Solving Third Order Ordinary Differential Equations Using One-Step Block Method with Four Equidistant Generalized Hybrid Points

Oluwaseun Adeyeye and Zurni Omar

Abstract—**The application of a hybrid block method to solving third order ordinary differential equations is considered in this article. The hybrid method is developed for a set of equidistant hybrid points using a new generalized linear block method (GLBM). The equations for the GLBM takes a similar form as the conventional linear multistep method, however the form produces the needed family of schemes required to simultaneously evaluate the solution of the third order ordinary differential equations at individual grid points in a self-starting mode. The hybrid block method obtained using GLBM is investigated and the block method possesses good basic property of a numerical method which is displayed in the numerical results obtained. Furthermore, the comparison to works of the past authors shows that the new hybrid block gives impressive results in terms of error and consistency particularly for large intervals.**

Index Terms—**Hybrid Block Method, Generalized Linear Block Methods, Third Order, One-Step, Ordinary Differential Equations.**

I. Introduction

The direct numerical approximation of general third order initial value problems (IVPs) of the form (1) below

$$y''' = f(x, y, y', y''), y(x_n) = a, y'(x_n) = b, y''(x_n) = c \quad (1)$$

(where a, b and c are given constants) have been vastly considered in literature by several authors such as [1]–[7] amongst others. These authors focused on the direct solution of (1) due to the shortcomings of reduction to a system of three first order initial value problems which include both human and computational burden [8], [9].

Hybrid block methods are one of numerical methods adopted for directly approximating (1). It is seen to perform favourably well when numerically approximating solutions to initial value problems as it combines the advantages of block method and overcoming the zero stability barrier in linear multistep method [7].

Some authors who have presented hybrid block methods include [1], [7], [10], [11], however the generalized form for equidistant hybrid points was not attempted. In addition, when considering large intervals, the numerical results obtained in comparison to the exact solution can also be improved. This informs the motivation for this article to develop a new block method with generalized equidistant hybrid points $r\xi$. A new linear block method is introduced in the methodology and detailed explanations on the application is discussed in the following sections.

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II. Development of the Block Method

Consider the generalized linear block method in the equation below

$$y_{n+r\xi} = \sum_{i=0}^{2} \frac{(r\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{4} (\phi_{\xi i} f_{n+ri}), \quad \xi = 1,2,3,4 \quad (2)$$

whose corresponding derivatives are

$$y_{n+r\xi}^{(a)} = \sum_{i=0}^{3-(a+1)} \frac{(r\xi h)^i}{i!} y_n^{(i+a)} + \sum_{i=0}^{4} (\omega_{\xi i a} f_{n+ri}); \quad a = 1_{(\xi=1,2,3,4)}, a = 2_{(\xi=1,2,3,4)} \quad (3)$$

where r is the distance between consecutive hybrid points, ξ represents the hybrid points which in this article is $\xi = 1, 2, 3, 4$. These equations presented in equations (2) and (3) above are used to develop the hybrid block method for solving third order ordinary differential equations.

Expanding (2) and (3) yields

$$\begin{aligned}
y_{n+r} &= y_n + rhy'_n + \frac{(rh)^2}{2!} y''_n \\
&+ (\phi_{10} f_n + \phi_{11} f_{n+r} + \phi_{12} f_{n+2r} + \phi_{13} f_{n+3r} + \phi_{14} f_{n+4r}) \\
y_{n+2r} &= y_n + 2rhy'_n + \frac{(2rh)^2}{2!} y''_n \\
&+ (\phi_{20} f_n + \phi_{21} f_{n+r} + \phi_{22} f_{n+2r} + \phi_{23} f_{n+3r} + \phi_{24} f_{n+4r}) \\
y_{n+3r} &= y_n + 3rhy'_n + \frac{(3rh)^2}{2!} y''_n \\
&+ (\phi_{30} f_n + \phi_{31} f_{n+r} + \phi_{32} f_{n+2r} + \phi_{33} f_{n+3r} + \phi_{34} f_{n+4r}) \\
y_{n+4r} &= y_n + 4rhy'_n + \frac{(4rh)^2}{2!} y''_n \\
&+ (\phi_{40} f_n + \phi_{41} f_{n+r} + \phi_{42} f_{n+2r} + \phi_{43} f_{n+3r} + \phi_{44} f_{n+4r})
\end{aligned} \quad (4)$$

$$\begin{aligned}
y'_{n+r} &= y'_n + (rh)y''_n + (\omega_{101} f_n + \omega_{111} f_{n+r} \\
&+ \omega_{121} f_{n+2r} + \omega_{131} f_{n+3r} + \omega_{141} f_{n+4r}) \\
y'_{n+2r} &= y'_n + (2rh)y''_n + (\omega_{201} f_n + \omega_{211} f_{n+r} \\
&+ \omega_{221} f_{n+2r} + \omega_{231} f_{n+3r} + \omega_{241} f_{n+4r}) \\
y'_{n+3r} &= y'_n + (3rh)y''_n + (\omega_{301} f_n + \omega_{311} f_{n+r} \\
&+ \omega_{321} f_{n+2r} + \omega_{331} f_{n+3r} + \omega_{341} f_{n+4r}) \\
y'_{n+4r} &= y'_n + (4rh)y''_n + (\omega_{401} f_n + \omega_{411} f_{n+r} \\
&+ \omega_{421} f_{n+2r} + \omega_{431} f_{n+3r} + \omega_{441} f_{n+4r})
\end{aligned}$$

$$\begin{aligned}
y''_{n+r} &= y''_n + (\omega_{102} f_n + \omega_{112} f_{n+r} + \omega_{122} f_{n+2r} \\
&+ \omega_{132} f_{n+3r} + \omega_{142} f_{n+4r}) \\
y''_{n+2r} &= y''_n + (\omega_{202} f_n + \omega_{212} f_{n+r} + \omega_{222} f_{n+2r} \\
&+ \omega_{232} f_{n+3r} + \omega_{242} f_{n+4r}) \\
y''_{n+3r} &= y''_n + (\omega_{302} f_n + \omega_{312} f_{n+r} + \omega_{322} f_{n+2r} \\
&+ \omega_{332} f_{n+3r} + \omega_{342} f_{n+4r}) \\
y''_{n+4r} &= y''_n + (\omega_{402} f_n + \omega_{412} f_{n+r} + \omega_{422} f_{n+2r} \\
&+ \omega_{432} f_{n+3r} + \omega_{442} f_{n+4r})
\end{aligned} \quad (5)$$

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To obtain the unknown ϕ and ω coefficients, Taylor expansion as discussed in [12] is adopted. Now, the Taylor expansion for $y_{n+r}^{(n)} = y^{(n)}(x_n + rh)$ about x_n is defined as

$$\begin{aligned} &y^{(n)}(x_n + rh) = y^{(n)}(x_n) + rhy^{(n+1)}(x_n) \\ &+\tfrac{(rh)^2}{2!}y^{(n+2)}(x_n) + \dots \end{aligned} \tag{6}$$

where $y^{(q)(x_n)} = \frac{d^q y}{dx^q}|_{x=x_n}, q = 1, 2, \dots$

Expanding each individual term in the first expression of equation (4) using equation (6) above gives

$$\begin{aligned} &y_n + (rh)y_n' + \tfrac{(rh)^2}{2!}y_n'' + \tfrac{(rh)^3}{3!}y_n''' + \tfrac{(rh)^4}{4!}y_n^{iv} + \tfrac{(rh)^5}{5!}y_n^{v} \\ &+\tfrac{(rh)^6}{6!}y_n^{vi} + \tfrac{(rh)^7}{7!}y_n^{vii} + \dots = y_n + (rh)y_n' + \tfrac{(rh)^2}{2!}y_n'' \\ &+(\phi_{10}y_n''' + \phi_{11}(y_n''' + (rh)y_n^{iv} + \tfrac{(rh)^2}{2!}y_n^{v} + \tfrac{(rh)^3}{3!}y_n^{vi} \\ &+\tfrac{(rh)^4}{4!}y_n^{vii} + \dots) + \phi_{12}(y_n''' + (2rh)y_n^{iv} + \tfrac{(2rh)^2}{2!}y_n^{v} \\ &+\tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \dots) + \phi_{13}(y_n''' + (3rh)y_n^{iv} \\ &+\tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi} + \tfrac{(3rh)^4}{4!}y_n^{vii} + \dots) + \phi_{14}(y_n''' \\ &+(4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v} + \tfrac{(4rh)^3}{3!}y_n^{vi} + \tfrac{(4rh)^4}{4!}y_n^{vii} + \dots) \end{aligned}$$

which can be written in the matrix form $Ax_1 = B_1$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{(rh)^1}{1!} & \frac{(2rh)^1}{1!} & \frac{(3rh)^1}{1!} & \frac{(4rh)^1}{1!} \\ 0 & \frac{(rh)^2}{2!} & \frac{(2rh)^2}{2!} & \frac{(3rh)^2}{2!} & \frac{(4rh)^2}{2!} \\ 0 & \frac{(rh)^3}{3!} & \frac{(2rh)^3}{3!} & \frac{(3rh)^3}{3!} & \frac{(4rh)^3}{3!} \\ 0 & \frac{(rh)^4}{4!} & \frac{(2rh)^4}{4!} & \frac{(3rh)^4}{4!} & \frac{(4rh)^4}{4!} \end{pmatrix},$$

$$x_1 = \begin{pmatrix} \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{14} \end{pmatrix}, B_1 = \begin{pmatrix} \frac{(rh)^3}{3!} \\ \frac{(rh)^4}{4!} \\ \frac{(rh)^5}{5!} \\ \frac{(rh)^6}{6!} \\ \frac{(rh)^7}{7!} \end{pmatrix}$$

Adopting matrix inverse approach, the $\phi-$coefficients are obtained to be

$$\begin{aligned} &[\phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \phi_{14}]^T \\ &= \left[\tfrac{113r^3h^3}{1120}, \tfrac{107r^3h^3}{1008}, -\tfrac{103r^3h^3}{1080}, \tfrac{43r^3h^3}{1680}, -\tfrac{47r^3h^3}{10080}\right]^T \end{aligned} \tag{7}$$

Similarly, expanding each individual term in the second expression of equation (4) using equation (6) yields

$$\begin{aligned} &y_n + (2rh)y_n' + \tfrac{(2rh)^2}{2!}y_n'' + \tfrac{(2rh)^3}{3!}y_n''' + \tfrac{(2rh)^4}{4!}y_n^{iv} + \tfrac{(2rh)^5}{5!}y_n^{v} \\ &+\tfrac{(2rh)^6}{6!}y_n^{vi} + \tfrac{(2rh)^7}{7!}y_n^{vii} + \dots = y_n + (2rh)y_n' + \tfrac{(2rh)^2}{2!}y_n'' \\ &+(\phi_{20}y_n''' + \phi_{21}(y_n''' + (rh)y_n^{iv} + \tfrac{(rh)^2}{2!}y_n^{v} + \tfrac{(rh)^3}{3!}y_n^{vi} \\ &+\tfrac{(rh)^4}{4!}y_n^{vii} + \dots) + \phi_{22}(y_n''' + (2rh)y_n^{iv} + \tfrac{(2rh)^2}{2!}y_n^{v} \\ &+\tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \dots) + \phi_{23}(y_n''' + (3rh)y_n^{iv} \\ &+\tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi} + \tfrac{(3rh)^4}{4!}y_n^{vii} + \dots) + \phi_{24}(y_n''' \\ &+(4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v} + \tfrac{(4rh)^3}{3!}y_n^{vi} + \tfrac{(4rh)^4}{4!}y_n^{vii} + \dots) \end{aligned}$$

with corresponding matrix form $Ax_2 = B_2$ where

$$x_2 = \begin{pmatrix} \phi_{20} \\ \phi_{21} \\ \phi_{22} \\ \phi_{23} \\ \phi_{24} \end{pmatrix}, B_2 = \begin{pmatrix} \frac{(2rh)^3}{3!} \\ \frac{(2rh)^4}{4!} \\ \frac{(2rh)^5}{5!} \\ \frac{(2rh)^6}{6!} \\ \frac{(2rh)^7}{7!} \end{pmatrix}$$

The resulting $\phi-$coefficients from matrix inverse approach are

$$\begin{aligned} &[\phi_{20}, \phi_{21}, \phi_{22}, \phi_{23}, \phi_{24}]^T \\ &= \left[\tfrac{331r^3h^3}{630}, \tfrac{332r^3h^3}{315}, -\tfrac{8r^3h^3}{21}, \tfrac{52r^3h^3}{315}, -\tfrac{19r^3h^3}{630}\right]^T \end{aligned} \tag{8}$$

Considering the third expression of equation (4), individual terms are expanded using equation (6) to obtain

$$\begin{aligned} &y_n + (3rh)y_n' + \tfrac{(3rh)^2}{2!}y_n'' + \tfrac{(3rh)^3}{3!}y_n''' + \tfrac{(3rh)^4}{4!}y_n^{iv} \\ &+\tfrac{(3rh)^5}{5!}y_n^{v} + \tfrac{(3rh)^6}{6!}y_n^{vi} + \tfrac{(3rh)^7}{7!}y_n^{vii} + \dots = y_n + (3rh)y_n' \\ &+\tfrac{(3rh)^2}{2!}y_n'' + (\phi_{30}y_n''' + \phi_{31}(y_n''' + (rh)y_n^{iv} + \tfrac{(rh)^2}{2!}y_n^{v} \\ &+\tfrac{(rh)^3}{3!}y_n^{vi} + \tfrac{(rh)^4}{4!}y_n^{vii} + \dots) + \phi_{32}(y_n''' + (2rh)y_n^{iv} \\ &+\tfrac{(2rh)^2}{2!}y_n^{v} + \tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \dots) + \phi_{33}(y_n''' \\ &+(3rh)y_n^{iv} + \tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi} + \tfrac{(3rh)^4}{4!}y_n^{vii} + \dots) \\ &+\phi_{34}(y_n''' + (4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v} + \tfrac{(4rh)^3}{3!}y_n^{vi} \\ &+\tfrac{(4rh)^4}{4!}y_n^{vii} + \dots) \end{aligned}$$

having matrix form $Ax_3 = B_3$ with

$$x_3 = \begin{pmatrix} \phi_{30} \\ \phi_{31} \\ \phi_{32} \\ \phi_{33} \\ \phi_{34} \end{pmatrix}, B_3 = \begin{pmatrix} \frac{(3rh)^3}{3!} \\ \frac{(3rh)^4}{4!} \\ \frac{(3rh)^5}{5!} \\ \frac{(3rh)^6}{6!} \\ \frac{(3rh)^7}{7!} \end{pmatrix}$$

and

$$\begin{aligned} &[\phi_{30}, \phi_{31}, \phi_{32}, \phi_{33}, \phi_{34}]^T \\ &= \left[\tfrac{1431r^3h^3}{1120}, \tfrac{1863r^3h^3}{560}, -\tfrac{243r^3h^3}{560}, \tfrac{45r^3h^3}{112}, -\tfrac{81r^3h^3}{1120}\right]^T \end{aligned} \tag{9}$$

Likewise, considering the fourth and last expression of equation (4), individual terms are expanded using equation (6) to obtain

$$\begin{aligned} &y_n + (4rh)y_n' + \tfrac{(4rh)^2}{2!}y_n'' + \tfrac{(4rh)^3}{3!}y_n''' + \tfrac{(4rh)^4}{4!}y_n^{iv} \\ &+\tfrac{(4rh)^5}{5!}y_n^{v} + \tfrac{(4rh)^6}{6!}y_n^{vi} + \tfrac{(4rh)^7}{7!}y_n^{vii} + \dots = y_n \\ &+(4rh)y_n' + \tfrac{(4rh)^2}{2!}y_n'' + (\phi_{40}y_n''' + \phi_{41}(y_n''' + (rh)y_n^{iv} \\ &+\tfrac{(rh)^2}{2!}y_n^{v} + \tfrac{(rh)^3}{3!}y_n^{vi} + \tfrac{(rh)^4}{4!}y_n^{vii} + \dots) + \phi_{42}(y_n''' \\ &+(2rh)y_n^{iv} + \tfrac{(2rh)^2}{2!}y_n^{v} + \tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \dots) \\ &+\phi_{43}(y_n''' + (3rh)y_n^{iv} + \tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi} \\ &+\tfrac{(3rh)^4}{4!}y_n^{vii} + \dots) + \phi_{44}(y_n''' + (4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v} \\ &+\tfrac{(4rh)^3}{3!}y_n^{vi} + \tfrac{(4rh)^4}{4!}y_n^{vii} + \dots) \end{aligned}$$

written in the matrix form $Ax_4 = B_4$,

$$x_4 = \begin{pmatrix} \phi_{40} \\ \phi_{41} \\ \phi_{42} \\ \phi_{43} \\ \phi_{44} \end{pmatrix}, B_4 = \begin{pmatrix} \frac{(4rh)^3}{3!} \\ \frac{(4rh)^4}{4!} \\ \frac{(4rh)^5}{5!} \\ \frac{(4rh)^6}{6!} \\ \frac{(4rh)^7}{7!} \end{pmatrix}$$

and The $\phi-$coefficients are obtained as

$$\begin{aligned} &[\phi_{40}, \phi_{41}, \phi_{42}, \phi_{43}, \phi_{44}]^T \\ &= \left[\tfrac{248r^3h^3}{105}, \tfrac{2176r^3h^3}{315}, \tfrac{32r^3h^3}{105}, \tfrac{128r^3h^3}{105}, -\tfrac{8r^3h^3}{63}\right]^T \end{aligned} \tag{10}$$

Moving on to obtaining the $\omega-$coefficients in equation (5). Expanding each individual term in the first expression of equation (5) using equation (6) above gives

$$y'_n + (rh)y''_n + \frac{(rh)^2}{2!}y'''_n + \frac{(rh)^3}{3!}y^{iv}_n + \frac{(rh)^4}{4!}y^{v}_n + \frac{(rh)^5}{5!}y^{vi}_n + \frac{(rh)^6}{6!}y^{vii}_n + \dots = y'_n + (rh)y''_n + (\omega_{101}y'''_n + \omega_{111}(y'''_n + (rh)y^{iv}_n + \frac{(rh)^2}{2!}y^{v}_n + \frac{(rh)^3}{3!}y^{vi}_n + \frac{(rh)^4}{4!}y^{vii}_n + \dots) + \omega_{121}(y'''_n + (2rh)y^{iv}_n + \frac{(2rh)^2}{2!}y^{v}_n + \frac{(2rh)^3}{3!}y^{vi}_n + \frac{(2rh)^4}{4!}y^{vii}_n + \dots) + \omega_{131}(y'''_n + (3rh)y^{iv}_n + \frac{(3rh)^2}{2!}y^{v}_n + \frac{(3rh)^3}{3!}y^{vi}_n + \frac{(3rh)^4}{4!}y^{vii}_n + \dots) + \omega_{141}(y'''_n + (4rh)y^{iv}_n + \frac{(4rh)^2}{2!}y^{v}_n + \frac{(4rh)^3}{3!}y^{vi}_n + \frac{(4rh)^4}{4!}y^{vii}_n + \dots)$$

which can be written in the matrix form $Ax_5 = B_5$ where

$$x_5 = \begin{pmatrix} \omega_{101} \\ \omega_{111} \\ \omega_{121} \\ \omega_{131} \\ \omega_{141} \end{pmatrix}, B_5 = \begin{pmatrix} \frac{(rh)^2}{2!} \\ \frac{(rh)^3}{3!} \\ \frac{(rh)^4}{4!} \\ \frac{(rh)^5}{5!} \\ \frac{(rh)^6}{6!} \end{pmatrix}$$

and The matrix inverse approach is also adopted to obtain the $\omega-$coefficients as

$$[\omega_{101}, \omega_{111}, \omega_{121}, \omega_{131}, \omega_{141}]^T = \left[\frac{367r^2h^2}{1440}, \frac{3r^2h^2}{8}, -\frac{47r^2h^2}{240}, \frac{29r^2h^2}{360}, -\frac{7r^2h^2}{480}\right]^T \tag{11}$$

For the second expression in equation (5), each individual term are likewise expanded using equation (6) to obtain

$$y'_n + (2rh)y''_n + \frac{(2rh)^2}{2!}y'''_n + \frac{(2rh)^3}{3!}y^{iv}_n + \frac{(2rh)^4}{4!}y^{v}_n + \frac{(2rh)^5}{5!}y^{vi}_n + \frac{(2rh)^6}{6!}y^{vii}_n + \dots = y'_n + (2rh)y''_n + (\omega_{201}y'''_n + \omega_{211}(y'''_n + (rh)y^{iv}_n + \frac{(rh)^2}{2!}y^{v}_n + \frac{(rh)^3}{3!}y^{vi}_n + \frac{(rh)^4}{4!}y^{vii}_n + \dots) + \omega_{221}(y'''_n + (2rh)y^{iv}_n + \frac{(2rh)^2}{2!}y^{v}_n + \frac{(2rh)^3}{3!}y^{vi}_n + \frac{(2rh)^4}{4!}y^{vii}_n + \dots) + \omega_{231}(y'''_n + (3rh)y^{iv}_n + \frac{(3rh)^2}{2!}y^{v}_n + \frac{(3rh)^3}{3!}y^{vi}_n + \frac{(3rh)^4}{4!}y^{vii}_n + \dots) + \omega_{241}(y'''_n + (4rh)y^{iv}_n + \frac{(4rh)^2}{2!}y^{v}_n + \frac{(4rh)^3}{3!}y^{vi}_n + \frac{(4rh)^4}{4!}y^{vii}_n + \dots)$$

which can be written in the matrix form $Ax_6 = B_6$ where

$$x_6 = \begin{pmatrix} \omega_{201} \\ \omega_{211} \\ \omega_{221} \\ \omega_{231} \\ \omega_{241} \end{pmatrix}, B_6 = \begin{pmatrix} \frac{(2rh)^2}{2!} \\ \frac{(2rh)^3}{3!} \\ \frac{(2rh)^4}{4!} \\ \frac{(2rh)^5}{5!} \\ \frac{(2rh)^6}{6!} \end{pmatrix}$$

and

$$[\omega_{201}, \omega_{211}, \omega_{221}, \omega_{231}, \omega_{241}]^T = \left[\frac{53r^2h^2}{90}, \frac{8r^2h^2}{5}, -\frac{r^2h^2}{3}, \frac{8r^2h^2}{45}, -\frac{r^2h^2}{30}\right]^T \tag{12}$$

Moving to the third expression in equation (5). In same manner, equation (6) and (7) are adopted to expand its individual terms to obtain

$$y'_n + (3rh)y''_n + \frac{(3rh)^2}{2!}y'''_n + \frac{(3rh)^3}{3!}y^{iv}_n + \frac{(3rh)^4}{4!}y^{v}_n + \frac{(3rh)^5}{5!}y^{vi}_n + \frac{(3rh)^6}{6!}y^{vii}_n + \dots = y'_n + (3rh)y''_n + (\omega_{301}y'''_n + \omega_{311}(y'''_n + (rh)y^{iv}_n + \frac{(rh)^2}{2!}y^{v}_n + \frac{(rh)^3}{3!}y^{vi}_n + \frac{(rh)^4}{4!}y^{vii}_n + \dots) + \omega_{321}(y'''_n + (2rh)y^{iv}_n + \frac{(2rh)^2}{2!}y^{v}_n + \frac{(2rh)^3}{3!}y^{vi}_n + \frac{(2rh)^4}{4!}y^{vii}_n + \dots) + \omega_{331}(y'''_n + (3rh)y^{iv}_n + \frac{(3rh)^2}{2!}y^{v}_n + \frac{(3rh)^3}{3!}y^{vi}_n + \frac{(3rh)^4}{4!}y^{vii}_n + \dots) + \omega_{341}(y'''_n + (4rh)y^{iv}_n + \frac{(4rh)^2}{2!}y^{v}_n + \frac{(4rh)^3}{3!}y^{vi}_n + \frac{(4rh)^4}{4!}y^{vii}_n + \dots)$$

in the matrix form $Ax_7 = B_7$ where

$$x_7 = \begin{pmatrix} \omega_{301} \\ \omega_{311} \\ \omega_{321} \\ \omega_{331} \\ \omega_{341} \end{pmatrix}, B_7 = \begin{pmatrix} \frac{(3rh)^2}{2!} \\ \frac{(3rh)^3}{3!} \\ \frac{(3rh)^4}{4!} \\ \frac{(3rh)^5}{5!} \\ \frac{(3rh)^6}{6!} \end{pmatrix}$$

and

$$[\omega_{301}, \omega_{311}, \omega_{321}, \omega_{331}, \omega_{341}]^T = \left[\frac{147r^2h^2}{160}, \frac{117r^2h^2}{40}, \frac{27r^2h^2}{80}, \frac{3r^2h^2}{8}, -\frac{9r^2h^2}{160}\right]^T \tag{13}$$

Now consider the fourth expression in equation (5), which is the last expression for the first derivative schemes. Individual terms are also expanded using equation (6) to obtain

$$y'_n + (4rh)y''_n + \frac{(4rh)^2}{2!}y'''_n + \frac{(4rh)^3}{3!}y^{iv}_n + \frac{(4rh)^4}{4!}y^{v}_n + \frac{(4rh)^5}{5!}y^{vi}_n + \frac{(4rh)^6}{6!}y^{vii}_n + \dots = y'_n + (4rh)y''_n + (\omega_{401}y'''_n + \omega_{411}(y'''_n + (rh)y^{iv}_n + \frac{(rh)^2}{2!}y^{v}_n + \frac{(rh)^3}{3!}y^{vi}_n + \frac{(rh)^4}{4!}y^{vii}_n + \dots) + \omega_{421}(y'''_n + (2rh)y^{iv}_n + \frac{(2rh)^2}{2!}y^{v}_n + \frac{(2rh)^3}{3!}y^{vi}_n + \frac{(2rh)^4}{4!}y^{vii}_n + \dots) + \omega_{431}(y'''_n + (3rh)y^{iv}_n + \frac{(3rh)^2}{2!}y^{v}_n + \frac{(3rh)^3}{3!}y^{vi}_n + \frac{(3rh)^4}{4!}y^{vii}_n + \dots) + \omega_{441}(y'''_n + (4rh)y^{iv}_n + \frac{(4rh)^2}{2!}y^{v}_n + \frac{(4rh)^3}{3!}y^{vi}_n + \frac{(4rh)^4}{4!}y^{vii}_n + \dots)$$

which can be written in the matrix form $Ax_8 = B_8$ where

$$x_8 = \begin{pmatrix} \omega_{401} \\ \omega_{411} \\ \omega_{421} \\ \omega_{431} \\ \omega_{441} \end{pmatrix}, B_8 = \begin{pmatrix} \frac{(4rh)^2}{2!} \\ \frac{(4rh)^3}{3!} \\ \frac{(4rh)^4}{4!} \\ \frac{(4rh)^5}{5!} \\ \frac{(4rh)^6}{6!} \end{pmatrix}$$

with resulting $\omega-$coefficients obtained to be

$$[\omega_{401}, \omega_{411}, \omega_{421}, \omega_{431}, \omega_{441}]^T = \left[\frac{56r^2h^2}{45}, \frac{64r^2h^2}{15}, \frac{16r^2h^2}{15}, \frac{64r^2h^2}{45}, 0\right]^T \tag{14}$$

The fifth expression in equation (5) is the first expression for the second derivative schemes. Expanding each individual term in this fifth expression of equation (5) using equation (6) above gives

$$y''_n + (rh)y'''_n + \frac{(rh)^2}{2!}y^{iv}_n + \frac{(rh)^3}{3!}y^{v}_n + \frac{(rh)^4}{4!}y^{vi}_n + \frac{(rh)^5}{5!}y^{vii}_n + \dots = y''_n + (\omega_{102}y'''_n + \omega_{112}(y'''_n + (rh)y^{iv}_n + \frac{(rh)^2}{2!}y^{v}_n + \frac{(rh)^3}{3!}y^{vi}_n + \frac{(rh)^4}{4!}y^{vii}_n + \dots) + \omega_{122}(y'''_n + (2rh)y^{iv}_n + \frac{(2rh)^2}{2!}y^{v}_n + \frac{(2rh)^3}{3!}y^{vi}_n + \frac{(2rh)^4}{4!}y^{vii}_n + \dots) + \omega_{132}(y'''_n + (3rh)y^{iv}_n + \frac{(3rh)^2}{2!}y^{v}_n + \frac{(3rh)^3}{3!}y^{vi}_n + \frac{(3rh)^4}{4!}y^{vii}_n + \dots) + \omega_{142}(y'''_n + (4rh)y^{iv}_n + \frac{(4rh)^2}{2!}y^{v}_n + \frac{(4rh)^3}{3!}y^{vi}_n + \frac{(4rh)^4}{4!}y^{vii}_n + \dots)$$

implying the matrix form $Ax_9 = B_9$ with

$$x_9 = \begin{pmatrix} \omega_{102} \\ \omega_{112} \\ \omega_{122} \\ \omega_{132} \\ \omega_{142} \end{pmatrix}, B_9 = \begin{pmatrix} \frac{(rh)^1}{1!} \\ \frac{(rh)^2}{2!} \\ \frac{(rh)^3}{3!} \\ \frac{(rh)^4}{4!} \\ \frac{(rh)^5}{5!} \end{pmatrix}$$

The $\omega-$coefficients are obtained to be

$$[\omega_{102}, \omega_{112}, \omega_{122}, \omega_{132}, \omega_{142}]^T = \left[\frac{251rh}{720}, \frac{323rh}{320}, -\frac{11rh}{30}, \frac{53rh}{360}, -\frac{19rh}{720}\right]^T \tag{15}$$

Similarly, expanding each individual term in the sixth expression in equation (5) using equation (6) yields

$$\begin{aligned}&y_n'' + (2rh)y_n''' + \tfrac{(2rh)^2}{2!}y_n^{iv} + \tfrac{(2rh)^3}{3!}y_n^{v} + \tfrac{(2rh)^4}{4!}y_n^{vi}\\ &+\tfrac{(2rh)^5}{5!}y_n^{vii} + \cdots = y_n'' + (\omega_{202}y_n''' + \omega_{212}(y_n''' + (rh)y_n^{iv}\\ &+\tfrac{(rh)^2}{2!}y_n^{v} + \tfrac{(rh)^3}{3!}y_n^{vi} + \tfrac{(rh)^4}{4!}y_n^{vii} + \ldots) + \omega_{222}(y_n'''\\ &+(2rh)y_n^{iv} + \tfrac{(2rh)^2}{2!}y_n^{v} + \tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \ldots)\\ &+\omega_{232}(y_n''' + (3rh)y_n^{iv} + \tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi}\\ &+\tfrac{(3rh)^4}{4!}y_n^{vii} + \ldots) + \omega_{242}(y_n''' + (4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v}\\ &+\tfrac{(4rh)^3}{3!}y_n^{vi} + \tfrac{(4rh)^4}{4!}y_n^{vii} + \ldots)\end{aligned}$$

with corresponding matrix form $Ax_{10} = B_{10}$ where

$$x_{10} = \begin{pmatrix}\omega_{202}\\ \omega_{212}\\ \omega_{222}\\ \omega_{232}\\ \omega_{242}\end{pmatrix}, B_{10} = \begin{pmatrix}\frac{(2rh)^1}{1!}\\ \frac{(2rh)^2}{2!}\\ \frac{(2rh)^3}{3!}\\ \frac{(2rh)^4}{4!}\\ \frac{(2rh)^5}{5!}\end{pmatrix}$$

and

$$\begin{aligned}&[\omega_{202}, \omega_{212}, \omega_{222}, \omega_{232}, \omega_{242}]^T\\ &= \left[\tfrac{29rh}{90}, \tfrac{62rh}{45}, \tfrac{4rh}{15}, \tfrac{2rh}{360}, -\tfrac{rh}{90}\right]^T\end{aligned} \tag{16}$$

For the seventh expression in equation (5), individual terms are expanded using equation (6) to obtain

$$\begin{aligned}&y_n'' + (3rh)y_n''' + \tfrac{(3rh)^2}{2!}y_n^{iv} + \tfrac{(3rh)^3}{3!}y_n^{v} + \tfrac{(3rh)^4}{4!}y_n^{vi}\\ &+\tfrac{(3rh)^5}{5!}y_n^{vii} + \cdots = y_n'' + (\omega_{302}y_n''' + \omega_{312}(y_n''' + (rh)y_n^{iv}\\ &+\tfrac{(rh)^2}{2!}y_n^{v} + \tfrac{(rh)^3}{3!}y_n^{vi} + \tfrac{(rh)^4}{4!}y_n^{vii} + \ldots) + \omega_{322}(y_n'''\\ &+(2rh)y_n^{iv} + \tfrac{(2rh)^2}{2!}y_n^{v} + \tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \ldots)\\ &+\omega_{332}(y_n''' + (3rh)y_n^{iv} + \tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi} + \tfrac{(3rh)^4}{4!}y_n^{vii}\\ &+\ldots) + \omega_{342}(y_n''' + (4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v} + \tfrac{(4rh)^3}{3!}y_n^{vi}\\ &+\tfrac{(4rh)^4}{4!}y_n^{vii} + \ldots)\end{aligned}$$

written in the matrix form $Ax_{11} = B_{11}$ where

$$x_{11} = \begin{pmatrix}\omega_{302}\\ \omega_{312}\\ \omega_{322}\\ \omega_{332}\\ \omega_{342}\end{pmatrix}, B_{11} = \begin{pmatrix}\frac{(3rh)^1}{1!}\\ \frac{(3rh)^2}{2!}\\ \frac{(3rh)^3}{3!}\\ \frac{(3rh)^4}{4!}\\ \frac{(3rh)^5}{5!}\end{pmatrix}$$

and

$$\begin{aligned}&[\omega_{302}, \omega_{312}, \omega_{322}, \omega_{332}, \omega_{342}]^T\\ &= \left[\tfrac{27rh}{80}, \tfrac{51rh}{40}, \tfrac{9rh}{10}, \tfrac{21rh}{40}, -\tfrac{3rh}{80}\right]^T\end{aligned} \tag{17}$$

Finally, for the eighth expression in equation (5) which is the last expression for the derivative schemes, individual terms are also expanded using equation (6) as given

$$\begin{aligned}&y_n'' + (4rh)y_n''' + \tfrac{(4rh)^2}{2!}y_n^{iv} + \tfrac{(4rh)^3}{3!}y_n^{v} + \tfrac{(4rh)^4}{4!}y_n^{vi}\\ &+\tfrac{(4rh)^5}{5!}y_n^{vii} + \cdots = y_n'' + (\omega_{402}y_n''' + \omega_{412}(y_n''' + (rh)y_n^{iv}\\ &+\tfrac{(rh)^2}{2!}y_n^{v} + \tfrac{(rh)^3}{3!}y_n^{vi} + \tfrac{(rh)^4}{4!}y_n^{vii} + \ldots) + \omega_{422}(y_n'''\\ &+(2rh)y_n^{iv} + \tfrac{(2rh)^2}{2!}y_n^{v} + \tfrac{(2rh)^3}{3!}y_n^{vi} + \tfrac{(2rh)^4}{4!}y_n^{vii} + \ldots)\\ &+\omega_{432}(y_n''' + (3rh)y_n^{iv} + \tfrac{(3rh)^2}{2!}y_n^{v} + \tfrac{(3rh)^3}{3!}y_n^{vi}\\ &+\tfrac{(3rh)^4}{4!}y_n^{vii} + \ldots) + \omega_{442}(y_n''' + (4rh)y_n^{iv} + \tfrac{(4rh)^2}{2!}y_n^{v}\\ &+\tfrac{(4rh)^3}{3!}y_n^{vi} + \tfrac{(4rh)^4}{4!}y_n^{vii} + \ldots)\end{aligned}$$

with corresponding matrix form $Ax_{12} = B_{12}$ where

$$x_{12} = \begin{pmatrix}\omega_{402}\\ \omega_{412}\\ \omega_{422}\\ \omega_{432}\\ \omega_{442}\end{pmatrix}, B_{12} = \begin{pmatrix}\frac{(4rh)^1}{1!}\\ \frac{(4rh)^2}{2!}\\ \frac{(4rh)^3}{3!}\\ \frac{(4rh)^4}{4!}\\ \frac{(4rh)^5}{5!}\end{pmatrix}$$

and

$$\begin{aligned}&[\omega_{402}, \omega_{412}, \omega_{422}, \omega_{432}, \omega_{442}]^T\\ &= \left[\tfrac{14rh}{45}, \tfrac{64rh}{45}, \tfrac{8rh}{15}, \tfrac{64rh}{45}, \tfrac{14rh}{45}\right]^T\end{aligned} \tag{18}$$

Substituting the values obtained for the unknown coefficients in equations (7), (8), (9) and (10), into equation (4) gives the block form

$$I_4Y_{nr} = A_1Z_n + A_2Z_{n1} + A_3Z_{n2} + B_1F_n + B_2F_{nr} \tag{19}$$

where

$$I_4 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, Y_{nr} = \begin{pmatrix}y_{n+r}\\ y_{n+2r}\\ y_{n+3r}\\ y_{n+4r}\end{pmatrix},$$

$$A_1 = \begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}, Z_n = \begin{pmatrix}y_{n-3r}\\ y_{n-2r}\\ y_{n-r}\\ y_n\end{pmatrix}$$

$$A_2 = \begin{pmatrix}0&0&0&rh\\0&0&0&2rh\\0&0&0&3rh\\0&0&0&4rh\end{pmatrix}, Z_{n1} = \begin{pmatrix}y'_{n-3r}\\ y'_{n-2r}\\ y'_{n-r}\\ y'_n\end{pmatrix},$$

$$A_3 = \begin{pmatrix}0&0&0&\frac{(rh)^2}{2!}\\0&0&0&\frac{(2rh)^2}{2!}\\0&0&0&\frac{(3rh)^2}{2!}\\0&0&0&\frac{(4rh)^2}{2!}\end{pmatrix}, Z_{n2} = \begin{pmatrix}y''_{n-3r}\\ y''_{n-2r}\\ y''_{n-r}\\ y''_n\end{pmatrix}$$

$$B_1 = \begin{pmatrix}0&0&0&\frac{113r^3h^3}{1120}\\0&0&0&\frac{331r^3h^3}{630}\\0&0&0&\frac{1431r^3h^3}{1120}\\0&0&0&\frac{248r^3h^3}{105}\end{pmatrix}, F_n = \begin{pmatrix}f_{n-3r}\\ f_{n-2r}\\ f_{n-r}\\ f_n\end{pmatrix},$$

$$B_2 = \begin{pmatrix}\frac{107r^3h^3}{1008} & -\frac{103r^3h^3}{1680} & \frac{43r^3h^3}{1680} & -\frac{47r^3h^3}{10080}\\ \frac{332r^3h^3}{315} & -\frac{8r^3h^3}{21} & \frac{52r^3h^3}{315} & -\frac{19r^3h^3}{630}\\ \frac{1863r^3h^3}{560} & -\frac{243r^3h^3}{560} & \frac{45r^3h^3}{112} & -\frac{81r^3h^3}{1120}\\ \frac{2176r^3h^3}{315} & \frac{32r^3h^3}{105} & \frac{128r^3h^3}{105} & -\frac{8r^3h^3}{63}\end{pmatrix},$$

$$\text{and} \quad F_{nr} = \begin{pmatrix}f_{n+r}\\ f_{n+2r}\\ f_{n+3r}\\ f_{n+4r}\end{pmatrix}.$$

Substituting the values obtained for the unknown coefficients in equations (10)-(18), into equation (5) gives the block form

$$\begin{aligned}I_4Y_{nr}^{(1)} &= A_1Z_{n1} + A_2Z_{n2} + B_1^{(1)}F_n + B_2^{(1)}F_{nr}\\ I_4Y_{nr}^{(2)} &= A_1Z_{n2} + B_1^{(2)}F_n + B_2^{(2)}F_{nr}\end{aligned} \tag{20}$$

where

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Y_{nr}^{(1)} = \begin{pmatrix} y'_{n+r} \\ y'_{n+2r} \\ y'_{n+3r} \\ y'_{n+4r} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Z_{n1} = \begin{pmatrix} y'_{n-3r} \\ y'_{n-2r} \\ y'_{n-r} \\ y'_{n} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & rh \\ 0 & 0 & 0 & 2rh \\ 0 & 0 & 0 & 3rh \\ 0 & 0 & 0 & 4rh \end{pmatrix}, Z_{n2} = \begin{pmatrix} y''_{n-3r} \\ y''_{n-2r} \\ y''_{n-r} \\ y''_{n} \end{pmatrix},$$

$$B_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & \frac{367r^2h^2}{1440} \\ 0 & 0 & 0 & \frac{53r^2h^2}{90} \\ 0 & 0 & 0 & \frac{147r^2h^2}{160} \\ 0 & 0 & 0 & \frac{56r^2h^2}{45} \end{pmatrix}, F_n = \begin{pmatrix} f_{n-3r} \\ f_{n-2r} \\ f_{n-r} \\ f_{n} \end{pmatrix},$$

$$B_2^{(1)} = \begin{pmatrix} \frac{3r^2h^2}{8} & -\frac{47r^2h^2}{240} & \frac{29r^2h^2}{360} & -\frac{7r^2h^2}{480} \\ \frac{8r^2h^2}{5} & -\frac{r^2h^2}{3} & \frac{8r^2h^2}{45} & -\frac{r^2h^2}{30} \\ \frac{117r^2h^2}{40} & \frac{27r^2h^2}{80} & \frac{3r^2h^2}{8} & -\frac{9r^2h^2}{160} \\ \frac{64r^2h^2}{15} & \frac{16r^2h^2}{15} & \frac{64r^2h^2}{45} & 0 \end{pmatrix},$$

$$F_{nr} = \begin{pmatrix} f_{n+r} \\ f_{n+2r} \\ f_{n+3r} \\ f_{n+4r} \end{pmatrix}, Y_{nr}^{(2)} = \begin{pmatrix} y''_{n+r} \\ y''_{n+2r} \\ y''_{n+3r} \\ y''_{n+4r} \end{pmatrix},$$

$$B_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & \frac{251rh}{720} \\ 0 & 0 & 0 & \frac{29rh}{90} \\ 0 & 0 & 0 & \frac{27rh}{80} \\ 0 & 0 & 0 & \frac{14rh}{45} \end{pmatrix}, \quad \text{and}$$

$$B_2^{(2)} = \begin{pmatrix} \frac{323rh}{360} & -\frac{11rh}{30} & \frac{53rh}{360} & -\frac{19rh}{720} \\ \frac{62rh}{45} & \frac{4rh}{15} & \frac{2rh}{45} & -\frac{rh}{90} \\ \frac{51rh}{40} & \frac{9rh}{10} & \frac{21rh}{40} & -\frac{3rh}{80} \\ \frac{64rh}{45} & \frac{8rh}{15} & \frac{64rh}{45} & \frac{14rh}{45} \end{pmatrix}.$$

III. Analysis of the Block Method

In this section, the properties investigated for the block method will be limited to the properties needed to ensure the block method is convergent. As conventionally known, a linear multistep method is convergent iff it is consistent and zero-stable [13]

Definition 3.1: A linear multistep method is *consistent* if it has order $p \geq 1$.

A. Order of the Block Method

To obtain the order of the block method, the y and $f-$ values in equation (19) are expanded to obtain

$$\begin{aligned} &\textstyle\sum_{i=0}^{7} \frac{(r\xi h)^i}{i!} y_n^{(i)} - \sum_{i=0}^{2} \frac{(r\xi h)^i}{i!} y_n^{(i)} \\ &- \left(\sum_{j=0}^{4} \phi_{\xi j} \left(\sum_{i=3}^{7} \frac{(r\xi h)^i}{i!} y_n^{(i)}\right)\right); \xi = 1,2,3,4 \\ &= [0,0,0,0] \end{aligned} \tag{21}$$

This gives the order of the block method to be of order $[5,5,5,5]^T$ with error constants $[\frac{139r^8}{40320}, \frac{r^8}{45}, \frac{243r^8}{4480}, \frac{32r^8}{315}]$

B. Zero-Stability of the Block Method

To analyze the block method (19) for zero stability, the roots of the first characteristic polynomial $\rho(r) = |rI_4 - A_1|$ must be simple or less than one. This implies that

$$\rho(r) = |rI_4 - A_1|$$
$$= \left| r \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = r^3(r-1)$$

which has roots $r = 0,0,0,1$ and this implies that the block method is zero-stable.

C. Convergence of the Block Method

The block method is consistent and zero-stable, hence the block method is convergent.

IV. Numerical Examples

Problem 1: $y''' + e^x = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 3.$
Exact solution: $y(x) = 2x^2 - e^x + 2$ with $h = 0.1$.
This third order initial value problem was solved by [7] using an hybrid block method of order 5.

Problem 2: $y''' + 4y' = x, \quad y(0) = y'(0) = 0, \quad y''(0) = 1.$
Exact solution: $y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$
This third order initial value problem was solved by [11] using a combination of predictor of order 6 and the corrector of order 7 and also [2].

Problem 3: $y''' + y = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1, [0,1].$
Exact solution: $y(x) = e^{-x} + 2$ with $h = 0.1$.
This third order initial value problem was solved by [5] and [7] using block methods of order 8 and 5 respectively. The maximum error at the end of the interval is considered

Problem 4: $y''' - xy'' + x^2y^2 = xsinx - cosx + x^2sin^2x, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, [0,1].$
Exact solution: $y(x) = sinx$ with $h = 0.1$.
This nonlinear initial value problem in third order ordinary differential equation was sourced from [14].

Problem 5: $y''' = \frac{3}{8y^5}, \quad y(x_0) = 0, \quad y'(x_0) = \frac{1}{2}, \quad y''(x_0) = -\frac{1}{4}$ where $0 \leq x \leq 2$.
Exact solution: $y(x) = \sqrt{1+x}$ with $h = 0.1$.
This special nonlinear initial value problem in third order ordinary differential equation was sourced from [15]. The authors did not provide a numerical approximation of this problem, thus Table 5 displays only the numerical solution using the hybrid block method in comparison to the exact solution.

Table 1: Comparison of results with [7] for solving Problem 1

x	Exact Solution	Computed Solution with $r = \frac{1}{4}$	Error [7]	Error (Hybrid Block Method)
0.1	0.91482908192435231	0.91482908192436851	1.110223E-14	1.620926E-14
0.2	0.85859724183983022	0.85859724183989627	1.607603E-13	6.605827E-14
0.3	0.83014119242399698	0.83014119242414974	6.310508E-13	1.527667E-13
0.4	0.82817530235872971	0.82817530235900927	1.623146E-12	2.795542E-13
0.5	0.85127872929987181	0.85127872930032200	3.359091E-12	4.501954E-13
0.6	0.89788119960949109	0.89788119961015944	6.084133E-12	6.683543E-13
0.7	0.96624729252952335	0.96624729253046260	1.006994E-11	9.392487E-13
0.8	1.05445907150753240	1.05445907150879940	1.561595E-11	1.266987E-12
0.9	1.16039688884305030	1.16039688884470800	2.305356E-11	1.657785E-12
1.0	1.28171817154095450	1.28171817154307190	3.274958E-11	2.117417E-12

Table 2: Comparison of results for solving Problem 2

Awoyemi (2003)				Zanariah et al (2012)			Hybrid Block Method with $r = \frac{1}{4}$		
Step-size	b	TS	Maximum Error	b	TS	Maximum Error	b	TS	Maximum Error
0.025	5.0	200	3.94E-6	5.0	46	4.66E-7	5.0	46	1.20E-10
					56	9.14E-8		56	3.69E-11
					88	1.53E-10		88	2.44E-12
0.025	10.0	400	3.80E-6	10.0	61	4.66E-7	10.0	61	5.54E-09
					91	2.43E-8		91	5.04E-10
					136	1.53E-10		136	4.53E-11
0.025	15.0	600	2.29E-6	15.0	76	4.66E-7	15.0	76	2.67E-08
					110	2.63E-8		110	2.91E-09
					180	1.54E-10		180	1.52E-10
0.025	20.0	800	1.30E-6	20.0	91	4.66E-7	20.0	91	5.29E-08
					129	2.63E-8		129	6.54E-09
					204	1.28E-9		204	4.19E-10

Table 3: Comparison of results with [5] and [7] for solving Problem 3

Maximum Error [5]	8.200535E-11
Maximum Error [7]	3.443523E-12
Maximum Error (Hybrid Block Method with $r = \frac{1}{4}$)	1.184275E-12

Table 4: Comparison of results for solving Problem 4

x	Exact Solution	Computed Solution with $r = \frac{1}{4}$	Error (Hybrid Block Method)
0.1	0.099833416646828155	0.099833416646827489	6.661338E-16
0.2	0.198669330795061220	0.198669330795057300	3.913536E-15
0.3	0.295520206661339600	0.295520206661327170	1.243450E-14
0.4	0.389418342308650520	0.389418342308621660	2.886580E-14
0.5	0.479425538604203010	0.479425538604146990	5.601075E-14
0.6	0.564642473395035370	0.564642473394938450	9.692247E-14
0.7	0.644217687237691020	0.644217687237536360	1.546541E-13
0.8	0.717356090899522680	0.717356090899290090	2.325917E-13
0.9	0.783326909627483300	0.783326909627148680	3.346212E-13
1.0	0.841470984807896390	0.841470984807431990	4.644063E-13

Table 5: Comparison of results for solving Problem 5

x	Exact Solution	Computed Solution with $r = \frac{1}{4}$	Error (Hybrid Block Method)
0.1	1.048808848170151600	1.048808848176159900	6.008305E-12
0.2	1.095445115010332100	1.095445115032145100	2.181300E-11
0.3	1.140175425099138100	1.140175425143080500	4.394241E-11
0.4	1.183215956619923200	1.183215956690615500	7.069234E-11
0.5	1.224744871391588900	1.224744871492748900	1.011600E-10
0.6	1.264911064067351800	1.264911064202180800	1.348290E-10
0.7	1.303840481040529700	1.303840481211909800	1.713800E-10
0.8	1.341640786499873800	1.341640786710470700	2.105969E-10
0.9	1.378404875209022100	1.378404875461343400	2.523213E-10
1.0	1.414213562373095100	1.414213562669522500	2.964273E-10
1.1	1.449137674618943700	1.449137674961753900	3.428102E-10
1.2	1.483239697419132600	1.483239697810508700	3.913760E-10
1.3	1.516575088810310000	1.516575089252350800	4.420408E-10
1.4	1.549193338482966800	1.549193338977691300	4.947245E-10
1.5	1.581138830084189800	1.581138830633542500	5.493528E-10
1.6	1.612451549659710000	1.612451550265563600	6.058536E-10
1.7	1.643167672515498400	1.643167673179656700	6.641583E-10
1.8	1.673320053068151300	1.673320053792351400	7.242000E-10
1.9	1.702938636592640200	1.702938637378554900	7.859147E-10
2.0	1.732050807568877400	1.732050808418116800	8.492393E-10

Problems 1 to 5 have considered sample third order ODEs ranging from linear to nonlinear problems. The results displayed of each Problem are displayed in Tables 1 to 5 respectively and the hybrid block methods shows impressive accuracy when compares to other authors.

For Problem 5, [15] displayed the maximum error for solving Problem 5 using a five-step explicit method as 1.02939×10^{-7} over the defined interval.

Therefore, the new hybrid block method has better accuracy with its maximum error of 8.492393×10^{-10}.

To further show the usability of the hybrid block method, certain benchmark models of third order ODEs are considered as discussed in the following sections.

V. Application to Solve Nonlinear Genesio Equation

The section considers the non-linear chaotic system from [16]

$$x''' + Ax'' + Bx' - f(x(t)) = 0 \qquad (22)$$

with

$$f(x(t)) = -Cx(t) + x^2(t) \qquad (23)$$

subject to the following conditions:

$$x(0) = 0.2, \quad x'(0) = -0.3, x''(0) = 0.1, \quad t \in [0, b]$$

where $A = 1.2$, $B = 2.92$ and $C = 6$ are positive constants satisfying $AB < C$ to guarantee the existence of the solution of (22). The new hybrid block method is adopted to solve (22) in self-starting mode where the block method simultaneously integrates (22) at all grid points. The numerical results obtained from the new hybrid block method are compared with the solutions obtained in [17]–[19]. Table 6 shows the comparison in the numerical approximation of x at the end points $b = 1$ and $b = 4$. It is the observed that the new hybrid block method obtains convergent results as [17] variable step three-point block multistep method and [19] block hybrid collocation method. Figure 2 displays the numerical solutions for the nonlinear Genesio equation (22) in the interval [0, 4.5] as depicted in the separate works of [18] and [19]. The numerical approximations obtained by the new hybrid block method is further seen to be in agreement with the authors.

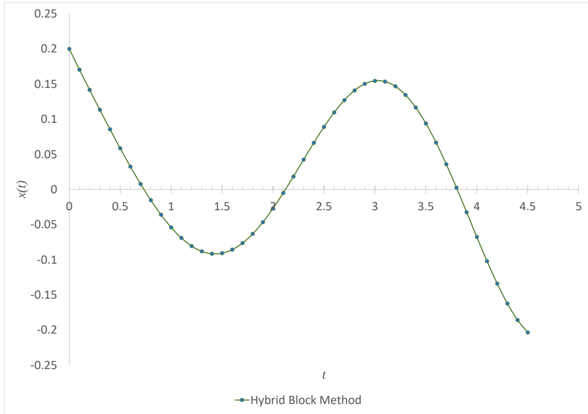

Fig. 2. Solutions obtained for (22) using Hybrid Block Method

Table 6: Comparison of results for the nonlinear Genesio equation (22)

Mehrkanoon (2011)			Yap et al (2014)			Hybrid Block Method with $r = \frac{1}{4}$		
b	TS	x	b	TS	x	b	TS	x
1.0	17	-0.054005	1.0	4	-0.05400408324564678	1.0	4	-0.0540040856662849158
	26	-0.054003		34	-0.05400408355473926		17	-0.0540040835551079960
							26	-0.0540040835547795220
							34	-0.0540040835547572970
4.0	23	-0.067921	4.0	13	-0.06763059062408930	4.0	13	-0.0676306051720177990
	45	-0.067692		133	-0.06763060515900272		23	-0.0676306051595603060
							45	-0.0676306051591478860
							133	-0.0676306051591406140

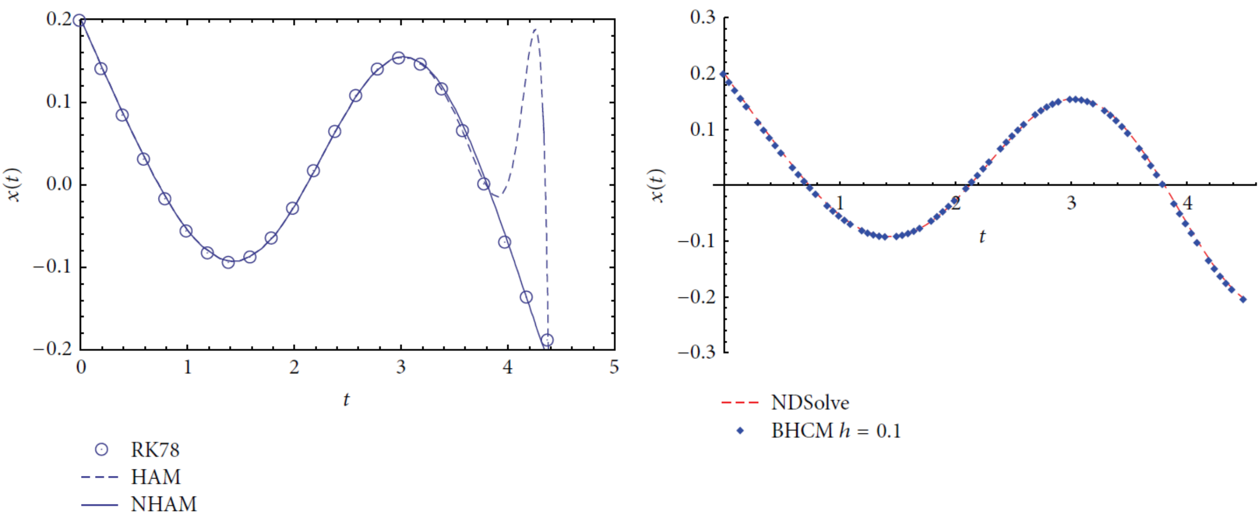

Fig. 1. Solutions obtained for (22) using seven- and eight-order Runge Kutta method (RK78), homotopy analysis method (HAM), new variant of HAM (NHAM) [18] and block hybrid collocation method [19]

VI. Application to Solve Problem in Thin Film Flow

Consider the problem concerned with the flow of thin films of viscous fluid with a free surface in which surface tension effects play a role typically leading to third-order ordinary differential equations governing the shape of the free surface of the fluid, $y = y(x)$. One of such equation is the fluid dynamics problem formulated as an autonomous third order ODE

$$y''' = f(y) \tag{24}$$

where

$$\begin{aligned} &f(y) = -1 + y^{-2}, \\ &f(y) = -1 + (1 + \delta + \delta^2)y^{-2} - (\delta + \delta^2)y^{-3}, \\ &f(y) = y^{-2} - y^{-3}, \\ &f(y) = y^{-2}. \end{aligned} \tag{25}$$

Numerical methods for solving third order ODEs have been extended to solve these resultant third order ODE problem in thin film flow [20] of the form

$$y''' = y^{-k} \tag{26}$$

with initial conditions $y(0) = y'(0) = y''(0) = 1$ for the cases $k = 2$ and $k = 3$.

[20] applied the three-stage fifth order Runge Kutta method to solve the third order physical problem (27) directly while [21] adopted the seven-stage fifth-order Runge-Kutta. The new hybrid block method is also applied to solve (27) directly. The results are presented in the following tables in comparison to [21] and [20].

Table 7: Numerical results for problem in Thin Film Flow (27) with $h = 0.01$, $k = 2$

x	Exact Solution	Error [21]	Error [20]	Error (Hybrid Block Method)
0.0	1.000000000	1.0000000000	1.0000000000	1.0000000000
0.2	1.221211030	1.2212100045	1.2212100045	1.2212100045
0.4	1.488834893	1.4888347799	1.4888347799	1.4888347799
0.6	1.807361404	1.8073613977	1.8073613977	1.8073613977
0.8	2.179819234	2.1798192339	2.1798192339	2.1798192339
1.0	2.608275822	2.6082748676	2.6082748676	2.6082748676

Table 7 demonstrates that the new hybrid block method is suitable to solve the thin film flow model with $h = 0.01$ and $k = 2$. This is observed in the convergent results between the [21], [20] and the new hybrid block method.

The next case considered if for $k = 3$ with $h = 0.01$. This case actual has no analytic solution. Table 8 shows the numerical results.

Table 8: Numerical results for problem in Thin Film Flow (27) with $h = 0.01$, $k = 3$

x	Error [21]	Error [20]	Error (Hybrid Block Method)
0.0	1.0000000000	1.0000000000	1.0000000000
0.2	1.2211551424	1.2211551423	1.2211551424
0.4	1.4881052842	1.4881052838	1.4881052842
0.6	1.8042625481	1.8042625471	1.8042625481
0.8	2.1715227981	2.1715227960	2.1715227981
1.0	2.5909582591	2.5909582556	2.5909582591

As a result of the inability to obtain the exact solution to (27) for $k = 3$, comparison is made between the adopted approaches. Further convergence is also displayed by the new hybrid block method in Table 8. The results obtained by the hybrid block method are exactly same to [21] although the three-stage fifth order Runge Kutta method by [20] also gave close results.

VII. Conclusion

From the tables above, the hybrid block method has shown better accuracy with less and equal number of steps in comparison to [2] and [11] respectively. Also, in comparison to previous existing methods having equal and higher order [5], [7], [15], the hybrid block method has also competed favourably. The properties of convergence and consistency can also been seen from the numerical results. In addition, the suitability of the hybrid block method in application to the nonlinear Genesio equation and the physical problem modelling thin film flow was also investigated. Convergence in solution and improved accuracy were properties displayed by the hybrid block method when solving these additional models. Thus, this new generalized hybrid block method suitable for numerical approximation of third order initial value problems.

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