Existence of Positive Solutions for a Kind of Fractional Multi-point Boundary Value Problems at Resonance

Tingting Xue, Xiaolin Fan*, and Jiabo Xu

***Abstract*—The paper studies the existence of positive solutions for a class of fractional multi-point boundary value problems under different resonant conditions. By using Leggett-Williams norm-type theorem, some new existence results are obtained. Finally, an example is provided to show the application of the main results.**

***Index Terms*—multi-point boundary value problem, Leggett-Williams norm-type theorem, resonant, positive solution.**

I. Introduction

FRACTIONAL boundary value problems (FBVPs for short) arise from the studies about models of fluid flow, aerodynamics, electrical networks, polymer rheology, biology chemical physics, economics, control theory, signal and image processing research, etc. At present, more and more scholars are interested in this field, see [1-12]. For example, Ates and Zegeling [12] investigated the fractional-order advection-diffusion reaction boundary value problems:

$$\begin{cases} \varepsilon^C D^\alpha u + \gamma u' + f(u) = S(x), \quad x \in [0,1], \\ u(0) = u_L, \ u(1) = u_R, \end{cases}$$

where $1 < \alpha \le 2$, $0 < \varepsilon \le 1$, $\gamma \in \mathbb{R}$, ${}^C D^\alpha$ is the Caputo fractional derivative.

In the last two decades, many valuable results have been obtained by using various methods for the existence and multiplicity of solutions for FBVPs. For example, Bai [13] studied the existence and multiplicity of positive solutions for the FBVPs by means of some fixed-point theorems on cone. Liang [14] considered the existence of positive solutions by lower and upper solution method and fixed-point theorems. Jiang [15] discussed the solvability of FBVPS at resonance by using the coincidence degree theory due to Mawhin (see [16]), and so on. It is worth noting that the Leggett-Williams norm-type theorem is also an effective tool in determining the existence of positive solutions for FBVPs at resonance. In [17], Infante and Zima first studied the existence of positive solutions for the following BVPs at resonance:

$$\begin{cases} -x''(t) = f(t, x(t)), \quad t \in (0,1), \\ x'(0) = 0, \ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

where $m > 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\beta_i \ge 0, i = \overline{1, m-2}$, and $\sum_{i=1}^{m-2} \beta_i = 1$. Due to Leggett-Williams norm-type theorem, the existence of positive solutions was obtained by O'Regan and Zima [18]. Later, Yang [19], Jiang [20, 21] and other scholars made further research on this kind of problem, see [22-25]. Yang and Wang [26] studied the existence of positive solutions for the following FBVPs

$$\begin{cases} -{}^C D_{0+}^\alpha x(t) = f(t, x(t)), \ t \in [0,1], \\ x(0) = 0, \qquad x'(0) = x'(1), \end{cases}$$

where ${}^C D_{0+}^\alpha$ is the Caputo fractional derivative, $1 < \alpha \le 2$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Chen and Tang [27] considered the following FBVPs:

$$\begin{cases} D_{0+}^\alpha x(t) = f(t, x(t)), \ t \in [0, \infty), \\ x(0) = x'(0) = x''(0) = 0, \ D_{0+}^{\alpha-1} x(0) = \lim_{t\to\infty} D_{0+}^{\alpha-1} x(t), \end{cases}$$

where D_{0+}^α is the Riemann-Liouville fractional derivative, $3 < \alpha < 4$, and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous.

However, as far as we know, the fractional differential equations with m-point boundary value conditions at resonance have not been considered. Inspired by the above papers, we study the following problem:

$$\begin{cases} -{}^C D_{0+}^\alpha x(t) = f(t, x(t), x'(t)), \quad t \in (0,1), \\ x(0) = \sum_{i=1}^{m-2} \gamma_i x(\xi_i), \ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1}$$

where ${}^C D_{0+}^\alpha$ is the Caputo fractional derivative, $1 < \alpha \le 2, 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \gamma_i, \beta_i \ge 0, i = \overline{1, m-2}, \sum_{i=1}^{m-2} \gamma_i (1 - \xi_i) < 1, \sum_{i=1}^{m-2} \beta_i \xi_i < 1, f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with conditions

(i) $\sum_{i=1}^{m-2} \gamma_i = \sum_{i=1}^{m-2} \beta_i = 1$,

(ii) $\sum_{i=1}^{m-2} \gamma_i \neq 1, \sum_{i=1}^{m-2} \beta_i \neq 1, \sum_{i=1}^{m-2} \gamma_i \xi_i (1 - \sum_{i=1}^{m-2} \beta_i) + (1 - \sum_{i=1}^{m-2} \gamma_i)(1 - \sum_{i=1}^{m-2} \beta_i \xi_i) = 0$.

Let us emphasize the contribution of our article: firstly, as far as we know, there is no paper which considered existence of positive solutions for FBVPs (1) with different resonant conditions (i) and (ii), so our article enriches some existing results. Secondly, since the m-point boundary value problems studied in this paper are more complex, the following

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Tingting Xue is with School of Public Education, Xinjiang Institute of Engineering, Urumqi, 830000 PR China. And she is also with School of Mathematics, China University of Mining and Technology, Xuzhou, 221116 PR China.

Xiaolin Fan is corresponding author, with School of Public Education, Xinjiang Institute of Engineering, Urumqi, 830000 PR China; e-mail: mxjhfxl@163.com.

Jiabo Xu is with School of Public Education, Xinjiang Institute of Engineering, Urumqi, 830000 PR China.

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difficulties are brought: (1) new space and norm need to be constructed; (2) it is difficult to construct the projection operator; (3) it brings difficulties to the estimation of the priori bounds, mainly in verifying the boundedness of Ω_0.

II. Preliminaries

To facilitate understanding, we present some concepts and lemmas in the article. For more details, please refer to the references hereunder (see [28-31]).

Definition 2.1 ([32]). Let X, Y be real Banach spaces, and $L : \mathrm{dom}L \subset X \to Y$ be a linear map. If $\dim \mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$ and $\mathrm{Im}L$ is a closed subset in Y, then the map L is a Fredholm operator with index zero. If there exists the continuous projections $P : X \to X$ and $Q : Y \to Y$ satisfying $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Ker}Q = \mathrm{Im}L$, then $L\mid_{\mathrm{dom}L\cap\mathrm{Ker}P}$: $\mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is reversible. We denote the inverse of this map by K_P, i.e. $K_P = L_P^{-1}$ and $K_{P,Q} = K_P(I-Q)$. Moreover, since $\dim\mathrm{Im}Q = \mathrm{codim}\mathrm{Im}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$. It is known that the operator equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I-Q)Nx,$$

where $N : X \to Y$ is a nonlinear operator. If Ω is an open bounded subset of X and $\mathrm{dom}L \cap \Omega \neq \varnothing$, then the map N is L-compact on $\overline{\Omega}$ when $QN : \overline{\Omega} \to Y$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Let C be a cone in X. Then C induces a partial order in X by

$$x \le y \quad \text{iff} \quad y - x \in C.$$

Lemma 2.1 ([18]). Let C be a cone in X. Then for every $u \in C\backslash\{0\}$ there exists a positive number $\sigma(u)$ such that $\|x+u\| \ge \sigma(u)\|x\|$ for all $x \in C$. Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_P(I-Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.2 ([18]). Let C be a cone in X and Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega_1} \subset \Omega_2$ and $C\cap(\overline{\Omega_2}\backslash\Omega_1) \neq \varnothing$. Assume that the following conditions are satisfied:

(1) $L : \mathrm{dom}L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on every bounded subset of X,

(2) $Lx \neq \lambda Nx$ for every $(x,\lambda) \in [C \cap \partial\Omega_2 \cap \mathrm{dom}L] \times (0,1)$,

(3) γ maps subsets of $\overline{\Omega_2}$ into bounded subsets of C,

(4) $\deg([I-(P+JQN)\gamma]|_{\mathrm{Ker}L}, \mathrm{Ker}L\cap\Omega_2, 0) \neq 0$,

(5) there exists $u_0 \in C\backslash\{0\}$ such that $\|x\| \le \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0)\cap\partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \le x\}$ for some $\mu > 0$ and $\sigma(u_0)$ is such that $\|x+u_0\| \ge \sigma(u_0)\|x\|$ for every $x \in C$,

(6) $(P+JQN)\gamma(\partial\Omega_2) \subset C$,

(7) $\Psi_\gamma(\overline{\Omega_2}\backslash\Omega_1) \subset C$,

then the equation $Lx = Nx$ has at least one solution in $C\cap(\overline{\Omega_2}\backslash\Omega_1)$.

Definition 2.2 ([34]). The Riemann-Liouville fractional integral of order $\alpha(\alpha > 0)$ for the function $x : (0,+\infty) \to \mathbb{R}$ is defined as:

$$I_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\mathrm{d}s,$$

provided that the right-hand side integral is defined on $(0,+\infty)$.

Definition 2.3 ([34]) The Captuo fractional derivative of order $\alpha(\alpha > 0)$ for the function $x : (0,+\infty) \to \mathbb{R}$: is defined as:

$$\begin{aligned}{}^{C}D_{0+}^{\alpha}x(t) &= I_{0+}^{n-\alpha}\frac{d^n x(t)}{dt^n} \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)\mathrm{d}s,\end{aligned}$$

where $n = [\alpha]+1$, provided that the right-hand side integral is defined on $(0,+\infty)$.

Lemma 2.3 ([34]) If $n-1 < \alpha \le n$, then the solution of the fractional differential equation ${}^{C}D_{0+}^{\alpha}x(t) = 0$ is

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0,1,\cdots,n-1,\ n = [\alpha]+1$.

Lemma 2.4 ([34]) Let $n-1 < \alpha \le n$, if ${}^{C}D_{0+}^{\alpha}x(t) \in C[0,1]$, then

$$I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0,1,\cdots,n-1,\ n = [\alpha]+1$.

III. Main result

Take the Banach spaces $X = C^1[0,1]$, $Y = C[0,1]$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$, $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max\limits_{t\in[0,1]}|x(t)|$.

Define the linear operator $L : \mathrm{dom}L \subset X \to Y$ by

$$Lx = -{}^{C}D_{0+}^{\alpha}x(t), \tag{2}$$

where

$$\begin{aligned}\mathrm{dom}L = \{&x \in X \mid {}^{C}D_{0+}^{\alpha}x(t) \in Y,\ x(0) \\ &= \sum_{i=1}^{m-2}\gamma_i x(\xi_i),\ x(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i)\}\end{aligned}$$

and $N : X \to Y$ by

$$Nx(t) = f(t,x(t),x'(t)),\ \forall t \in [0,1].$$

Then FBVPs (1) can be written by the operator equation

$$Lx = Nx,\ x \in \mathrm{dom}L.$$

3.1 FBVPs (1) with resonant condition (i)

For convenience, let $\xi_0 = 0$, $\xi_{m-1} = 1$, $\gamma_0 = \gamma_{m-1} = \beta_0 = \beta_{m-1} = 0$ and the function $G(s)$, $s \in [0,1]$ as follow:

$$\begin{aligned}G(s) = (1-s)^{\alpha-1} + \frac{\sum\limits_{i=0}^{m-1}\beta_i\xi_i - 1}{\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}\sum_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1} \\ -\sum_{i=k}^{m-1}\beta_i(\xi_i-s)^{\alpha-1},\ \xi_{k-1} \le s \le \xi_k,\ k = \overline{1,m-1}.\end{aligned}$$

Denote the function $U(t,s)$ as follow:

$$U(t,s)=\begin{cases}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(2t-1)\sum\limits_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}+\frac{G(s)}{\int_0^1 G(s)ds}\\ \times[1-\frac{1}{\Gamma(\alpha+2)}-\frac{(2t-1)\sum\limits_{i=k}^{m-1}\gamma_i(\xi_i^{\alpha}-(\xi_i-1)^{\alpha})}{2\Gamma(\alpha+1)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}+\frac{t^{\alpha}}{\Gamma(\alpha+1)}]\\ \qquad \xi_{k-1}\le s\le\xi_k,\ 0\le t<s\le 1,\\ \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(2t-1)\sum\limits_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{G(s)}{\int_0^1 G(s)ds}\\ \times[1-\frac{1}{\Gamma(\alpha+2)}-\frac{(2t-1)\sum\limits_{i=k}^{m-1}\gamma_i(\xi_i^{\alpha}-(\xi_i-1)^{\alpha})}{2\Gamma(\alpha+1)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}+\frac{t^{\alpha}}{\Gamma(\alpha+1)}],\\ \qquad \xi_{k-1}\le s\le\xi_k,\ 0\le s\le t\le 1.\end{cases}$$

It is easy to check that $U(t,s)>0,\ t\in[1/2,1],\ s\in[0,1]$. Set positive number

$$\kappa=\min\{1,\ \min_{s\in[0,1]}\frac{\int_0^1 G(s)ds}{G(s)},\ \frac{1}{\max\limits_{t,s\in[0,1]}U(t,s)}\}$$

Theorem 3.1 Let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be continuous. Suppose that:

(H_1) there exist nonnegative functions $a,b,c\in C[0,1]$ with $\frac{2b_1(\alpha+1)}{\Gamma(\alpha+1)}+\frac{2c_1}{\Gamma(\alpha)}<1$ such that

$$|f(t,u,v)|\le a(t)+b(t)|u|+c(t)|v|,$$

for all $(t,u,v)\in[0,1]\times\mathbb{R}\times\mathbb{R}$, where $a_1=\|a\|_\infty$, $b_1=\|b\|_\infty$, $c_1=\|c\|_\infty$,

(H_2) there exist two constants $B_1,\ B_2>0$ such that for all $t\in[0,1]$, if $|u|>B_1$ or $|v|>B_2$, then

$$f(t,u,v)<0,$$

(H_3) $f(t,u,v)>-\kappa u$, for all $(t,u,v)\in[0,1]\times[0,\infty)\times\mathbb{R}$,

(H_4) there exist $r\in(0,\infty),\ t_0\in[1/2,1],\ a\in(0,1],\ M\in(0,1)$ and continuous functions $g:[0,1]\to[0,\infty)$, $h:(0,r]\to[0,\infty)$ such that $f(t,u,v)\ge g(t)h(u)$ for $[t,u,v]\in[0,1]\times(0,r]\times\mathbb{R}$, and $h(u)/u^a$ is non-increasing on $(0,r]$ with

$$\frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)ds\ge\frac{1-M}{M^a},$$

then FBVPs (1) has at least one positive solution.

To prove the above theorem, we begin with some useful lemmas.

Lemma 3.2 Let L be defined by (2), then

$$\mathrm{Ker}L=\{x\in X|x(t)=c,\ \forall t\in[0,1],\ c\in\mathbb{R}\},$$

$$\mathrm{Im}L=\{y\in Y|\int_0^1 G(s)y(s)ds=0\},$$

and the linear continuous projector operators $P:X\to X$ and $Q:Y\to Y$ can be defined as

$$Px(t)=\int_0^1 x(s)ds,\quad \forall t\in[0,1],$$

$$Qy(t)=\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)y(s)ds.$$

Moreover, the operator $K_P:\mathrm{Im}L\to\mathrm{dom}L\cap\mathrm{Ker}P$ is given by

$$K_Py(t)=\int_0^1 k(t,s)y(s)ds,\ \forall t\in[0,1],$$

where

$$k(t,s)=\begin{cases}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(2t-1)\sum\limits_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},\\ \qquad \xi_{k-1}\le s\le\xi_k,\ 0\le s\le t\le 1,\\ \frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(2t-1)\sum\limits_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i},\\ \qquad \xi_{k-1}\le s\le\xi_k,\ 0\le t<s\le 1.\end{cases}$$

Lemma 3.3 If $\Omega\subset X$ is an open bounded subset and $\mathrm{dom}L\cap\overline{\Omega}\ne\varnothing$, then N is L-compact on $\overline{\Omega}$.

Proof By the continuity of f, there exists a constant $A>0$ such that $|f(t,x(t),x'(t))|\le A,\ x\in\overline{\Omega}$. Then, we have

$$\begin{aligned}|QNx(t)|&=|\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,x(s),x'(s))ds|\\ &\le\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)|f(s,x(s),x'(s))|ds\\ &\le\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)Ads=A.\end{aligned}$$

So, $QN:X\to Y$ is bounded. For $\forall x\in\overline{\Omega}$, one has

$$\begin{aligned}&|K_P(I-Q)Nx(t)|\\ &=|\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-s)^{\alpha}(I-Q)Nx(s)ds-\\ &\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(I-Q)Nx(s)ds\\ &+\frac{(2t-1)}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}\sum_{i=0}^{m-1}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}(I-Q)Nx(s)ds|\\ &\le\frac{1}{\Gamma(\alpha+1)}\int_0^1|Nx(s)|ds+\frac{1}{\Gamma(\alpha+1)}\int_0^1|QNx(s)|ds\\ &\ +\frac{1}{\Gamma(\alpha)}\int_0^t|Nx(s)|ds+\frac{1}{\Gamma(\alpha)}\int_0^t|QNx(s)|ds\\ &\ +\frac{1}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}\int_0^1|Nx(s)|ds\\ &\ +\frac{1}{2\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}\int_0^1|QNx(s)|ds\\ &\le 2A(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)})+\frac{A}{\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}\end{aligned}$$

Therefore, $K_P(I-Q)N(\overline{\Omega})$ is bounded. For $\forall\varepsilon>$

$0,\ x \in \overline{\Omega},\ 0 \le t_1 < t_2 \le 1, |t_2 - t_1| < \delta$, we get

$$
\begin{aligned}
&|K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)| \\
&= |-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})(I-Q)Nx(s)ds \\
&-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nx(s)ds \\
&+\frac{(t_2-t_1)}{\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}\sum_{i=0}^{m-1}\gamma_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}(I-Q)Nx(s)ds| \\
&\le \frac{2A}{\Gamma(\alpha)}\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})ds \\
&\quad +\frac{2A}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+\frac{2A\delta}{\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i} \\
&\le \frac{2A}{\Gamma(\alpha+1)}(t_2^\alpha-t_1^\alpha)+\frac{2A\delta}{\Gamma(\alpha)\sum\limits_{i=0}^{m-1}\gamma_i\xi_i}.
\end{aligned}
$$

Since t^α is uniformly continuous on $[0,1]$, we see that $K_P(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous. Hence, $K_P(I-Q)N : X \to X$ is compact.

Lemma 3.4 Suppose (H_1), (H_2) hold, then

$$\Omega_0 = \{x \in \mathrm{dom}L : Lx = \lambda Nx, \lambda \in (0,1)\}$$

is bounded.

Proof Let $x \in \Omega_0$, then $Nx \in \mathrm{Im}L$, we have

$$\int_0^1 G(s)f(s,x(s),x'(s))ds = 0.$$

By the integral mean value theorem and (H_2), there exist two constants $\varepsilon_1, \varepsilon_2 \in (0,1)$, such that $|x(\varepsilon_1)| \le B_1$ and $|x'(\varepsilon_2)| \le B_2$. Then, by $x(t) = I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x(t) + c_0 + c_1 t$, one has

$$
\begin{aligned}
x'(t) &= I_{0+}^{\alpha-1}\,{}^{C}D_{0+}^{\alpha}x(t) + c_1 \\
&= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\,{}^{C}D_{0+}^{\alpha}x(s)\,\mathrm{d}s + c_1.
\end{aligned}
$$

Let $t = \varepsilon_2$, then

$$x'(\varepsilon_2) = \frac{1}{\Gamma(\alpha-1)}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\alpha-2}\,{}^{C}D_{0+}^{\alpha}x(s)\,\mathrm{d}s + c_1.$$

Since $|x'(\varepsilon_2)| \le B_2$, we get

$$
\begin{aligned}
|c_1| &\le |x'(\varepsilon_2)| + \frac{1}{\Gamma(\alpha-1)}\int_0^{\varepsilon_2}(\varepsilon_2-s)^{\alpha-2}\left|{}^{C}D_{0+}^{\alpha}x(s)\right|\mathrm{d}s \\
&\le B_2 + \frac{\varepsilon_2^{\alpha-1}}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty \\
&\le B_2 + \frac{1}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x'\|_\infty &\le \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\left|{}^{C}D_{0+}^{\alpha}x(s)\right|\mathrm{d}s + |c_1| \\
&\le \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty + B_2 + \frac{1}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty \\
&\le \frac{2}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty + B_2.
\end{aligned}
$$

Let $t = \varepsilon_1$, then

$$x(\varepsilon_1) = \frac{1}{\Gamma(\alpha)}\int_0^{\varepsilon_1}(\varepsilon_1-s)^{\alpha-1}\,{}^{C}D_{0+}^{\alpha}x(s)\,\mathrm{d}s + c_0 + c_1\varepsilon_1.$$

From $|x(\varepsilon_1)| \le B_1$, we obtain

$$
\begin{aligned}
|c_0| &\le |x(\varepsilon_1)| + \frac{1}{\Gamma(\alpha)}\int_0^{\varepsilon_1}(\varepsilon_1-s)^{\alpha-1}\left|{}^{C}D_{0+}^{\alpha}x(s)\right|\mathrm{d}s + |c_1| \\
&\le B_1 + \frac{\varepsilon_1^\alpha}{\Gamma(\alpha+1)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty + B_2 + \frac{1}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty \\
&\le B_1 + B_2 + \frac{1+\alpha}{\Gamma(\alpha+1)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|x\|_\infty &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|{}^{C}D_{0+}^{\alpha}x(s)\right|\mathrm{d}s + |c_0| + |c_1| \\
&\le \frac{t^\alpha}{\Gamma(\alpha+1)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty + B_1 + 2B_2 \\
&\quad + \frac{1}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty + \frac{1+\alpha}{\Gamma(\alpha+1)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty \\
&\le B_1 + 2B_2 + \frac{2(\alpha+1)}{\Gamma(\alpha+1)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty.
\end{aligned}
$$

Furthermore, by $Lx = \lambda Nx$, we have $-{}^{C}D_{0+}^{\alpha}x(t) = \lambda f(t, x(t), x'(t))$. Combining (H_1) and $\lambda \in (0,1)$, one has $|{}^{C}D_{0+}^{\alpha}x(t)| \le a_1 + b_1|x| + c_1|x'|$. Hence,

$$
\begin{aligned}
\left\|{}^{C}D_{0+}^{\alpha}x(t)\right\|_\infty &\le a_1 + b_1\|x\|_\infty + c_1\|x'\|_\infty \\
&\le a_1 + b_1(B_1 + 2B_2 + \frac{2(\alpha+1)}{\Gamma(\alpha+1)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty) \\
&\quad + c_1(\frac{2}{\Gamma(\alpha)}\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty + B_2) \\
&\le a_1 + b_1B_1 + 2b_1B_2 + c_1B_2 \\
&\quad + (\frac{2b_1(\alpha+1)}{\Gamma(\alpha+1)} + \frac{2c_1}{\Gamma(\alpha)})\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty.
\end{aligned}
$$

On account of $\frac{2b_1(\alpha+1)}{\Gamma(\alpha+1)} + \frac{2c_1}{\Gamma(\alpha)} < 1$, there exists a constant $D_1 > 0$ such that $\left\|{}^{C}D_{0+}^{\alpha}x\right\|_\infty \le D_1$. Then,

$$\|x\|_\infty \le B_1 + 2B_2 + \frac{2(\alpha+1)}{\Gamma(\alpha+1)}D_1 := D_2,$$

$$\|x'\|_\infty \le B_2 + \frac{2}{\Gamma(\alpha)}D_1 := D_3.$$

Therefore, Ω_0 is bounded.

Next we will give the main proof of Theorem 3.1.

Proof of Theorem 3.1 Let $C = \{x \in X : x(t) \ge 0,\ t \in [0,1]\}$, $\Omega_1 = \{x \in X : r > |x(t)| > M\|x\|_X,\ t \in [0,1]\}$, and $\Omega_2 = \{x \in X : \|x\|_X < R\}$, where $R = \max\{D_2, D_3\} + 1$. Note that Ω_1, Ω_2 are open bounded subsets of X and

$$\overline{\Omega_1} = \{x \in X : r \ge |x(t)| \ge M\|x\|_\infty,$$

where, $t \in [0,1]\} \subset \Omega_2,\ C \cap \overline{\Omega_2} \setminus \Omega_1 \ne \emptyset$. By Lemma 3.2, 3.3, and 3.4, we find that the conditions (1) and (2) of Lemma 2.2 are fulfilled.

Let $\gamma x(t) = |x(t)|$ for $x \in X$ and operator $J = I$. Then we see that γ is a retraction and maps subsets of $\overline{\Omega_2}$ into bounded subsets of C. Clearly, (3) of Lemma 2.2 holds.

For $x \in \mathrm{Ker}L \cap \Omega_2$, one has $x(t) \equiv c$, set

$$H(c,\lambda) = c - \lambda|c| - \frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|c|,0)ds,$$

where $\lambda \in [0,1]$. Suppose $H(c,\lambda) = 0$. By (H_3), we obtain

$$\begin{aligned} c &= \lambda|c| + \frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|c|,0)ds \\ &\geq \lambda|c| - \frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)\kappa|c|ds = \lambda|c|(1-\kappa) \geq 0. \end{aligned}$$

Thus $H(c,\lambda) = 0$ implies $c \geq 0$. Clearly, $H(R,0) \neq 0$. Moreover, if $H(R,\lambda) = 0$, $\lambda \in (0,1]$, we get

$$0 \leq R(1-\lambda) = \frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,R,0)ds,$$

which contradicts to condition (H_2). Hence $H(x,\lambda) \neq 0$ for $x \in \partial\Omega_2,\ \lambda \in [0,1]$. Therefore,

$$\begin{aligned} &\deg([I-(P+JQN)\gamma]|_{\mathrm{Ker}L}, \mathrm{Ker}L \cap \Omega_2, 0) \\ &\quad = \deg(H(x,1), \mathrm{Ker}L \cap \Omega_2, 0) \\ &\quad = \deg(H(x,0), \mathrm{Ker}L \cap \Omega_2, 0) \\ &\quad = \deg(I, \mathrm{Ker}L \cap \Omega_2, 0) = 1 \neq 0. \end{aligned}$$

Then, (4) of Lemma 2.2 holds.

Let $x \in \overline{\Omega_2} \setminus \Omega_1,\ t \in [0,1]$. By (H_3), we get

$$\begin{aligned} \Psi_\gamma x(t) &= \int_0^1 |x(t)|dt + \int_0^1 k(t,s)[f(s,|x(s)|,|x'(s)|) \\ &\quad - \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(\tau)f(\tau,|x(\tau)|,|x'(\tau)|)d\tau]ds \\ &\quad + \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|,|x'(s)|)ds \\ &= \int_0^1 |x(t)|dt + \int_0^1 U(t,s)f(s,|x(s)|,|x'(s)|)ds \\ &\geq \int_0^1 |x(s)|ds - \kappa\int_0^1 U(t,s)|x(s)|ds \\ &= \int_0^1 (1-\kappa U(t,s))|x(s)|ds \geq 0. \end{aligned}$$

So, $\Psi_\gamma(\overline{\Omega_2}\backslash\Omega_1) \subset C$. For $x \in \partial\Omega_2$, one has

$$\begin{aligned} &(P+JQN)\gamma x \\ &= \int_0^1 |x(s)|ds + \frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|,|x'(s)|)ds \\ &\geq \int_0^1 (1 - \frac{\kappa G(s)}{\int_0^1 G(s)ds})|x(s)|ds \geq 0, \end{aligned}$$

thus $(P+JQN)\gamma(\partial\Omega_2) \subset C$. Clearly, (6), (7) of Lemma 2.2 hold.

Let $u_0(t) \equiv 1,\ t \in [0,1]$ and $\sigma(u_0) = 1$, then $u_0 \in C\backslash\{0\},\ C(u_0) = \{x \in C | x(t) > 0, t \in [0,1]\}$. For $x \in C(u_0) \cap \partial\Omega_1$, one has $x(t) > 0,\ t \in [0,1],\ 0 < \|x\|_X \leq r$ and $x(t) \geq M\|x\|_X,\ t \in [0,1]$. So, by (H_4), we have

$$\begin{aligned} (\Psi x)(t_0) &= \int_0^1 x(s)ds + \int_0^1 U(t_0,s)f(s,x(s),|x'(s)|)ds \\ &\geq M\|x\|_X + \int_0^1 U(t_0,s)g(s)h(x(s))ds \\ &= M\|x\|_X + \int_0^1 U(t_0,s)g(s)\frac{h(x(s))}{x^a(s)}x^a(s)ds \\ &\geq M\|x\|_X + \frac{h(r)}{r^a}\int_0^1 U(t_0,s)g(s)M^a\|x\|_X^a ds \\ &\geq M\|x\|_X + (1-M)\|x\|_X = \|x\|_X. \end{aligned}$$

Then $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, that is, (5) of Lemma 2.2 holds. By Lemma 2.2, the equation $Lx = Nx$ has at least a solution x, which implies FBVPs (1) with resonant condition (i) has at least one positive solution in X.

3.2 FBVPs (1) with resonant condition (ii)

The definitions of the cone C and sets $\Omega_1,\ \Omega_2$ are similar with that in Section 3.1. Define function $\rho(t),\ t \in [0,1]$:

$$\rho(t) = \sum_{i=0}^{m-1}\gamma_i\xi_i + (1 - \sum_{i=0}^{m-1}\gamma_i)t$$

and positive numbers

$$\rho_1 = \begin{cases} \sum\limits_{i=0}^{m-1}\gamma_i\xi_i, & \sum\limits_{i=0}^{m-1}\gamma_i < 1, \\ \sum\limits_{i=0}^{m-1}\gamma_i\xi_i + 1 - \sum\limits_{i=0}^{m-1}\gamma_i, & \sum\limits_{i=0}^{m-1}\gamma_i > 1, \end{cases}$$

$$\rho_2 = \begin{cases} \sum\limits_{i=0}^{m-1}\gamma_i\xi_i + 1 - \sum\limits_{i=0}^{m-1}\gamma_i, & \sum\limits_{i=0}^{m-1}\gamma_i < 1, \\ \sum\limits_{i=0}^{m-1}\gamma_i\xi_i, & \sum\limits_{i=0}^{m-1}\gamma_i > 1, \end{cases}$$

$$\rho_3 = \begin{cases} \dfrac{\sum_{i=0}^{m-1}\gamma_i\xi_i}{\sum_{i=0}^{m-1}\gamma_i\xi_i + 1 - \sum_{i=0}^{m-1}\gamma_i}, & \sum\limits_{i=0}^{m-1}\gamma_i < 1, \\ \dfrac{\sum_{i=0}^{m-1}\gamma_i\xi_i + 1 - \sum_{i=0}^{m-1}\gamma_i}{\sum_{i=0}^{m-1}\gamma_i\xi_i}, & \sum\limits_{i=0}^{m-1}\gamma_i > 1, \end{cases}$$

$$\rho_4 = \sum_{i=0}^{m-1}\gamma_i\xi_i + \frac{1}{2}(1 - \sum_{i=0}^{m-1}\gamma_i),\ \rho_5 = \frac{\rho_1}{\rho_4}.$$

Clearly, $\rho(t) \in [\rho_1, \rho_2],\ t \in [0,1],\ \rho_3 \in (0,1)$. Denote the function $U_1(t,s)$ as follow:

$$\mathrm{U_1(t,s)} = \begin{cases} \frac{(2t-1)}{2\rho_4}[\frac{(1-\sum_{i=0}^{m-1}\gamma_i)(1-s)^\alpha}{\Gamma(\alpha+1)} + \frac{\sum_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}] + \frac{G(s)}{\int_0^1 G(s)ds} \\ \times[1 - \frac{1}{\Gamma(\alpha+2)} - \frac{(2t-1)}{2\rho_4}(\frac{1-\sum_{i=0}^{m-1}\gamma_i}{\Gamma(\alpha+2)} + \frac{\sum_{i=k}^{m-1}\gamma_i(\xi_i^\alpha-(\xi_i-1)^\alpha)}{\Gamma(\alpha+1)}) \\ + \frac{t^\alpha}{\Gamma(\alpha+1)}] + \frac{(1-s)^\alpha}{\Gamma(\alpha+1)},\ \xi_{k-1} \leq s \leq \xi_k,\ 0 \leq t \leq s \leq 1, \\ \frac{(2t-1)}{2\rho_4}[\frac{(1-\sum_{i=0}^{m-1}\gamma_i)(1-s)^\alpha}{\Gamma(\alpha+1)} + \frac{\sum_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}] + \frac{G(s)}{\int_0^1 G(s)ds} \\ \times[1 - \frac{1}{\Gamma(\alpha+2)} - \frac{(2t-1)}{2\rho_4}(\frac{1-\sum_{i=0}^{m-1}\gamma_i}{\Gamma(\alpha+2)} + \frac{\sum_{i=k}^{m-1}\gamma_i(\xi_i^\alpha-(\xi_i-1)^\alpha)}{\Gamma(\alpha+1)}) \\ + \frac{t^\alpha}{\Gamma(\alpha+1)}] + \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ \xi_{k-1} \leq s \leq \xi_k,\ 0 \leq s \leq t \leq 1, \end{cases}$$

and positive number

$$\kappa_1=\min\{1,\ \min_{s\in[0,1]}\frac{\rho_5\int_0^1 G(s)ds}{G(s)},\ \min_{t,s\in[0,1]}\frac{\rho_5}{U_1(t,s)}\}.$$

Theorem 3.5 Under assumption $(\mathrm{H}_1),(\mathrm{H}_2)$, there exists a constant $R\in(0,\infty)$ such that $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous and
(H_5) $f(t,u,v)<0$, for $(t,u,v)\in[0,1]\times[\rho_3R,R]\times\mathbb{R}$,
(H_6) $f(t,u,v)>-\kappa u$, for all $(t,u,v)\in[0,1]\times[0,R]\times\mathbb{R}$,
(H_7) there exist $r\in(0,R),\ t_0\in[1/2,1],\ a\in(0,1],\ M\in(0,1)$ and continuous functions $g:[0,1]\to[0,\infty),\ h:(0,r]\to[0,\infty)$ such that $f(t,u,v)\geq g(t)h(u)$ for $(t,u,v)\in[0,1]\times(0,r]\times\mathbb{R}$, and $h(u)/u^a$ is non-increasing on $(0,r]$ with

$$\frac{h(r)}{r^a}\int_0^1 U_1(t_0,s)g(s)ds\geq\frac{1-M\rho_5}{M^a},$$

then FBVPs (1) has at least one positive solution.

In order to prove the above theorem, we first give a useful lemma.

Lemma 3.6 Let L be defined by (2), then

$$\mathrm{Ker}L=\{x\in X|x(t)=c\rho(t),\ \forall t\in[0,1],\ c\in\mathbb{R}\},$$
$$\mathrm{Im}L=\{y\in Y|\int_0^1 G(s)y(s)ds=0\},$$

and the linear continuous projector operators $P:X\to X$ and $Q:Y\to Y$ can be defined as

$$Px(t)=\frac{\rho(t)}{\rho_4}\int_0^1 x(s)ds,\quad \forall t\in[0,1],\ x\in X,$$
$$Qy(t)=\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)y(s)ds.$$

Moreover, the operator $K_P:\mathrm{Im}L\to\mathrm{dom}L\cap\mathrm{Ker}P$ is given by

$$K_Py(t)=\int_0^1 k_1(t,s)y(s)ds,\ \forall t\in[0,1],$$

where

$$\mathrm{k}_1(\mathrm{t,s})=\begin{cases}\frac{(2t-1)}{2\rho_4}[\frac{(1-\sum_{i=0}^{m-1}\gamma_i)(1-s)^\alpha}{\Gamma(\alpha+1)}+\frac{\sum_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}]\\ +\frac{(1-s)^\alpha}{\Gamma(\alpha+1)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ \xi_{k-1}\le s\le\xi_k,\ 0\le s\le t\le 1,\\ \frac{(2t-1)}{2\rho_4}[\frac{(1-\sum_{i=0}^{m-1}\gamma_i)(1-s)^\alpha}{\Gamma(\alpha+1)}+\frac{\sum_{i=k}^{m-1}\gamma_i(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}]\\ +\frac{(1-s)^\alpha}{\Gamma(\alpha+1)},\ \xi_{k-1}\le s\le\xi_k,\ 0\le t<s\le 1.\end{cases}$$

Next we will give the main proof of Theorem 3.5.

Proof of Theorem 3.5 We claim that the conditions (H_1), (H_2) ensure that (1)-(3) of Lemma 2.2 are satisfied. The proof is same with that in Section 3.1. For $x\in\mathrm{Ker}L\cap\Omega_2$, let

$$H(x,\lambda)=x-\lambda|x|-\frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x|,|x'|)ds,$$

Suppose $x\in\partial\Omega_2\cap\mathrm{Ker}L$ and $H(x,\lambda)=0$, we have $x=\frac{R\rho(t)}{\rho_2}$ and $\|x\|_\infty=R$.

From the definition of ρ_3, one has $\rho_3R\le x(t)\le R$, which implies $f(t,x,x')<0$. It contradicts to

$$0\le(1-\lambda)x=\frac{\lambda}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x|,|x'|)ds,$$

thus $H(x,\lambda)\neq 0$ for $x\in\partial\Omega_2,\ \lambda\in[0,1]$, then

$$\begin{aligned}&\deg([I-(P+JQN)\gamma]|_{\mathrm{Ker}L},\mathrm{Ker}L\cap\Omega_2,0)\\&=\deg(H(c,1),\mathrm{Ker}L\cap\Omega_2,0)\\&=\deg(H(c,0),\mathrm{Ker}L\cap\Omega_2,0)\\&=\deg(I,\mathrm{Ker}L\cap\Omega_2,0)=1\neq 0.\end{aligned}$$

So (4) of Lemma 2.2 holds.

Let $x\in\overline{\Omega_2}\setminus\Omega_1,\ t\in[0,1]$. By (H_6), we get

$$\begin{aligned}\Psi_\gamma x(t)=&\frac{\rho(t)}{\rho_4}\int_0^1|x(t)|dt\\&+\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|,|x'(s)|)ds\\&+\int_0^1 k_1(t,s)[f(s,|x(s)|,|x'(s)|)\\&-\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(\tau)f(\tau,|x(\tau)|,|x'(\tau)|)d\tau]ds\\=&\frac{\rho(t)}{\rho_4}\int_0^1|x(t)|dt+\int_0^1 U_1(t,s)f(s,|x(s)|,|x'(s)|)ds\\\geq&\frac{\rho_1}{\rho_4}\int_0^1|x(s)|ds-\kappa_1\int_0^1 U_1(t,s)|x(s)|ds\\=&\int_0^1(\rho_5-\kappa_1U_1(t,s))|x(s)|ds\geq 0.\end{aligned}$$

So, $\Psi_\gamma(\overline{\Omega_2}\setminus\Omega_1)\subset C$. For $x\in\partial\Omega_2$, one has

$$\begin{aligned}(P+JQN)\gamma x=&\frac{\rho(t)}{\rho_4}\int_0^1|x(s)|ds\\&+\frac{1}{\int_0^1 G(s)ds}\int_0^1 G(s)f(s,|x(s)|,|x'(s)|)ds\\\geq&\int_0^1(\frac{\rho_1}{\rho_4}-\frac{\kappa_1G(s)}{\int_0^1 G(s)ds})|x(s)|ds\geq 0\\=&\int_0^1(\rho_5-\frac{\kappa_1G(s)}{\int_0^1 G(s)ds})|x(s)|ds\geq 0,\end{aligned}$$

thus $(P+JQN)\gamma(\partial\Omega_2)\subset C$. Clearly, (6), (7) of Lemma 2.2 hold.

Let $u_0(t)\equiv 1,\ t\in[0,1]$ and $\sigma(u_0)=1$, then $u_0\in C\backslash\{0\},\ C(u_0)=\{x\in C|x(t)>0,t\in[0,1]\}$. For $x\in C(u_0)\cap\partial\Omega_1$, we have $x(t)>0,\ t\in[0,1],\ 0<\|x\|_X\le r$ and $x(t)\geq M\|x\|_X,\ t\in[0,1]$. Hence, by (H_7), one has

$$\begin{aligned}&(\Psi x)(t_0)\\&=\frac{\rho(t)}{\rho_4}\int_0^1 x(s)ds+\int_0^1 U_1(t_0,s)f(s,x(s),|x'(s)|)ds\\&\geq M\rho_5\|x\|_X+\int_0^1 U_1(t_0,s)g(s)h(x(s))ds\\&=M\rho_5\|x\|_X+\int_0^1 U_1(t_0,s)g(s)\frac{h(x(s))}{x^a(s)}x^a(s)ds\\&\geq M\rho_5\|x\|_X+\frac{h(r)}{r^a}\int_0^1 U_1(t_0,s)g(s)M^a\|x\|_X^a ds\\&\geq M\rho_5\|x\|_X+(1-M\rho_5)\|x\|_X=\|x\|_X.\end{aligned}$$

Then $\|x\| \le \sigma(u_0)\,\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, that is, (5) of Lemma 2.2 holds. By Lemma 2.2, the equation $Lx = Nx$ has at least a solution x, which means FBVPs (1) with resonant condition (ii) has at least one positive solution in X.

Corollary 3.7 In order to make the results of this paper more comprehensive, we consider the existence of solutions when FBVPs (1) does not satisfy both resonance conditions (i) and (ii). That is, when $0 < \sum_{i=1}^{m-2}\gamma_i < 1$ and $0 < \sum_{i=1}^{m-2}\beta_i < 1$ are satisfied, we have the unique solution of FBVPs (1) expressed as the following integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s), x'(s))ds$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} (t-s)^{\alpha-1} + \Delta_1 + \Delta_2, \ 0 \le s \le t \le 1, \ s \le \xi_i, \\ \Delta_1 + \Delta_2, \ 0 \le t \le s \le \xi_i < 1, \\ (t-s)^{\alpha-1} + \Delta_2, \ 0 < \xi_i \le s \le t \le 1, \\ \Delta_2, \ 0 \le t \le s \le 1, \ s \ge \xi_i, \end{cases}$$

$$\Delta_1 = \frac{\sum\gamma_i\sum\beta_i\xi_i - \sum\gamma_i\xi_i\sum\beta_i - \sum\beta_i}{(1-\sum\gamma_i)(1-\sum\beta_i\xi_i) + (1-\sum\beta_i)\sum\gamma_i\xi_i} \cdot (\xi_i - s)^{\alpha-1}$$

$$\Delta_2 = \frac{\sum\gamma_i\xi_i + 1 - \sum\gamma_i}{(1-\sum\gamma_i)(1-\sum\beta_i\xi_i) + (1-\sum\beta_i)\sum\gamma_i\xi_i} \cdot (1-s)^{\alpha-1}$$

According to the properties of $G(t,s)$, we obtain

$$0 < G(t,s) < \frac{\Delta_3 + \sum\gamma_i\xi_i + 1 - \sum\gamma_i}{\Delta_3}, \ \forall s, t \in (0,1).$$

where, $\Delta_3 = (1 - \sum\gamma_i)(1 - \sum\beta_i\xi_i) + (1 - \sum\beta_i)\sum\gamma_i\xi_i$. Define the operator $T : C[0,1] \to C[0,1]$ as $Tx(t) = \int_0^1 G(t,s) f(s, x(s), x'(s))ds$. Then by using Schaefer's fixed point theorem, we get the operator T has a fixed point, which implies FBVPs (1) has at least one solution.

Corollary 3.8 If $0 < \sum_{i=1}^{m-2}\gamma_i < 1, \ 0 < \sum_{i=1}^{m-2}\beta_i < 1$, and there exists a constant $k > 0$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \le k(|u_1 - u_2| + |v_1 - v_2|)$$

for each $t \in [0,1]$ and all $u_1, \ v_1 \ u_2, \ v_2 \in \mathbb{R}$, then by using the Banach contraction mapping principle, we obtain FBVPs(1) has a unique solution.

IV. Example

Example 4.1. Consider the following FBVPs

$$\begin{cases} -{}^C D_{0+}^{\frac{3}{2}} x(t) = -\frac{1}{3}(t^2 - t - 1)(x-1)(x-3)\sqrt{(x-3)^2 + 1}, \\ x(0) = \frac{1}{2}x(\frac{1}{4}) + \frac{1}{2}x(\frac{1}{2}), \ x(1) = \frac{1}{3}x(\frac{1}{4}) + \frac{2}{3}x(\frac{1}{2}), \end{cases} \tag{3}$$

where $t \in (0,1), \ \alpha = \frac{3}{2}, \ \gamma_1 = \gamma_2 = \frac{1}{2}, \ \beta_1 = \frac{1}{3}, \ \beta_2 = \frac{2}{3}, \ \xi_1 = \frac{1}{4}, \ \xi_2 = \frac{1}{2}$. By a simple computation, we obtain

$$\int_0^1 G(s)ds = 0.24, \ \kappa = 0.338, \ \int_0^1 U(0,s)ds = 1.$$

Let $B = \frac{6}{5}, \ r = \frac{1}{2}, \ t_0 = 0, \ a = 1, \ M = \frac{1}{2}$ and $g(t) = -\frac{1}{3}(t^2 - t - 1), \ h(x) = \sqrt{(x-3)^2 + 1}$. It is easy to verify that

$$\frac{1}{3} \le g(t) \le \frac{5}{12}, \ t \in [0,1], \ (\mathrm{x}-1)(\mathrm{x}-3) \ge -x, \ x \in [0, \frac{6}{5}]$$

and

(1) $\frac{2b_1}{\Gamma(\alpha+1)} = \frac{2\times\frac{5}{12}}{\Gamma(\frac{5}{2})} = \frac{10}{9\sqrt{\pi}} < 1$,
(2) $f(t, B) < 0$, for all $t \in [0,1]$,
(3) $f(t,x) > -0.338x$, for all $(t,x) \in [0,1] \times [0,\infty)$,
(4) $f(t,x) \ge g(t)h(x)$ for all $[t,x] \in [0,1] \times (0, \frac{1}{2}]$ and $\frac{h(x)}{x} = \frac{\sqrt{(x-3)^2+1}}{x}$ is non-increasing on $(0, \frac{1}{2}]$ with

$$\begin{aligned} \frac{h(r)}{r^a}\int_0^1 U(0,s)g(s)ds &\ge \frac{\sqrt{29}}{3}\int_0^1 U(0,s)ds \\ &= \frac{\sqrt{29}}{3} \ge 1 = \frac{1-M}{M^a}. \end{aligned}$$

So the conditions of Theorem 3.1 are satisfied, that is, FBVPs (3) has at least one positive solution.

V. Conclusion

In this paper, we study the existence of positive solutions of problem at resonance. In the past, there have been many studies on the solutions of resonance problems and the positive solutions of non-resonance problems, but few studies on the positive solutions of resonance problems. Therefore, we use Leggett-Williams norm-type theorem to study the existence of positive solutions for a class of fractional multi-point boundary value problems (1) with different resonant conditions (i) and (ii), and obtain some new existence results (see Theorem 3.1, 3.5). Since the multi-point boundary value problem is more general and the positive solution has more practical significance, some existing results are generalized in this paper. In addition, since fractional derivatives are non-local, it is more difficult to study fractional order problems than integer order ones.

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