The Generalized L_p-mixed Volume and the Generalized L_p-mixed Projection Body

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***Abstract*—In this paper, based on three classical concept of the mixed surface area measure, the mixed volumes and the mixed projection bodies, we introduce three new concepts of the generalized L_p-mixed surface area measure, the generalized L_p-mixed volumes and the generalized L_p-mixed projection bodies of convex bodies. In addition, some important inequalities, such as, the Aleksandrov-Fenchel inequality for the generalized L_p-mixed volumes, the Aleksandrov-Fenchel inequality for the generalized L_p-mixed projection bodies and the Brunn-Minkowski inequality for polar of L_p-mixed projection bodies are established, respectively. We also give a generalization of Pythagorean inequality and Loomis-Whitney inequality for L_p-mixed volumes, respectively.**

***Index Terms*—Convex body, surface area measure; L_p-mixed surface area measure, L_p-mixed volume; projection body, L_p-mixed projection body.**

I. Introduction

THE mixed volume is a central part of the Brunn-Minkowski theory of convex bodies. The monograph by Schneider [80] introduced the mixed volume and closely related mixed area measures, establish their fundamental properties. In the early 1960s, Firey [25] defined the Minkowski-Firey L_p-additions of convex bodies for each $p \geq 1$ and also established the L_p-Brunn-Minkowski inequality. Based on the L_p-additions, Lutwak [51] defined the L_p-mixed volume of two convex bodies and established the famous L_p-Minkowski mixed volume inequality. In the mid 1990s, study on the volume of Minkowski-Firey L_p-additions in [51] and [52] leads to an L_p-Brunn-Minkowski theory. The rapidly developing L_p-Brunn-Minkowski theory of convex bodies is a natural extension of the Brunn-Minkowski theory (see, e.g., [2], [3], [4], [6], [8], [14], [15], [16], [17], [18], [21], [22], [23], [30], [31], [32], [33], [35], [40], [41], [42], [43], [44], [51], [52], [56], [58], [59], [60], [61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [72], [73], [74], [75], [76], [79], [83], [85], [87], [91]).

The study of projection bodies or zonoids has a long history [36]. An article [10] first considered this problem, since then, considerable attention has been paid to the projection bodies [5], [11], [13], [19], [28], [46], [47], [71], [78], [80], [86], [89]. The related applications appeared in [86], [7], [81], [88]. The projection bodies topic has been focus on the intense study [1], [12], [20], [29], [45], [46], [47], [48], [49], [50], [77], [82].

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbf{R}^n$; $\mathcal{K}^n_o, \mathcal{K}^n_c$ denote the subset of $\mathcal{K}^n$ containing the origin in their interiors and the subset of $\mathcal{K}^n$ that contains the centered (centrally symmetric with respect to the origin) bodies, respectively.

For $K_1, \cdots, K_n \in \mathcal{K}^n$, Aleksandrov-Fenchel inequality [80] is

$$V(K_1, \cdots, K_n)^r \geq \prod_{j=1}^{r} V(\underbrace{K_j \cdots, K_j}_{r}, K_{r+1}, \cdots, K_n), \quad (1)$$

the equality condition in the Aleksandrov-Fenchel inequality is not hold. The special case holds

$$V(\underbrace{K \cdots, K}_{n-j-1}, \underbrace{B \cdots, B}_{j}, L)^{n-j} \geq W_j(K)^{n-j-1} W_j(L), \quad (2)$$

with equality if and only if K and L are homothetic.

If we take $j = 0$ in (2), then Minkowski inequality for mixed volumes is

$$V(\underbrace{K \cdots, K}_{n-1}, L)^n \geq V(K)^{n-1} V(L), \quad (3)$$

with equality if and only if K and L are homothetic.

In 1993, the Aleksandrov-Fenchel inequality and Brunn-Minkowski inequality for the mixed projection bodies have been established by Lutwak [45]. If $K, L \in \mathcal{K}^n$, then

$$V(\Pi(K+L))^{\frac{1}{n(n-1)}} \geq V(\Pi K)^{\frac{1}{n(n-1)}} + V(\Pi L)^{\frac{1}{n(n-1)}}, \quad (4)$$

with equality if and only if K and L are homothetic.

If $K_1, \cdots, K_{n-1} \in \mathcal{K}^n$, then

$$V(\Pi(K_1, \cdots, K_{n-1}))^r \geq \prod_{j=1}^{r} V(\Pi(\underbrace{K_j \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})). \quad (5)$$

In particular, taking $K_{n-i} = \cdots = K_{n-1} = B$ ($i = 0, 1, \cdots, n-2$) in (5), and denotes

$$\Pi_i(K_1, \cdots, K_{n-1-i}) = \Pi(K_1, \cdots, K_{n-1-i}, \underbrace{B, \cdots, B}_{i}).$$

So, we have

$$V(\Pi_i(K_1, \cdots, K_{n-1-i}))^r \geq \prod_{j=1}^{r} V(\Pi_i(\underbrace{K_j \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1-i})). \quad (6)$$

In 2004, Leng et al. [38] established the Aleksandrov-Fenchel inequality for the polar of projection bodies as follows

$$V(\Pi^*(K_1, \cdots, K_{n-1}))^r \leq \prod_{j=1}^{r} V(\Pi^*(\underbrace{K_j \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1})), \quad (7)$$

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with equality if $K_1, \cdots, K_{n-1}$ are homothetic. In particular, taking $K_{n-i} = \cdots = K_{n-1} = B$ $(i = 0, 1, \cdots, n-2)$ in (7), and denotes

$$\Pi_i^*(K_1, \cdots, K_{n-1-i}) = \Pi^*(K_1, \cdots, K_{n-1-i}, \underbrace{B, \cdots, B}_{i}).$$

Therefore, we have

$$\begin{aligned} & V(\Pi_i^*(K_1, \cdots, K_{n-1-i}))^r \\ \leq & \prod_{j=1}^{r} V(\Pi_i^*(\underbrace{K_j \cdots, K_j}_{r}, K_{r+1}, \cdots, K_{n-1-i})), \quad (8) \end{aligned}$$

with equality if $K_1, \cdots, K_{n-1}$ are homothetic n-balls.

Based on the concepts of classical mixed surface area, mixed volume and mixed projection body involving a plurality of convex bodies, the purpose of this paper first is to introduce three notions of the generalized L_p-mixed surface area measure, generalized L_p-mixed volume and generalized L_p-mixed projection bodies, respectively. In addition, we will establish the Aleksandrov-Fenchel inequality for the generalized L_p-mixed volume, Aleksandrov-Fenchel inequality for the generalized L_p-mixed projection bodies and Brunn-Minkowski inequality for polars of L_p-mixed projection bodies of convex bodies, respectively. Our findings further enrich the architecture of L_p-Brunn-Minkowski theory.

First at all, we introduce the abbreviation

$$(\underbrace{K_1 \cdots, K_1}_{r_1}, \cdots, \underbrace{K_m \cdots, K_m}_{r_m}) := (K_1[r_1], \cdots, K_m[r_m]).$$

The following is our main results.

Theorem 1. If $p \geq 1$ and $K_1, \cdots, K_n \in \mathcal{K}_o^n$, then

$$\begin{aligned} & V_p(K_1, \cdots, K_n)^r \\ \geq & \prod_{j=1}^{r} V(K_j[r], K_{r+1}, \cdots, K_n)^p \\ \times & \prod_{j=1}^{n-1} V(K_1, \cdots, K_{n-1}, K_j)^{\frac{(1-p)r}{n-1}}. \end{aligned} \quad (9)$$

Theorem 2. If $p \geq 1$ and $K_1, \cdots, K_n \in \mathcal{K}_o^n$, then

$$\begin{aligned} & V(\Pi_p(K_1, \cdots, K_{n-1}))^{\frac{p}{n}} \\ \geq & n^{1-p} \prod_{j=1}^{r} V(\Pi(K_j[r], K_{r+1} \cdots, K_{n-1}))^{\frac{p}{nr}} \\ \times & \prod_{j=1}^{n-1} V(K_1, \cdots, K_{n-1}, K_j)^{\frac{1-p}{n-1}}. \end{aligned} \quad (10)$$

Theorem 3. If $p \geq 1$ and $K_1, \cdots, K_n \in \mathcal{K}_o^n$, then

$$\begin{aligned} & V(\Pi_p^*(K_1, \cdots, K_{n-1})) \\ \leq & n^{\frac{(p-1)n}{p}} \prod_{j=1}^{r} V(\Pi^*(K_j[r], K_{r+1} \cdots, K_{n-1}))^{\frac{1}{r}} \\ \times & \prod_{j=1}^{n-1} V(K_1, \cdots, K_{n-1}, K_j)^{\frac{(p-1)n}{(n-1)p}}. \end{aligned} \quad (11)$$

with equality if and only if K_i is the line segment joining $-\lambda_i u$ and $\lambda_i u$, where $\lambda_i > 0$ $(i = 1, \cdots, n-1)$.

Theorem 4. If $K, L, K_2, \cdots, K_{n-1} \in \mathcal{K}_o^n$, $\lambda, \mu \geq 0$ (not both zero), $p \geq 1, i \in \mathbf{R}$ and $C = (K_2, \cdots, K_{n-1})$, then

$$\begin{aligned} & 4\widetilde{V}_{p,i}(\Pi^*(\lambda K + \mu L, C))^{\frac{1}{n-i}} \\ \leq & \lambda \widetilde{V}_{p,i}(\Pi^*(K, C))^{\frac{1}{n-i}} + \mu \widetilde{V}_{p,i}(\Pi^*(L, C))^{\frac{1}{n-i}}, \end{aligned} \quad (12)$$

with equality if and only if $\Pi(K, C) = \Pi(L, C)$.

Next, we use the methods to give a generalization of Pythagorean inequality for mixed volumes obtained by Firey [24].

Theorem 5. Let $p > 0$ and $K_1, \cdots, K_{n-1} \in \mathcal{K}_o^n$. Assume that $u_1, \cdots, u_m$ is a sequence of unit vectors in $\mathbf{R}^n$ and $c_1, \cdots, c_m$ be a sequence of positive numbers satisfying

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \quad (13)$$

where I_n is the identity mapping on $\mathbf{R}^n$. Then, for any $u \in S^{n-1}$,

$$\begin{aligned} & V_p(K_1, \cdots, K_{n-1}, [u])^{\frac{2}{p}} \\ \leq & \sum_{i=1}^{m} c_i V_p(K_1, \cdots, K_{n-1}, [u_i])^{\frac{2}{p}}, \end{aligned} \quad (14)$$

where $[u]$ denotes the unit segments in the direction u, with equality if and only if

$$\frac{|u \cdot u_1|}{V_p(K_1, \cdots, K_{n-1}, [u_1])} = \cdots = \frac{|u \cdot u_m|}{V_p(K_1, \cdots, K_{n-1}, [u_m])}.$$

The classical Loomis-Whitney inequality [39] shows the relation between the volume of a convex body and the geometric mean of its shadows. The Loomis-Whitney inequality is one of the fundamental inequalities in convex geometry and has been studied intensively. We generalize the Loomis-Whitney inequality to the following form of L_p-mixed volume associated with John basis.

Theorem 6. Suppose that $K_1, \cdots, K_{n-1} \in \mathcal{K}^n$, $\{u_i\}_{i=1}^m$ is a sequence of unit vectors in $\mathbf{R}^n$, and $\{c_i\}_{i=1}^m$ is a sequence of positive numbers such that $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$. Then for $p \geq 1$,

$$\begin{aligned} & \prod_{i=1}^{m} \Bigg(\prod_{j=1}^{n-2} v(K_j^{u_i}, K_1^{u_i}, \cdots, K_{n-2}^{u_i})^{\frac{p-1}{p(n-2)}} \\ \times & \; v_p(K_1^{u_i}, \cdots, K_{n-1}^{u_i})^{\frac{1}{p}} \Bigg)^{c_i} \\ \geq & \; n^{\frac{n(1-p)}{n}} V([K_1, \cdots, K_{n-1}])^{n-1}. \end{aligned} \quad (15)$$

Contents of the paper. For our studies, we state some relevant knowledge for the convex geometric analysis in Section 2. In Section 3, we propose two new concepts for the generalized L_p-mixed volumes and generalized L_p-mixed quermassintegrals, discuss some of their related properties. Simultaneously, we introduce a new concept for the L_p-mixed projection bodies. Section 4, we prove Theorems 1-6 which stated in the beginning of this paper, respectively. As an application, we give a generalization of Pythagorean inequality for mixed volumes, which has been obtained by Firey [24]. In addition, we established a generalized inequality of Loomis-Whitney inequality.

II. Background material

The setting for this paper is n-dimensional Euclidean space $\mathbf{R}^n(n\geq 2)$. Let u denotes unit vector, and B denotes unit ball centered at the origin, the surface of B is S^{n-1}. For $u\in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. $[u]$ denotes the line segment joining $\{\lambda u:|\lambda|\leq\frac{1}{2}\}$. Let $V(K)$ denote the n-dimensional volume of a body K, and let $x\cdot y$ denote the usual inner product for x and y in $\mathbf{R}^n$. For $x\in\mathbf{R}^n\backslash\{o\}$, the notation $x\otimes x$ represents the linear operator of the rank 1 on $\mathbf{R}^n$ that takes y to $(x\cdot y)x$. Let $\mathrm{GL}(n)$ denote non-singular affine (or linear) transformation group, ϕ^t denotes the transpose of ϕ, and ϕ^{-t} denotes the inverse of the transpose of ϕ

A. Support function, radial function, polar of convex body and Minkowski linear combination

Let $h(K,\cdot):\mathbf{R}^n\to(0,\infty)$ denote the support function of $K\in\mathcal{K}^n$, defined by $h(K,x)=\max\{x\cdot y:y\in K\}$. If $\phi\in\mathrm{GL}(n)$, then for the support function of the image $\phi K=\{\phi x:x\in K\}$, we easily have

$$h_{\phi K}(x)=h_K(\phi^t x). \tag{16}$$

For $K,L\in\mathcal{K}^n$, Hausdorff metric δ of K and L is defined by

$$\delta(K,L)=\sup\{|h_K(u)-h_L(u)|:u\in S^{n-1}\}.$$

For $K\in\mathcal{K}^n$ and a nonnegative scalar λ, $\lambda K=\{\lambda x:x\in K\}$. For $K_i\in\mathcal{K}^n,\lambda_i\geq 0(i=1,\cdots,r)$, Minkowski linear combination $\sum_{i=1}^r\lambda_iK_i\in\mathcal{K}^n$ is defined by

$$\sum_{i=1}^r\lambda_iK_i=\left\{\sum_{i=1}^r\lambda_ix_i\in\mathcal{K}^n:x_i\in K_i,i=1,\cdots,r\right\}.$$

It is trivial to verify that

$$h\bigg(\sum_{i=1}^r\lambda_iK_i,\cdot\bigg)=\sum_{i=1}^r\lambda_ih(K_i,\cdot). \tag{17}$$

For $K,L\in\mathcal{K}_o^n,p\geq 1$ and $\lambda,\mu\geq 0$ (not both zero), the Firey L_p-combination $\lambda\cdot K+_p\mu\cdot L\in\mathcal{K}_o^n$ is defined by (see [25])

$$h(\lambda\cdot K+_p\mu\cdot L,\cdot)^p=\lambda h(K,\cdot)^p+\mu h(L,\cdot)^p. \tag{18}$$

If $K\in\mathcal{K}_o^n$, we define the polar body of $K\in\mathcal{K}_o^n$, $K^*=\{x\in\mathbf{R}^n:x\cdot y\leq 1,\text{for all }y\in K\}$. Obviously, for $\phi\in\mathrm{GL}(n)$, $(\phi K)^*=\phi^{-t}K^*$. If $K\in\mathcal{K}_o^n$ we have that $K^*\in\mathcal{K}_o^n$ and $(K^*)^*=K$ (see [80]).

If K is a compact star-shaped (about the origin) in $\mathbf{R}^n$, its radial function, $\rho(K,\cdot):\mathbf{R}^n\backslash\{o\}\to[0,\infty)$ is defined by $\rho(K,x)=\max\{\lambda\geq 0:\lambda x\in K\}$. If ρ_K is positive and continuous, then K is called a star body. Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbf{R}^n$. Obviously, for $x\neq 0$ and $\phi\in\mathrm{GL}(n)$, $\rho_{\phi K}(x)=\rho_K(\phi^{-1}x)$.

Together the support function, the radial function with polar body, it follows obviously that for $K\in\mathcal{K}_o^n$

$$\rho(K,\cdot)^{-1}=h(K^*,\cdot)\ \text{ and }\ h(K,\cdot)^{-1}=\rho(K^*\cdot). \tag{19}$$

B. Mixed volumes, L_p-mixed volumes, mixed surface area measure and L_p-mixed quermassintegrals

If $K_1,\cdots,K_r\in\mathcal{K}^n$ and $\lambda_1,\cdots,\lambda_r$ are nonnegative real numbers, then the volume of $\lambda_1K_1+\cdots+\lambda_rK_r$ is a homogeneous polynomial in $\lambda_1,\cdots,\lambda_r$ (see [80])

$$V(\lambda_1K_1+\cdots+\lambda_rK_r)=\sum_{i_1,\cdots,i_n}\lambda_{i_1}\cdots\lambda_{i_n}V_{i_1\cdots i_n}, \tag{20}$$

where the sum is taken over all n-tuples $(i_1,\cdots,i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1,\cdots,i_n}$ depends only on the bodies $K_{i_1},\cdots,K_{i_n}$ and is uniquely determined by (20), it is called the mixed volume of $K_{i_1},\cdots,K_{i_n}$ and is written as $V(K_{i_1},\cdots,K_{i_n})$.

Associated with $K_1,\cdots,K_{n-1}\in\mathcal{K}^n$ is Borel measure, $S(K_1,\cdots,K_{n-1})$, on S^{n-1}, called the mixed surface area measure of $K_1,\cdots,K_{n-1}$, which has the property that for each $L\in\mathcal{K}^n$ (see [37]),

$$\begin{aligned}&V(K_1,\cdots,K_{n-1},L)\\ =\ &\frac{1}{n}\int_{S^{n-1}}h(L,u)\mathrm{d}S(K_1,\cdots,K_{n-1};u).\end{aligned} \tag{21}$$

For $\lambda,\mu>0$, if K_1 is replaced by $\lambda K_1+\mu L_1$, then we have

$$\begin{aligned}&S(\lambda K_1+\mu L_1,\cdots,K_{n-1};\cdot)\\ =\ &\lambda S(K_1,\cdots,K_{n-1};\cdot)\\ &+\mu S(L_1,\cdots,K_{n-1};\cdot).\end{aligned} \tag{22}$$

An important fact [26] is

$$\int_{S^{n-1}}u\mathrm{d}S(K_1,\cdots,K_{n-1};u)=0.$$

We noted that the mixed area measure $S(K_1,\cdots,K_{n-1};\cdot)$ also satisfies the hypothesis of Minkowski's existence theorem. Thus, for $K_1,\cdots,K_{n-1}\in\mathcal{K}^n$, there exists a convex body denoted by $[K_1,\cdots,K_{n-1}]$, whose area function is $S(K_1,\cdots,K_{n-1};\cdot)$, namely,

$$S([K_1,\cdots,K_{n-1}],\cdot)=S(K_1,\cdots,K_{n-1};\cdot),$$

where $[K,\cdots,K]=K$.

A direct consequence of (21) is following

$$\begin{aligned}&V([K_1,\cdots,K_{n-1}][n-1],K_n)\\ =\ &V(K_1,\cdots,K_{n-1},K_n).\end{aligned} \tag{23}$$

Since

$$\begin{aligned}&V([K_1,\cdots,K_{n-1}])\\ =\ &V([K_1,\cdots,K_{n-1}][n-1],[K_1,\cdots,K_{n-1}]),\end{aligned}$$

(23) implies that

$$\begin{aligned}&V([K_1,\cdots,K_{n-1}])\\ =\ &V(K_1,\cdots,K_{n-1},[K_1,\cdots,K_{n-1}]),\end{aligned} \tag{24}$$

$$\begin{aligned}&V(K_1,\cdots,K_i,\cdots,K_j,\cdots,K_n)\\ =\ &V(K_1,\cdots,K_j,\cdots,K_i,\cdots,K_n).\end{aligned} \tag{25}$$

If $K_1=\cdots=K_{n-i-1}=K$ and $K_{n-i}=\cdots=K_{n-1}=B$, then $S(K_1,\cdots,K_{n-1},\cdot)$ is written as $S_i(K,\cdot)$, $V(K_1,\cdots,K_{n-1},L)$ is written as $W_i(K,L)$. If $L=K$,

$W_i(K,K)$ is written as $W_i(K)$ that is called ith quermass-integrals of convex body K; i.e.,

$$W_i(K)=\frac{1}{n}\int_{S^{n-1}}h(K,u)\mathrm{d}S_i(K,u). \quad (26)$$

We recall that $W_0(K)$ is $V(K)$.

In [37], Lutwak proved that if $K_1,\cdots,K_n\in\mathcal{K}^n$, and $\phi\in\mathrm{GL}(n)$, then

$$V(\phi K_1,\cdots,\phi K_n)=|\det\phi|V(K_1,\cdots,K_n). \quad (27)$$

Suppose $K,L\in\mathcal{K}_o^n$, then for $p\geq 1$, the L_p-mixed volume, $V_p(K,L)$, of K and L is defined by (see [51])

$$\frac{n}{p}V_p(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K+_p\varepsilon\cdot L)-V(K)}{\varepsilon}. \quad (28)$$

For $K\in\mathcal{K}_o^n$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that (see [51])

$$V_p(K,Q)=\frac{1}{n}\int_{S^{n-1}}h(Q,u)^p\mathrm{d}S_p(K,u), \quad (29)$$

for each $Q\in\mathcal{K}_o^n$. The measure $S_p(K,\cdot)$ is just the L_p-surface area measure of K, which is absolutely continuous with respect to classical surface area measure $S(K,\cdot)$, and process Radon-Nikodym derivative

$$\frac{\mathrm{d}S_p(K,\cdot)}{\mathrm{d}S(K,\cdot)}=h^{1-p}(K,\cdot), \quad (30)$$

it follows from (30) that $S_1(K,\cdot)$ is just $S(K,\cdot)$.

The L_p-Minkowski inequality was given by Lutwak [52]. If $K,L\in\mathcal{K}_o^n$ and $p\geq 1$, then

$$V_p(K,L)\geq V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \quad (31)$$

with equality for $p=1$ if and only if K and L are homothetic, for $p>1$ if and only if K and L are dilates.

For $K,L\in\mathcal{K}_o^n,\varepsilon>0$ and real $p\geq 1$, the L_p-mixed quermassintegrals, $W_{p,i}(K,L)(i=0,1,\ldots,n-1)$, of K and L is defined by (see [51])

$$\frac{n-i}{p}W_{p,i}(K,L)=\lim_{\varepsilon\to 0^+}\frac{W_i(K+_p\varepsilon\cdot L)-W_i(K)}{\varepsilon}. \quad (32)$$

The $W_{p,0}(K,L)$ is just L_p-mixed volume $V_p(K,L)$, namely $W_{p,0}(K,L)=V_p(K,L)$. In [51], Lutwak has shown that, for $p\geq 1$and each $K\in\mathcal{K}_o^n$, there exists a positive Borel measure $S_{p,i}(K,\cdot)$ $(i=0,1,\cdots,n-1)$ on S^{n-1}, such that the L_p-mixed quermassintegrals $W_{p,i}(K,L)$ has the following integral representation

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^p(v)\mathrm{d}S_{p,i}(K,v), \quad (33)$$

for all $L\in\mathcal{K}_o^n$. It turns out that the measure $S_{p,i}(K,\cdot)$ $(i=0,1,\ldots,n-1)$ on S^{n-1} is absolutely continuous with respect to $S_i(K,\cdot)$, and has the Radon-Nikodym derivative

$$\frac{\mathrm{d}S_{p,i}(K,\cdot)}{\mathrm{d}S_i(K,\cdot)}=h^{1-p}(K,\cdot), \quad (34)$$

where $S_i(K,\cdot)$ is a classical positive Borel measure on S^{n-1} (see [51]). Obviously, $S_{p,0}(K,\cdot)=S_p(K,\cdot)$. The Formula (34) has shown that, for $p\geq 1, i=0,1,\cdots,n-1$, and each $K\in\mathcal{K}_o^n$, there exists a positive Borel measure on S^{n-1}, by (see [51])

$$S_{p,i}(K,\omega)=\int_{\omega}h(K,u)^{1-p}\mathrm{d}S_i(K,u), \quad (35)$$

for each Borel $\omega\subset S^{n-1}$.

C. Dual mixed volume

If $K_i\in\mathcal{S}_o^n$ $(i=1,\cdots,n)$, then the dual mixed volume of $K_1,\cdots,K_n$ is defined by (see [53])

$$\widetilde{V}(K_1,\cdots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho(K_1,u)\cdots\rho(K_n,u)\mathrm{d}S(u), \quad (36)$$

where $\mathrm{d}S(u)$ denotes the area element of S^{n-1} at u. Note that $\widetilde{V}_i(K,L)=\widetilde{V}(K[n-i],L[i])$. Thus, if i is any real, then $\widetilde{V}_i(K,L)$ is said the dual mixed volume of $K,L\in\mathcal{S}_o^n$, and

$$\widetilde{V}_i(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}\rho(L,u)^i\mathrm{d}S(u). \quad (37)$$

In (37), let $L=B$ and we write $\widetilde{V}_i(K,B)=\widetilde{W}_i(K)$, together with $\rho(B,u)=1$ for all $u\in S^{n-1}$, the definition of dual quermassintegrals can be stated that: For $K\in\mathcal{S}_o^n, i\in\mathbf{R}$, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K is defined by (see [27])

$$\widetilde{W}_i(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}\mathrm{d}S(u). \quad (38)$$

We recall that the polar coordinate formula for volume of $K\in\mathcal{K}^n$ is $V(K)=\widetilde{W}_0(K)$.

III. The main concepts and their related properties

A. Generalized L_p-mixed surface area and generalized L_p-mixed volumes

In this section, we first proposed the two concepts of the generalized L_p-mixed surface area and the generalized L_p-mixed volume. Motivated by (35), we introduce the following definitions.

Definition 7. For $p>0$ and $K_1,\cdots,K_{n-1}\in\mathcal{K}_o^n$, the Borel measure $S_p(K_1,\cdots,K_{n-1};\cdot)$ on S^{n-1} is defined by

$$\begin{aligned}&S_p(K_1,\cdots,K_{n-1};\omega)\\ =\;&\int_{\omega}(h(K_1,u)\cdots h(K_{n-1},u))^{\frac{1-p}{n-1}}\\ &\times\mathrm{d}S(K_1,\cdots,K_{n-1};u),\end{aligned} \quad (39)$$

for each Borel $\omega\subset S^{n-1}$, where $S(K_1,\cdots,K_{n-1};\cdot)$ is the classical the mixed surface area measure of $K_1,\cdots,K_{n-1}$.

From (39), we easily obtain that

$$\frac{\mathrm{d}S_p(K_1,\cdots,K_{n-1};\cdot)}{\mathrm{d}S(K_1,\cdots,K_{n-1};\cdot)}=(h(K_1,\cdot)\cdots h(K_{n-1},\cdot))^{\frac{1-p}{n-1}}. \quad (40)$$

Associated with $K_1,\cdots,K_{n-1}\in\mathcal{K}_o^n$ is a Borel measure, $S_p(K_1,\cdots,K_{n-1};\cdot)$, on S^{n-1}, called the generalized L_p-mixed surface area measure of $K_1,\cdots,K_{n-1}$.

Taking $K_1=\cdots=K_{n-1}=K$ in (40), then (40) reduces to (30), where $S_p(K,\cdot):=S_p(K,\cdots,K;\cdot)$ and $S(K,\cdot):=S(K,\cdots,K;\cdot)$.

Definition 8. For $p>0$ and $K_1,\cdots,K_{n-1},L\in\mathcal{K}_o^n$, the generalized L_p-mixed volume, $V_p(K_1,\cdots,K_{n-1},L)$, is defined by

$$\begin{aligned}&V_p(K_1,\cdots,K_{n-1},L)\\ =\;&\frac{1}{n}\int_{S^{n-1}}h(L,u)^p\mathrm{d}S_p(K_1,\cdots,K_{n-1};u),\end{aligned} \quad (41)$$

where the Borel measure $S_p(K_1,\cdots,K_{n-1};\cdot)$ depends only on the bodies $K_1,\cdots,K_{n-1}$, and is uniquely determined by (39).

Some properties of $V_p(K_1,\cdots,K_n)$ are as follows

(i) (**Continuity**) If $K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n)$, then $V_p(K_1,\cdots,K_n)$ is continuous for p;

(ii) (**Positive definite property**) If $K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n)$, then $V_p(K_1,\cdots,K_n)>0$;

(iii) (**Positive definite homogeneity**) If $\lambda_i>0, K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n)$, then

$$\begin{aligned}&V_p(\lambda_1K_1,\cdots,\lambda_nK_n)\\=\;&(\lambda_1\cdots\lambda_{n-1})^{\frac{n-p}{n-1}}\lambda_n^pV_p(K_1,\cdots,K_n);\end{aligned}$$

(iv) (p**-additivity**) If $K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n), L\in\mathcal{K}_o^n$, and $\lambda,\mu\geq 0$(not both zero), then

$$\begin{aligned}&V_p(K_1,\cdots,K_{n-1},\lambda\cdot K_n+_p\mu\cdot L)\\=\;&\lambda V_p(K_1,\cdots,K_{n-1},K_n)\\&+\mu V_p(K_1,\cdots,K_{n-1},L);\end{aligned}$$

(v) (**Monotonicity**) If $K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n-1), K,L\in\mathcal{K}_o^n$, then

$$K\subset L\;\Rightarrow\;\begin{aligned}&V_p(K_1,\cdots,K_{n-1},K)\\&\leq V_p(K_1,\cdots,K_{n-1},L),\end{aligned}$$

with equality if and only if $h(K,u)=h(L,u)$ for all u in the support of the measure $S(K_1,\cdots,K_{n-1};\cdot)$.

Remark 9. The condition in (v) is in general not equivalent to $K=L$, since the support of $S(K_1,\cdots,K_{n-1};\cdot)$ can be a proper closed subset of the unit sphere.

It will be helpful to introduce some additional notation.

For $x\in\mathbf{R}^n$, let $\langle x\rangle=x/|x|$, whenever $x\neq 0$.

Definition 10. (see [56]) Given a measure $\mathrm{d}\mu(u)$ on S^{n-1}, a real $p>0$, and a $\phi\in\mathrm{GL}(n)$, define the measure $\mathrm{d}\mu^{(p)}(\phi u)$ on S^{n-1} by

$$\int_{S^{n-1}}f(u)\mathrm{d}\mu^{(p)}(\phi u)=\int_{S^{n-1}}|\phi^{-1}u|^pf(\langle\phi^{-1}u\rangle)\mathrm{d}\mu(u),$$

for each $f\in C(S^{n-1})$.

First note that for any convex bodies $K_1,\cdots,K_{n-1}$ and each $\phi\in\mathrm{GL}(n)$ for the classical mixed surface area measure we have

$$\begin{aligned}&\mathrm{d}S(\phi K_1,\cdots,\phi K_{n-1};u)\\=\;&|\det\phi|\mathrm{d}S^{(1)}(K_1,\cdots,K_{n-1};\phi^tu).\end{aligned}\tag{42}$$

To see this note that for any convex bodies $K_1,\cdots,K_{n-1}$ it follow from Definition 10, the homogeneity of h_Q, (16) and (27) that

$$\begin{aligned}&\int_{S^{n-1}}h_Q(u)\mathrm{d}S^{(1)}(K_1,\cdots,K_{n-1};\phi^tu)\\=\;&\int_{S^{n-1}}|\phi^{-t}u|h_Q(\langle\phi^{-t}u\rangle)\mathrm{d}S(K_1,\cdots,K_{n-1};u)\\=\;&\int_{S^{n-1}}h_Q(\phi^{-t}u)\mathrm{d}S(K_1,\cdots,K_{n-1};u)\\=\;&\int_{S^{n-1}}h_{\phi^{-1}Q}(u)\mathrm{d}S(K_1,\cdots,K_{n-1};u)\\=\;&|\det\phi^{-1}|\int_{S^{n-1}}h_Q(u)\mathrm{d}S(\phi K_1,\cdots,\phi K_{n-1};u).\end{aligned}$$

Proposition 11. If $K_1,\cdots,K_{n-1}\in\mathcal{K}_o^n$ and real $p>0$, then for $\phi\in\mathrm{GL}(n)$,

$$\begin{aligned}&\mathrm{d}S_p(\phi K_1,\cdots,\phi K_{n-1};u)\\=\;&|\det\phi|\mathrm{d}S_p^{(p)}(K_1,\cdots,K_{n-1};\phi^tu).\end{aligned}$$

Proof. If $f\in C(S^{n-1})$, then from Definition 7, (16), (42), Definition 10, the homogeneity of h_K, (42) again, and Definition 10 again, we have

$$\begin{aligned}&\int_{S^{n-1}}f(u)\mathrm{d}S_p(\phi K_1,\cdots,\phi K_{n-1};u)\\=\;&\int_{S^{n-1}}f(u)(h_{\phi K_1}(u)\cdots h_{\phi K_{n-1}}(u))^{\frac{1-p}{n-1}}\\&\times\mathrm{d}S(\phi K_1,\cdots,\phi K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}f(u)(h_{K_1}(\phi^tu)\cdots h_{K_{n-1}}(\phi^tu))^{\frac{1-p}{n-1}}\\&\times\mathrm{d}S^{(1)}(K_1,\cdots,K_{n-1};\phi^tu)\\=\;&|\det\phi|\int_{S^{n-1}}|\phi^{-t}u|f(\langle\phi^{-t}u\rangle)\\&\times\big(h_{K_1}(\phi^t\langle\phi^{-t}u\rangle)\cdots h_{K_{n-1}}(\phi^t\langle\phi^{-t}u\rangle)\big)^{\frac{1-p}{n-1}}\\&\times\mathrm{d}S(K_1,\cdots,K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}|\phi^{-t}u|^pf(\langle\phi^{-t}u\rangle)\\&\times\big(h_{K_1}(u)\cdots h_{K_{n-1}}(u)\big)^{\frac{1-p}{n-1}}\mathrm{d}S(K_1,\cdots,K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}|\phi^{-t}u|^pf(\langle\phi^{-t}u\rangle)\\&\times\mathrm{d}S_p(K_1,\cdots,K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}f(u)\mathrm{d}S_p^{(p)}(K_1,\cdots,K_{n-1};\phi^tu).\end{aligned}$$

An immediate result of Proposition 11 is:

Corollary 12. If $K_1,\cdots,K_{n-1},L\in\mathcal{K}_o^n$, real $p>0$ and $\phi\in\mathrm{GL}(n)$, then

$$\begin{aligned}&V_p(\phi K_1,\cdots,\phi K_{n-1},L)\\=\;&|\det\phi|V_p(K_1,\cdots,K_{n-1},\phi^{-1}L).\end{aligned}\tag{43}$$

Proof. From Definition 8, Proposition 11, Definition 10, the homogeneity of the support function, (16), and finally Definition 8 again, we have

$$\begin{aligned}&nV_p(\phi K_1,\cdots,\phi K_{n-1},L)\\=\;&\int_{S^{n-1}}h(L,u)^p\mathrm{d}S_p(\phi K_1,\cdots,\phi K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}h(L,u)^p\mathrm{d}S_p^{(p)}(K_1,\cdots,K_{n-1};\phi^tu)\\=\;&|\det\phi|\int_{S^{n-1}}|\phi^{-t}u|^ph(L,\langle\phi^{-t}u\rangle)^p\\&\times\mathrm{d}S_p(K_1,\cdots,K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}h(L,\phi^{-t}u)^p\mathrm{d}S_p(K_1,\cdots,K_{n-1};u)\\=\;&|\det\phi|\int_{S^{n-1}}h(\phi^{-1}L,u)^p\mathrm{d}S_p(K_1,\cdots,K_{n-1};u)\\=\;&n|\det\phi|V_p(K_1,\cdots,K_{n-1},\phi^{-1}L).\end{aligned}$$

Corollary 12 shows that for $K_1,\cdots,K_n$ are convex bodies that contain the origin in their interiors, real $p>0$, and $\phi\in\mathrm{GL}(n)$,

$$V_p(\phi K_1,\cdots,\phi K_n)=|\det\phi|V_p(K_1,\cdots,K_n).\tag{44}$$

B. Generalized ith L_p-mixed surface area and generalized ith L_p-mixed quermassintegrals

Definition 13. Let $p > 0$, and $K_1 = \cdots = K_{n-1-i} \in \mathcal{K}_o^n, K_{n-i} = \cdots = K_{n-1} = B$ $(i = 0, 1, \cdots, n-2)$, define the Borel measure $S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot)$ on S^{n-1}, by

$$\begin{aligned} & S_{p,i}(K_1, \cdots, K_{n-1-i}; \omega) \\ = & \int_\omega (h(K_1, u) \cdots h(K_{n-1-i}, u))^{\frac{1-p}{n-1-i}} \\ & \times \mathrm{d}S_i(K_1, \cdots, K_{n-1-i}; u), \end{aligned} \tag{45}$$

for each Borel $\omega \subset S^{n-1}$, where we denote

$$S_i(K_1, \cdots, K_{n-1-i}; \cdot) := S(K_1, \cdots, K_{n-1-i}, B[i]; \cdot),$$

$$S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot) := S_p(K_1, \cdots, K_{n-1-i}, B[i]; \cdot).$$

From (45), it is easily to obtain that

$$\begin{aligned} & \frac{\mathrm{d}S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot)}{\mathrm{d}S_i(K_1, \cdots, K_{n-1-i}; \cdot)} \\ = & \big(h(K_1, \cdot) \cdots h(K_{n-1-i}, \cdot)\big)^{\frac{1-p}{n-1-i}}. \end{aligned} \tag{46}$$

Taking $K_1 = \cdots = K_{n-1-i} = K$ in (46), then (46) reduces to (34).

Let $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = B$, and introducing the abbreviation

$$W_{p,i}(K_1, \cdots, K_{n-1-i}, L) := V_p(K_1, \cdots, K_{n-1-i}, B[i], L).$$

Definition 14. For $p > 0$, and $K_1, \cdots, K_{n-1-i}, L \in \mathcal{K}_o^n$ $(i = 0, 1, \cdots, n-2)$, we define the generalized L_p-mixed quermassintegrals, $W_{p,i}(K_1, \cdots, K_{n-1-i}, L)$, by

$$\begin{aligned} & W_{p,i}(K_1, \cdots, K_{n-1-i}, L) \\ = & \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \mathrm{d}S_{p,i}(K_1, \cdots, K_{n-1-i}; u), \end{aligned} \tag{47}$$

where the Borel measure $S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot)$ depends only on the bodies $K_1, \cdots, K_{n-1-i}$, and is uniquely determined by (45), it is called the generalized ith L_p-mixed surface area measure of $K_1, \cdots, K_{n-1-i}$.

Remark 15. By Definition 14 with that (47), we can deduce that the Definition 8 with that (41) but not vice versa. Therefore, Definition 14 with that (47) extend some known ones in the sense of the Definition 8 with that (41).

C. Generalized L_p-mixed projection bodies

In this section, we first introduce the concept of generalized L_p-mixed projection body.

Definition 16. If $p > 0$ and $K_i \in \mathcal{K}_o^n$ $(i = 1, \cdots, n-1)$, then for $u \in S^{n-1}$, the generalized L_p-mixed projection body,$\Pi_p(K_1, \cdots, K_{n-1})$, of $K_i (i = 1, \cdots, n-1)$ is defined by

$$\begin{aligned} & h(\Pi_p(K_1, \cdots, K_{n-1}), u)^p \\ = & \frac{1}{2^p} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_p(K_1, \cdots, K_{n-1}; v), \end{aligned} \tag{48}$$

where $S_p(K_1, \cdots, K_{n-1}, \cdot)$ depends only on the bodies $K_1, \cdots, K_{n-1}$, and is uniquely determined by (40).

When $p = 1$, then (48) reduces to the following definition of mixed projection bodies introduced by Lutwak [57]:

$$\begin{aligned} & h(\Pi(K_1, \cdots, K_{n-1}), u) \\ = & \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \mathrm{d}S(K_1, \cdots, K_{n-1}; v). \end{aligned} \tag{49}$$

This is equivalent to

$$v(K_1^u, \cdots, K_{n-1}^u) = nV(K_1, \cdots, K_{n-1}, [u]). \tag{50}$$

Additional, it follow from (41) and (49) that

$$h(\Pi_p(K_1, \cdots, K_{n-1}), u)^p = nV_p(K_1, \cdots, K_{n-1}, [u]). \tag{51}$$

Remark 17. If $p \neq 1$, it follows that

$$v_p(K_1^u, \cdots, K_{n-1}^u) \neq nV_p(K_1, \cdots, K_{n-1}, [u]).$$

In fact, a function of K_1, the right-hand side is (under dilatation) homogeneous of degree $\frac{n-p}{n-1}$, while the left-hand side is homogeneous of degree $\frac{n-1-p}{n-2}$. Further, a function of K_{n-1}, the right-hand side is homogeneous of degree $\frac{n-p}{n-1}$, while the left-hand side is homogeneous of degree p. However, from (40), (41) and Hölder's inequality (58) in the back, we have that for $p \geq 1$

$$\begin{aligned} & v_p(K_1^u, \cdots, K_{n-1}^u) \\ \geq & nV(K_1, \cdots, K_{n-1}, [u])^p \\ & \times \prod_{j=1}^{n-2} V(K_j, K_1, \cdots, K_{n-2}, [u])^{\frac{1-p}{n-2}}, \end{aligned} \tag{52}$$

for $0 < p \leq 1$, the inequality (52) is reversed. Equality holds in either if and only if $p = 1$.

We use $\Pi_p^*(K_1, \cdots, K_{n-1})$ to denote the polar body of $\Pi_p(K_1, \cdots, K_{n-1})$ called the polar of generalized L_p-mixed projection body of $K_1, \cdots, K_{n-1}$.

D. Generalized ith L_p-mixed projection bodies

Definition 18. If $p > 0, K_1, \cdots, K_{n-1-i} \in \mathcal{K}_o^n$ $(i = 0, 1, \cdots, n-2)$, then the generalized ith L_p-mixed projection body of K_j $(j = 1, \cdots, n-1-i)$ is defined by $\Pi_{p,i}(K_1, \cdots, K_{n-1-i})$, and whose support function is given, for $u \in S^{n-1}$, by

$$\begin{aligned} & h(\Pi_{p,i}(K_1, \cdots, K_{n-1-i}), u)^p = \frac{1}{2^p} \int_{S^{n-1}} |u \cdot v|^p \\ & \times \mathrm{d}S_{p,i}(K_1, \cdots, K_{n-1-i}; v), u \in S^{n-1}, \end{aligned} \tag{53}$$

where $S_{p,i}(K_1, \cdots, K_{n-1-i}; \cdot)$ is uniquely determined by (46).

From Definition 14 and Definition 18, it follows that

$$\begin{aligned} & h(\Pi_{p,i}(K_1, \cdots, K_{n-1-i}), u)^p \\ = & nW_{p,i}(K_1, \cdots, K_{n-1-i}, [u]). \end{aligned} \tag{54}$$

If $K_1 = \cdots = K_{n-2-i} = K$ and $K_{n-1-i} = L$, then $\Pi_{p,i}(K_1, \cdots, K_{n-1-i})$ will be written as $\Pi_{p,i}(K, L)$. If $L = B$, then $\Pi_{p,i}(K, B)$ is called the ith L_p-mixed projection body of K and denoted by $\Pi_{p,i}K$. We write $\Pi_{p,0}K$ as $\Pi_p K$. It easily to see that

$$h(\Pi_{p,i}(K, L), u)^p = \frac{1}{2^p} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_{p,i}(K, L, v) \tag{55}$$

and

$$h(\Pi_{p,i}K, u)^p = \frac{1}{2^p} \int_{S^{n-1}} |u \cdot v|^p \mathrm{d}S_{p,i}(K, v). \tag{56}$$

The following property will be used later. If $K, L \in \mathcal{K}_o^n, K_2, \cdots, K_{n-1} \in \mathcal{K}_o^n$ and $C = (K_2, \cdots, K_{n-1})$, then

$$\Pi(\lambda K + \mu L, C) = \lambda \Pi(K, C) + \mu \Pi(L, C). \tag{57}$$

IV. The main results and their proofs

A. The Aleksandrov-Fenchel inequality for the generalized L_p-mixed volume of convex bodies

In this section, we prove the Aleksandrov-Fenchel inequality for the generalized L_p-mixed volume of convex bodies stated in the beginning of this paper.

Proof of Theorem 1. For $p=1$, Theorem 1 is just inequality (1) stated in the beginning of this paper, its proof was completed by Schneider (see [80], p.401).

For $p>1$, we use Hölder's inequality (see [34], p.140) to complete the proof.

Suppose that $f_i \in L_\omega^{\alpha_i}(E), 1<\alpha_i<\infty$ $(i=0,1,\cdots,m)$ are nonnegative functions, α_i satisfies

$$\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_m}=1.$$

Then $\prod_{i=1}^m f_i \in L_\omega^1(E)$, and

$$\begin{aligned}\int_E \prod_{i=1}^m f_i(x)\omega(x)\mathrm{d}\mu(x) &\le \prod_{i=1}^m ||f_i||_{\alpha_i,\omega}\\ &= \prod_{i=1}^m \Big(\int_E (f_i(x))^{\alpha_i}\omega(x)\mathrm{d}\mu(x)\Big)^{\frac{1}{\alpha_i}}, \quad (58)\end{aligned}$$

with equality if and only if there exist positive constants $\lambda_1,\cdots,\lambda_m$ such that $\lambda_1 f_1(x)^{\alpha_1}=\cdots=\lambda_m f_m(x)^{\alpha_m}$ for $x\in E$.

If $0<\alpha_1<1$ and $\alpha_2<0,\cdots,\alpha_m<0$, then inequality (58) is reverse (the conditions of the reverse inequality of (58) is given by the author of this article).

For $p>1$, the reverse Hölder's inequality, together with (41), (40) and (21), yields

$$\begin{aligned}&V_p(K_1,\cdots,K_n)\\ =\ & \frac{1}{n}\int_{S^{n-1}} h(K_n,u)^p \mathrm{d}S_p(K_1,\cdots,K_{n-1};u)\\ =\ & \frac{1}{n}\int_{S^{n-1}} h(K_n,u)^p\big(h(K_1,u)\cdots h(K_{n-1},u)\big)^{\frac{1-p}{n-1}}\\ &\times \mathrm{d}S(K_1,\cdots,K_{n-1};u)\\ \ge\ & \left(\frac{1}{n}\int_{S^{n-1}} h(K_n,u)\mathrm{d}S(K_1,\cdots,K_{n-1};u)\right)^p\\ &\times\prod_{j=1}^{n-1}\left(\frac{1}{n}\int_{S^{n-1}} h(K_j,u)\mathrm{d}S(K_1,\cdots,K_{n-1};u)\right)^{\frac{1-p}{n-1}}\\ =\ & V(K_1,\cdots,K_n)^p\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}.\end{aligned}$$

Using the Aleksandrov-Fenchel inequality (1), we have

$$\begin{aligned}V_p(K_1,\cdots,K_n) &\ge \prod_{j=1}^{r} V(K_j[r],K_{r+1},\cdots,K_n)^{\frac{p}{r}}\\ &\times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}.\end{aligned}$$

Similarly, we can prove the reverse inequality.

Taking $r=n-1$ in (9), we obtain

Corollary 19. If $p\ge 1, K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n)$, then

$$\begin{aligned}V_p(K_1,\cdots,K_n)^{n-1} &\ge \prod_{j=1}^{n-1} V(K_j[n-1],K_n)^p\\ &\times V(K_1,\cdots,K_{n-1},K_j)^{1-p}. \quad (59)\end{aligned}$$

Taking $r=n$ in (9), we obtain

Corollary 20. If $p\ge 1, K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n)$, then

$$\begin{aligned}V_p(K_1,\cdots,K_n)^n &\ge \prod_{i=1}^n V(K_i)^p\\ &\times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{(1-p)n}{n-1}}. \quad (60)\end{aligned}$$

Remark 21. In particular, when $p=1$ in (60), the result has proved by Lutwak [57]: If $K_i\in\mathcal{K}_o^n$ $(i=1,\cdots,n)$, then

$$V(K_1,\cdots,K_n)^n \ge V(K_1)\cdots V(K_n), \quad (61)$$

with equality if and only if $K_1,\cdots,K_n$ are homothetic.

Using the same argument as in Theorem 1, we immediately can get the following theorem.

Theorem 22. If $p\ge 1$, and $K_1,\cdots,K_{n-1-i}\in\mathcal{K}_o^n$ $(i=0,1,\cdots,n-1)$, $K_n\in\mathcal{K}_o^n$, then

$$\begin{aligned}&W_{p,i}(K_1,\cdots,K_{n-1-i},K_n)^r\\ \ge\ & \prod_{j=1}^{r} W_i(K_j[r],K_{r+1},\cdots,K_{n-1-i},K_n)^p\\ &\times\prod_{j=1}^{n-1-i} W_i(K_1,\cdots,K_{n-1-i},K_j)^{\frac{(1-p)r}{n-1-i}}. \quad (62)\end{aligned}$$

Taking $r=n-1-i$ in (62), we obtain that

Corollary 23. If $p\ge 1, K_1,\cdots,K_{n-1-i},K_n$ $(i=0,1,\cdots,n-1)\in\mathcal{K}_o^n$, then

$$\begin{aligned}&W_{p,i}(K_1,\cdots,K_{n-1-i},K_n)^{n-1-i}\\ \ge\ & \prod_{j=1}^{n-1-i} W_i(K_j[n-1-i],K_n)^p\\ &\times W_i(K_1,\cdots,K_{n-1-i},K_j)^{1-p}. \quad (63)\end{aligned}$$

Remark 24. Taking $r=n-i, K_1=\cdots=K_{n-1-i}=K$ and $K_n=L$ in (62), we can get the Minkowski inequality proved by Lutwak [51]: If $K,L\in\mathcal{K}_o^n, p\ge 1$, then for $i=0,1,\cdots,n-2$,

$$W_{p,i}(K,L)^{n-i}\ge W_i(K)^{n-i-p}W_i(L)^p, \quad (64)$$

with equality for $p>1$ if and only if K and L are dilates, for $p=1$ if and only if K and L are homothetic.

B. The Aleksandrov-Fenchel inequality for generalized L_p-mixed projection of convex bodies

In this section, we prove the Aleksandrov-Fenchel inequality for L_p-mixed projection bodies of convex bodies stated in the beginning of this paper.

Lemma 25. If $p\ge 1$ and $K_1,\cdots,K_{n-1}\in\mathcal{K}_o^n$, for $i=0,1,\cdots,n-1$, then

$$\begin{aligned}&W_i(\Pi_p(K_1,\cdots,K_{n-1}))^{\frac{p}{n-i}}\\ \ge\ & n^{1-p}W_i(\Pi(K_1,\cdots,K_{n-1}))^{\frac{p}{n-i}}\\ &\times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}. \quad (65)\end{aligned}$$

Proof. We only prove the first inequality with $p \geq 1$. From Definition 1, Definition 6, the reverse of Hölder's inequality (58), (49) and (21), it follows that

$$\begin{aligned} & h(\Pi_p(K_1,\cdots,K_{n-1}),u)^p \\ = & \frac{1}{2^p}\int_{S^{n-1}}|u\cdot v|^p\big(h_{K_1}(v)\cdots h_{K_{n-1}}(v)\big)^{\frac{1-p}{n-1}} \\ & \times \mathrm{d}S(K_1,\cdots,K_{n-1};v) \\ \geq & n^{1-p}\left(\frac{1}{2}\int_{S^{n-1}}|u\cdot v|\mathrm{d}S(K_1,\cdots,K_{n-1};v)\right)^p \\ & \times\left(\prod_{j=1}^{n-1}\left(\frac{1}{n}\int_{S^{n-1}}h_{K_j}(v)\mathrm{d}S(K_1,\cdots,K_{n-1};v)\right)\right)^{\frac{1-p}{n-1}} \\ = & n^{1-p}h(\Pi(K_1,\cdots,K_{n-1}),u)^p \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}. \end{aligned}$$

Namely,

$$\begin{aligned} & h(\Pi_p(K_1,\cdots,K_{n-1}),u)^p \\ \geq & n^{1-p}h(\Pi(K_1,\cdots,K_{n-1}),u)^p \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}. \end{aligned} \tag{66}$$

For each $Q \in \mathcal{K}_o^n$, integrating both sides of (66) for $\mathrm{d}S_{p,i}(Q,u)$ in $u \in S^{n-1}$, and by (33), we obtain

$$\begin{aligned} & W_{p,i}(Q,\Pi_p(K_1,\cdots,K_{n-1})) \\ \geq & n^{1-p}W_{p,i}(Q,\Pi(K_1,\cdots,K_{n-1})) \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}. \end{aligned}$$

Taking $Q = \Pi_p(K_1,\cdots,K_{n-1})$, we have

$$\begin{aligned} & W_i(\Pi_p(K_1,\cdots,K_{n-1})) \\ \geq & n^{1-p}W_{p,i}(\Pi_p(K_1,\cdots,K_{n-1},\Pi(K_1,\cdots,K_{n-1})) \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}. \end{aligned} \tag{67}$$

Using inequality (64) in (67), we have (65).

Taking $i = 0$ in (65) and using inequality (5), this has Theorem 2. Taking $r = n-1$ in (10), we obtain

Corollary 26. Let $K_1,\cdots,K_{n-1} \in \mathcal{K}_o^n$. If $p \geq 1$, then

$$\begin{aligned} & V(\Pi_p(K_1,\cdots,K_{n-1}))^{\frac{p}{n}} \geq n^{1-p}\prod_{j=1}^{n-1}V(\Pi K_j)^{\frac{p}{n(n-1)}} \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{1-p}{n-1}}. \end{aligned} \tag{68}$$

When $p = 1$ in (10), we obtain the Brunn-Minkowski inequality (5) for mixed projection bodies established by Lutwak [45]. From (53), (46), the reverse of Hölder's inequality (58), (31) and (6), the same argument can get

Theorem 27. If $p \geq 1$, $K_1,\cdots,K_{n-1-i} \in \mathcal{K}_o^n$ $(i = 0,1,\cdots,n-2)$, then

$$\begin{aligned} & V(\Pi_{p,i}(K_1,\cdots,K_{n-1-i}))^{\frac{p}{n}} \\ \geq & n^{1-p}\prod_{j=1}^{r}V(\Pi_i(K_j[r],K_{r+1}\cdots,K_{n-1-i}))^{\frac{p}{nr}} \\ & \times\prod_{j=1}^{n-1-i}W_i(K_1,\cdots,K_{n-1-i},K_j)^{\frac{1-p}{n-1-i}}. \end{aligned} \tag{69}$$

Now we established the generalized Aleksandrov-Fenchel inequality for the polar of L_p-mixed projection bodies.

Lemma 28. If $p \geq 1, K_1,\cdots,K_{n-1} \in \mathcal{K}_o^n$ and $i \in \mathbf{R}$, then

$$\begin{aligned} & \widetilde{W}_i(\Pi_p^*(K_1,\cdots,K_{n-1})) \\ \leq & n^{\frac{(p-1)(n-i)}{p}}\widetilde{W}_i(\Pi^*(K_1,\cdots,K_{n-1})) \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{(p-1)(n-i)}{(n-1)p}}, \end{aligned} \tag{70}$$

with equality if and only if K_j is the line segment $\lambda_j[u]$, where $\lambda_j > 0$ $(j = 1,\cdots,n-1)$.

Proof. From (66) we have

$$\begin{aligned} & \rho(\Pi_p^*(K_1,\cdots,K_{n-1}),u)^{n-i} \\ \leq & n^{\frac{(p-1)(n-i)}{p}}\rho(\Pi^*(K_1,\cdots,K_{n-1}),u)^{n-i} \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{(p-1)(n-i)}{(n-1)p}}. \end{aligned} \tag{71}$$

For each $Q \in \mathcal{K}_o^n$, integrate both sides of (71) for $u \in S^{n-1}$, and by (38) and the formula of the dual quermassintegrals (38), we immediately obtain (70).

From the condition of equality in Hölder inequality, we know that equality holds in inequality (70) if and only if K_j is the line segment $\lambda_j[u]$, where $\lambda_j > 0$ $(j = 1,\cdots,n-1)$.

Taking $i = 0$ in (70) and using inequality (7), we can obtain Theorem 3. Taking $r = n-2$ in (11), we obtain

Corollary 29. If $p \geq 1$ and $K_1,\cdots,K_{n-1} \in \mathcal{K}_o^n$, then

$$\begin{aligned} & V(\Pi_p^*(K_1,\cdots,K_{n-1})) \\ \leq & n^{\frac{(p-1)n}{p}}\prod_{j=1}^{n-2}V(\Pi^*K_j)^{\frac{1}{n-2}} \\ & \times\prod_{j=1}^{n-1}V(K_1,\cdots,K_{n-1},K_j)^{\frac{(p-1)n}{(n-1)p}}, \end{aligned} \tag{72}$$

with equality if and only if K_j is the line segment $\lambda_j[u]$, where $\lambda_j > 0$ $(i = 1,\cdots,n-1)$.

Taking $p = 1$ in (11), we obtain the Aleksandrov-Fenchel inequality (7) for the polars of mixed projection bodies established by Leng et al. [38].

From Definition 18, (46), the reverse of Hölder's inequality (58), (38), (8) and (21), and similar to the proof of Theorem 3, we have

Theorem 30. If $p \geq 1$ and $K_1,\cdots,K_{n-1-i} \in \mathcal{K}_o^n$ $(i =$

$0,1,\cdots,n-2$), then

$$\begin{aligned} & V(\Pi^*_{p,i}(K_1,\cdots,K_{n-1-i})) \\ \leq \ & n^{\frac{(p-1)n}{p}} \prod_{j=1}^{r} V(\Pi^*_i(K_j[r],K_{r+1}\cdots,K_{n-1-i}))^{\frac{1}{r}} \\ & \times \prod_{j=1}^{n-1-i} W_j(K_1,\cdots,K_{n-1-i},K_j)^{\frac{(p-1)n}{(n-1-i)p}}, \quad (73) \end{aligned}$$

with equality if and only if $K_1,\cdots,K_{n-1-i}$ ($i = 0,1,\cdots,n-2$) are line segment $2[u]$.

C. The Brunn-Minkowski inequality for polars of generalized L_p-mixed projection bodies

For $u \in S^{n-1}, b(K,u) := \frac{1}{2}(h(K,u)+h(K,-u))$ is defined to be half the width of K in the direction u. Two convex bodies K and L are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K,u) = \lambda b(L,u)$ for all $u \in S^{n-1}$. For $K \in \mathcal{K}^n$ and $p \in \text{int}K$, we use K^p to denote the polar reciprocal of K with respect to the unit sphere centered at p. The width integrals were first considered by Blaschke (see [9], p.85). The width integrals of index i is defined by Lutwak [54]. For $K \in \mathcal{K}^n, i \in \mathbf{R}$,

$$B_i(K) = \frac{1}{n}\int_{S^{n-1}} b(K,u)^{n-i}\mathrm{d}S(u). \quad (74)$$

The width-integral of index i is a map $B_i : \mathcal{K}^n \to \mathbf{R}$. It is positive, continuous, homogeneous of degree n-i and invariant under motion. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

The following result will be used later (see [54]),

$$b(K+L,u) = b(K,u) + b(L,u). \quad (75)$$

On the other hand, Lutwak [55] showed the notion of L_p-mixed width integrals. Let $K_1,\cdots,K_n \in \mathcal{K}^n$, and real number $p \neq 0$, then the L_p-mixed width integrals of $K_1,\cdots,K_n \in \mathcal{K}^n$, can is defined by

$$\begin{aligned} & B_p(K_1,\cdots,K_n) \\ = \ & \omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(K,u)^p\cdots b(K_n,u)^p\mathrm{d}S(u)\right)^{\frac{1}{p}}. \end{aligned} \quad (76)$$

And the L_p-mixed width integrals of index i of K is defined by

$$\begin{aligned} B_{p,i}(K) &= B_p(K[n-i],B[i]) \\ &= \omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(K,u)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p}}, \\ & \qquad p \neq 0. \end{aligned} \quad (77)$$

The generalized L_p-quasi dual mixed volume was given by Zhao [93]. Let $K_i \in \mathcal{S}^n_o$ $(i=1,\cdots,n)$ and $p > 0$, then the generalized L_p-quasi dual mixed volume of $K_1,\cdots,K_n$ is defined by

$$\begin{aligned} & \widetilde{V}_p(K_1,\cdots,K_n) \\ = \ & \omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho(K_1,u)^p\cdots\rho(K_n,u)^p\mathrm{d}S(u)\right)^{\frac{1}{p}}. \end{aligned} \quad (78)$$

Taking $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$ in (34), we write $\widetilde{V}_{p,i}(K,L) = \widetilde{V}_p(K[n-i],L[i])$. If i is any real, then $\widetilde{V}_{p,i}(K,L)$ is said the ith L_p-quasi dual mixed volume of $K,L \in \mathcal{S}^n_o$ (see [93]), and

$$\begin{aligned} & \widetilde{V}_{p,i}(K,L) \\ = \ & \omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho(K,u)^{p(n-i)}\rho(L,u)^{pi}\mathrm{d}S(u)\right)^{\frac{1}{p}}. \end{aligned} \quad (79)$$

In (79), let $L = B$ and we write $\widetilde{V}_{p,i}(K,B) = \widetilde{V}_{p,i}(K)$. Thereby, for $K \in \mathcal{S}^n_o, i \in \mathbf{R}$, the ith L_p-quasi dual quermassintegrals, $\widetilde{V}_{p,i}(K)$, of K is defined by

$$\widetilde{V}_{p,i}(K) = \omega_n\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho(K,u)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p}}. \quad (80)$$

Lutwak proved the result [54]: If $K \in \mathcal{K}^n$, then $B_{2n}(K) \leq V(K^p)$, with equality if and only if K is symmetric with respect to p. We first give a generalization of this inequality.

Lemma 31. If $K \in \mathcal{K}^n, p > 0, i < n$, then

$$B_{p,2n-i}(K) \leq \widetilde{V}_{p,i}(K^*), \quad (81)$$

with equality if and only if K is symmetric with respect to the origin.

Proof. From (77) and (75), we have

$$\begin{aligned} & B_{p,2n-i}(K)^{\frac{1}{n-i}} \\ = \ & \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} b(K,u)^{-p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ = \ & \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \left(\frac{1}{b(K,u)}\right)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ = \ & \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \left(\frac{2}{h(K,u)+h(K,-u)}\right)^{p(n-i)} \right. \\ & \left. \times\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ \leq \ & \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \left(\frac{1}{2h(K,u)}+\frac{1}{2h(K,-u)}\right)^{p(n-i)} \right. \\ & \left. \times\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ \leq \ & \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \left(\frac{1}{2h(K,u)}\right)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ & +\omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \left(\frac{1}{2h(K,-u)}\right)^{p(n-i)} \right. \\ & \left. \times\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ \leq \ & \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho(K^*,u)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ = \ & \widetilde{V}_{p,i}(K^*)^{\frac{1}{n-i}}, \ (i=0,1,\cdots,n-1), \end{aligned} \quad (82)$$

with equality if and only if K is centered.

From (82), we know that inequality (81) is proved.

Lemma 32. If $K,L \in \mathcal{K}^n_c,\ p > 0$ and $i < n$, then

$$4\widetilde{V}_{p,i}((K+L)^*)^{\frac{1}{n-i}} \leq \widetilde{V}_{p,i}(K^*)^{\frac{1}{n-i}} + \widetilde{V}_{p,i}(L^*)^{\frac{1}{n-i}}, \quad (83)$$

with equality if and only if $K = L$.

Proof. For $K, L \in \mathcal{K}_c^n, K+L \in \mathcal{K}_c^n$, it follow from (77) and (81) that (83) is equivalent to the following inequality

$$4B_{p,2n-i}(K+L)^{\frac{1}{n-i}} \leq B_{p,2n-i}(K)^{\frac{1}{n-i}} + B_{p,2n-i}(L)^{\frac{1}{n-i}}.$$

In fact, from Minkowski inequality, (77) and (75), we have

$$\begin{aligned} & B_{p,2n-i}(K)^{\frac{1}{n-i}} + B_{p,2n-i}(L)^{\frac{1}{n-i}} \\ = \;& \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\left(\frac{1}{b(K,u)}\right)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ & +\omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\left(\frac{1}{b(L,u)}\right)^{p(n-i)}\mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ \geq \;& \omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\left(\frac{1}{b(K,u)}+\frac{1}{b(L,u)}\right)^{p(n-i)} \right. \\ & \left. \times \mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ \geq \;& 4\omega_n^{\frac{1}{n-i}}\left(\frac{1}{n\omega_n}\int_{S^{n-1}}\left(\frac{1}{b(K,u)+b(L,u)}\right)^{p(n-i)} \right. \\ & \left. \times \mathrm{d}S(u)\right)^{\frac{1}{p(n-i)}} \\ = \;& 4B_{p,2n-i}(K+L)^{\frac{1}{n-i}}, \end{aligned}$$

with equality if and only if $b(K,u) = b(L,u), u \in S^{n-1}$. Since K and L are centered, it follows that $K = L$.

Noting that $\Pi(K,C)$ and $\Pi(L,C)$ are centered, from (57) and (83), we infer Theorem 4.

D. A generalization of Pythagorean inequality for mixed volumes

Pythagorean inequalities were given by Firey [24]:

$$\begin{aligned} & V(K_1,\cdots,K_{n-1},[e])^2 \\ \leq \;& \sum_{i=1}^{n} V(K_1,\cdots,K_{n-1},[e_i])^2, \end{aligned} \tag{84}$$

where $\{e_1,\cdots,e_n\}$ is an orthogonal basis in $\mathbf{R}^n$ and e is an arbitrary unit vector. Now, we generalize inequality (84) to John basis. Namely, we complete the proof of Theorem 5 stated in the beginning of this paper.

Proof of Theorem 5. From the support function of $\Pi_p(K_1,\cdots,K_{n-1})$ and (51), we get

$$h(u)^p = nV_p(K_1,\cdots,K_{n-1},[u]).$$

Together with [3]

$$u = \sum_{i=1}^{m} c_i(u\cdot u_i)u_i,$$

we have

$$\begin{aligned} & nV_p(K_1,\cdots,K_{n-1},[u]) \\ = \;& h\left(\sum_{i=1}^{m} c_i(u\cdot u_i)u_i\right)^p \\ = \;& \left(\sum_{i=1}^{m} c_i h((u\cdot u_i)u_i)\right)^p \\ = \;& \left(\sum_{i=1}^{m} c_i|u\cdot u_i| h\big(\mathrm{sgn}(u\cdot u_i)u_i\big)\right)^p \\ = \;& \left(n^{\frac{1}{p}}\sum_{i=1}^{m} c_i|u\cdot u_i| V_p(K_1,\cdots,K_{n-1},[u_i])^{\frac{1}{p}}\right)^p. \end{aligned} \tag{85}$$

Together Cauchy inequality with $||x||^2 = \sum_{i=1}^m c_i|x\cdot u_i|^2$, we have

$$\begin{aligned} & \sum_{i=1}^{m} c_i|u\cdot u_i| V_p(K_1,\cdots,K_{n-1},[u_i])^{\frac{1}{p}} \\ \leq \;& \left(\sum_{i=1}^{m} c_i|u\cdot u_i|^2\right)^{\frac{1}{2}} \\ & \times\left(\sum_{i=1}^{m} c_i V_p(K_1,\cdots,K_{n-1},[u_i])^{\frac{2}{p}}\right)^{\frac{1}{2}}. \end{aligned} \tag{86}$$

From (85) and (86), we have

$$\begin{aligned} & nV_p(K_1,\cdots,K_{n-1},[u]) \\ \leq \;& n\left(\sum_{i=1}^{m} c_i|u\cdot u_i|^2\right)^{\frac{p}{2}} \\ & \times\left(\sum_{i=1}^{m} c_i V_p(K_1,\cdots,K_{n-1},[u_i])^{\frac{2}{p}}\right)^{\frac{p}{2}} \\ = \;& n||u||^p\left(\sum_{i=1}^{m} c_i V_p(K_1,\cdots,K_{n-1},[u_i])^{\frac{2}{p}}\right)^{\frac{p}{2}}. \end{aligned} \tag{87}$$

From (87), the proof of inequality (14) is completed.

Remark 33. The equality in (13) implies $\sum_{i=1}^m c_i = n$. Clearly, the sequence $\{u_1,\cdots,u_m\}$ is just like a standard orthogonal basis such that for any $x \in \mathbf{R}^n$,

$$||x||^2 = \sum_{i=1}^{m} c_i|u_i\cdot x|^2. \tag{88}$$

Moreover, let $c_i = \frac{n}{m}$. Then $\{u_1,\cdots,u_m\}$ is called star-coordinates by Kawashima [36]. It is also easy to prove that its inertial ellipsoid is a ball (see [90], [92]).

Taking $p = 1$ in Theorem 5, then inequality (14) reduces to Leng's result [38]. Taking $K_1 = \cdots = K_{n-1-r} = K, K_{n-r} = \cdots = K_{n-1} = B$ in (14), it follows that

Corollary 34. If $p > 0, K \in \mathcal{K}_o^n$, then

$$w_{p,r}(K^u)^{\frac{2}{p}} \leq \sum_{i=1}^{m} c_i w_{p,r}(K^{u_i})^{\frac{2}{p}}. \tag{89}$$

In particular, let $r = 1$ to (89), we have **Corollary 35.** If $p > 0, K \in \mathcal{K}_o^n$, then

$$S_p(K^u)^{\frac{2}{p}} \leq \sum_{i=1}^{m} c_i S_p(K^{u_i})^{\frac{2}{p}}. \tag{90}$$

If $\{u_i\}_{i=1}^m$ is a standard orthogonal basis, then we can prove the generalized results of the obtained results by Firey [24].

E. Generalized Loomis-Whitney inequality

We require the following result on the zonotope. In fact, a zonotope is a Minkowski combination of line segments, and see [80]. A body in $\mathcal{K}^n$ being the limit (with respect to the Hausdorff metric) of zonotope is called a zonoid.

Lemma 36. Suppose that $\{u_i\}_{i=1}^m$ is a sequence of unit vectors in $\mathbf{R}^n$, and $\{c_i\}_{i=1}^m$ is a sequence of positive numbers such that $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$. If $\lambda_1, \cdots, \lambda_m$ are the sequence real numbers and $Z = \sum_{i=1}^m \lambda_i [u_i]$, then

$$V(Z) \geq \prod_{i=1}^m \left(\frac{\lambda_i}{c_i}\right)^{c_i}. \tag{91}$$

Proof of Theorem 6. Let $\lambda_1, \cdots, \lambda_m$ are the sequence real numbers and $Z = \sum_{i=1}^m \lambda_i [u_i]$. It follow from (52) and the property of mixed volume that

$$\begin{aligned} & \sum_{i=1}^m \lambda_i v_p(K_1^{u_i}, \cdots, K_{n-1}^{u_i})^{\frac{1}{p}} \\ & \times \prod_{j=1}^{n-2} V(K_j, K_1, \cdots, K_{n-2}, [u_i])^{\frac{p-1}{p(n-2)}} \\ \geq \ & n^{\frac{1}{p}} \sum_{i=1}^m \lambda_i V(K_1, \cdots, K_{n-1}, [u_i]) \\ = \ & n^{\frac{1}{p}} V(K_1, \cdots, K_{n-1}, Z). \end{aligned} \tag{92}$$

Together (23), Minkowski inequality (3) with Lemma 8, we have

$$\begin{aligned} & V(K_1, \cdots, K_{n-1}, Z) \\ = \ & V([K_1, \cdots, K_{n-1}][n-1], Z) \\ \geq \ & V([K_1, \cdots, K_{n-1}])^{\frac{n-1}{n}} V(Z)^{\frac{1}{n}} \\ \geq \ & V([K_1, \cdots, K_{n-1}])^{\frac{n-1}{n}} \prod_{i=1}^m \left(\frac{\lambda_i}{c_i}\right)^{\frac{c_i}{n}}. \end{aligned} \tag{93}$$

Together (92) with (93), we have

$$\begin{aligned} & \sum_{i=1}^m \lambda_i v_p(K_1^{u_i}, \cdots, K_{n-1}^{u_i})^{\frac{1}{p}} \\ & \times \prod_{j=1}^{n-2} V(K_j, K_1, \cdots, K_{n-2}, [u_i])^{\frac{p-1}{p(n-2)}} \\ \geq \ & n^{\frac{1}{p}} V([K_1, \cdots, K_{n-1}])^{\frac{n-1}{n}} \prod_{i=1}^m \left(\frac{\lambda_i}{c_i}\right)^{\frac{c_i}{n}}. \end{aligned} \tag{94}$$

Let

$$A = v_p(K_1^{u_i}, \cdots, K_{n-1}^{u_i})^{\frac{1}{p}},$$

$$B = \prod_{j=1}^{n-2} V(K_j, K_1, \cdots, K_{n-2}, [u_i])^{\frac{p-1}{p(n-2)}},$$

$$\lambda_i = \frac{c_i}{AB},$$

and note that $\sum_{i=1}^m c_i = n$, we obtain Theorem 6.

Taking $p = 1$ in (15), we have

Corollary 37. Suppose that $K_1, \cdots, K_{n-1} \in \mathcal{K}^n$, $\{u_i\}_{i=1}^m$ is a sequence of unit vectors in $\mathbf{R}^n$, and $\{c_i\}_{i=1}^m$ is a sequence of positive numbers such that $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$. Then

$$\prod_{i=1}^m v(K_1^{u_i}, \cdots, K_{n-1}^{u_i})^{c_i} \geq V([K_1, \cdots, K_{n-1}])^{n-1}. \tag{95}$$

Inequality (95) is established by Si and Leng [84].

In particular, let $K_1 = \cdots = K_{n-1} = K$, and note that $[K_1, \cdots, K_{n-1}] = K$, then inequality (95) reduces to the following Ball's Loomis-Whitney inequality for John basis [5].

Corollary 38. Suppose that $K \in \mathcal{K}^n$, $\{u_i\}_{i=1}^m$ is a sequence of unit vectors in $\mathbf{R}^n$, and $\{c_i\}_{i=1}^m$ is a sequence of positive numbers such that $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$. Then

$$\prod_{i=1}^m v(K^{u_i})^{c_i} \geq V(K)^{n-1}. \tag{96}$$

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