists a constant N with

$$0 < N < \frac{1}{\Theta}$$

with

$$|f(t,u) - f(t,v)| \le N|u-v|,\ \text{for } t\in[0,1],\ \text{and } (u,v)\in R. \qquad (17)$$

Then the fractional boundary value problem (1), (2) has a unique solution.

proof By (16), for $t \in [0,1]$, we have

$$\left|\int_0^t f(s,u(s))ds\right| \le \int_0^1 |f(s,u(s))|\,ds \le \int_0^1 h(s)ds = H.$$

From Lemma 2.3 and (17), and for any $u, v \in E$, we have

$$|T_0u(t) - T_0v(t)| = \Big|\phi_q\Big[\frac{a}{1-a}\int_0^\xi f(s,u(s))ds + \int_0^t f(s,u(s))ds\Big] - \phi_q\Big[\frac{a}{1-a}\int_0^\xi f(s,v(s))ds + \int_0^t f(s,v(s))ds\Big]\Big|$$

$$\le (q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big|\frac{a}{1-a}\int_0^\xi f(s,u(s))ds + \textstyle\int_0^t f(s,u(s))ds - \frac{a}{1-a}\int_0^\xi f(s,v(s))ds - \int_0^t f(s,v(s))ds\Big|$$

$$\le (q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big(\Big|\frac{a}{1-a}\int_0^\xi f(s,u(s))ds - \frac{a}{1-a}\int_0^\xi f(s,v(s))ds\Big| + \Big|\int_0^t f(s,u(s))ds - \int_0^t f(s,v(s))ds\Big| \Big)$$

$$\le (q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big(\frac{a}{1-a}\int_0^\xi |f(s,u(s)) - f(s,v(s))|\,ds + \textstyle\int_0^t \Big|f(s,u(s))ds - \int_0^t f(s,v(s))\Big|\,ds \Big)$$

$$\le (q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big(\frac{a\xi}{1-a} N\|u-v\| + tN\|u-v\| \Big)$$

$$= N(q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big(\frac{a\xi}{1-a} + t \Big)\|u-v\|.$$

Therefore,

$$|Tu(t) - Tv(t)| = |T_1(T_0u)(t) - T_1(T_0v)(t)|$$

$$= \Big| \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left((T_0u)(\tau) - (T_0v)(\tau)\right)d\tau + \frac{b}{(1-b)\Gamma(\alpha)}\int_0^\eta (\eta-\tau)^{\alpha-1}\left((T_0u)(\tau) - (T_0v)(\tau)\right)d\tau - \frac{1}{(1-b)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\left((T_0u)(\tau) - (T_0v)(\tau)\right)d\tau \Big|$$

$$\le \Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Big(\frac{a\xi}{1-a} + \tau\Big)d\tau + \left|\frac{b}{(1-b)\Gamma(\alpha)}\right| \textstyle\int_0^\eta (\eta-\tau)^{\alpha-1}\left(\tfrac{a\xi}{1-a} + \tau\right)d\tau + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right| \int_0^1 (1-\tau)^{\alpha-1}\left(\tfrac{a\xi}{1-a} + \tau\right)d\tau \Big) N(q-1)\left(\tfrac{H}{1-a}\right)^{q-2}\|u-v\|$$

$$= \Big[\frac{1}{\Gamma(\alpha)}\Big(\frac{t^{\alpha+1}}{\alpha(\alpha+1)} + \frac{a\xi}{1-a}\frac{t^\alpha}{\alpha}\Big) + \left|\frac{b}{(1-b)\Gamma(\alpha)}\right| \Big(\tfrac{\eta^{\alpha+1}}{\alpha(\alpha+1)} + \frac{a\xi}{1-a}\frac{\eta^\alpha}{\alpha}\Big) + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right| \Big(\tfrac{1}{\alpha(\alpha+1)} + \frac{a\xi}{1-a}\frac{1}{\alpha}\Big) \Big] N(q-1)\left(\tfrac{H}{1-a}\right)^{q-2}\|u-v\|$$

$$= N(q-1)\left(\tfrac{H}{1-a}\right)^{q-2} \Big(\frac{1}{\Gamma(\alpha)} + \left|\frac{b}{(1-b)\Gamma(\alpha)}\right| + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right| \Big) \Big(\tfrac{1}{\alpha(\alpha+1)} + \tfrac{a\xi}{1-a}\tfrac{1}{\alpha} \Big)\|u-v\|$$

$$= L_1\|u-v\|,$$

where

$$L_1 = N(q-1)\left(\frac{H}{1-a}\right)^{q-2}\left(\frac{1}{\Gamma(\alpha)}+\left|\frac{b}{(1-b)\Gamma(\alpha)}\right|+\left|\frac{1}{(1-b)\Gamma(\alpha)}\right|\right)$$

$$\left(\frac{1}{\alpha(\alpha+1)}+\frac{a\xi}{1-a}\frac{1}{\alpha}\right).$$

According to the condition (H_2) of this theorem, we can get that $0 < L_1 < 1$, then

$$\|Tu - Tv\| \le L_1\|u-v\|.$$

This implies that $T: E \to E$ is a contraction mapping. In view of the Banach contraction mapping principle, we get that T has a unique fixed point in E, that is the boundary value problem (1), (2) has a unique solution. The proof is completed.

IV. Example

Example We consider the following nonlinear fractional boundary value problem

$$(\phi_{\frac{3}{2}}({}^C D^{\frac{1}{2}}u(t)))' = \frac{\sqrt{\pi}}{32}3t^2cos^2\frac{u}{3}, \quad 0 < t < 1, \tag{18}$$

$$\phi_{\frac{3}{2}}({}^C D^{\frac{1}{2}}u(0)) = \frac{2}{3}\phi_{\frac{3}{2}}({}^C D^{\frac{1}{2}}u(\frac{1}{3})), \quad u(1) = 3u(\frac{2}{3}), \tag{19}$$

where

$$p = \frac{3}{2},\ \alpha = \frac{1}{2},\ \xi = \frac{1}{3},\ \eta = \frac{2}{3},\ a = \frac{2}{3},\ b = 3,$$

$$f(t,u) = \frac{\sqrt{\pi}}{32}3t^2cos^2\frac{u}{3} \in C([0,1]\times R, R).$$

Choose $h(t) = \frac{\sqrt{\pi}}{32}3t^2$, then $\int_0^1 h(t)dt = \int_0^1 \frac{\sqrt{\pi}}{32}3t^2 = \frac{\sqrt{\pi}}{32} > 0$, we obtain $H = \int_0^1 h(t)dt = \frac{\sqrt{\pi}}{32}$. Taking $N = \frac{\sqrt{\pi}}{32}$, in view of $\Gamma(\frac{1}{2}) = \sqrt{\pi},\ q = 3$, we have

$$0 < N < \frac{2}{3},$$

and

$$L_1 = N(q-1)\left(\tfrac{H}{1-a}\right)^{q-2}\left(\frac{1}{\Gamma(\alpha)}+\left|\frac{b}{(1-b)\Gamma(\alpha)}\right| + \left|\frac{1}{(1-b)\Gamma(\alpha)}\right|\right)\left(\tfrac{1}{\alpha(\alpha+1)}+\tfrac{a\xi}{1-a}\tfrac{1}{\alpha}\right) = \tfrac{3\sqrt{\pi}}{64} < 1.$$

We can check that the nonlinear term $f(t,u)$ satisfies:
(1) $|f(t,u)| \le \frac{\sqrt{\pi}}{32}3t^2 = h(t)$, for $(t,u) \in [0,1]\times R$, ;

(2)
$$\begin{aligned}|f(t,u)-f(t,v)| &= \left|\tfrac{\sqrt{\pi}}{32}3t^2cos^2\tfrac{u}{3} - \tfrac{\sqrt{\pi}}{32}3t^2cos^2\tfrac{v}{3}\right| \\ &\le \tfrac{\sqrt{\pi}}{32}3t^2\left|cos^2\tfrac{u}{3} - cos^2\tfrac{v}{3}\right| \\ &\le \tfrac{\sqrt{\pi}}{32}t^2|u-v| \le \tfrac{\sqrt{\pi}}{32}|u-v| = N|u-v|.\end{aligned}$$

for $t \in [0,1],\ u,v \in R$. Then all assumptions of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, fractional boundary value problem (18), (19) has a unique solution.

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