Finite Generalized Hankel Transformation on Different Spaces Extended to Class of Generalized Functions

V. R. Lakshmi Gorty, *Member, IAENG.*

***Abstract*—In this paper, the finite generalized Hankel transformation on spaces of generalized functions by developing a new procedure has been studied. The transformation is used by application of a generalized function to the kernel function and is extend to a larger space of generalized functions.**

***Index Terms*—generalized Hankel transformation, distributional spaces, operator.**

I. Introduction

Malgonde [1] investigated the following generalized Hankel transformation
$(h_{1,\mu,\alpha,\beta,\nu}f)(y) = F_1(y)$

$$= \nu\beta\ y^{-1-2\alpha+2\nu} \int_0^\infty \mathscr{J}_{\alpha,\beta,\nu,\mu}(xy)\ f(x)\,dx,\ (\mu \geq -1/2) \quad \text{(I.1)}$$

where $\mathscr{J}_{\alpha,\beta,\nu,\mu}(x) = x^\alpha J_\mu(\beta x^\nu)$, $J_\mu(x)$ being the Bessel function of the first kind of order μ. This transformation has been studied on distributional spaces by different authors [2], [3], [4]. Malgonde and Gorty [5] investigated the finite generalized Hankel-Clifford transformation and now extended to the variant of the classical modified finite generalized Hankel-transform type defined by
$(\hbar_{1,\alpha,\beta,\nu,\mu}f)(n) = F_{\alpha,\beta,\nu,\mu}(n)$

$$= \nu^2\beta^2 \int_0^a x^{2\nu-2\alpha-1} \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x)\, f(x)\,dx\,;\ m = 1,2,.. \quad \text{(I.2)}$$

for a function $f(x)$ defined on the interval $(0,1)$ is introduced in [6] as where $\mu \geq -\frac{1}{2}$ and $\lambda_1, \lambda_2, ...$ represent the positive zeros of $\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n a)$ [7] arranged in ascending order of magnitude.
Here

$$\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x) = (x\lambda_n)^\alpha J_\mu\left(\beta(x\lambda_n)^\nu\right). \quad \text{(I.3)}$$

The inversion theorem for the transformation (I.2) is given in [5].

Theorem 1.1: Let $f(t)$ be a function defined and absolutely integrable on $(0,a)$. Assume that $\mu \geq -\frac{1}{2}$ and

$$f(t) = \sum_{m=1}^{\infty} a_m \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m t) \quad \text{(I.4)}$$

set
$a_m =$

$$\frac{2\nu}{a^{2\nu-2\alpha}\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_m a)} \int_0^a t^{2\nu-2\alpha-1} \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m t) f(t)\,dt. \quad \text{(I.5)}$$

If $f(t)$ is of the bounded variation in $(a_1,a_2), (0 < a_1 < a_2 < a)$ and if $t \in (a_1,a_2)$, then the series $\sum_{m=1}^{\infty} a_m \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x)$ converges to $\frac{1}{2}[f(x+0) + f(x-0)]$.
Some orthogonal series expansions involving generalized functions have been discussed by Zemanian and motivated to this author to introduce the finite generalized Hankel transformation [4]. The same transformation has been extended to a class of generalized functions by Dube [8] developing a quiet different technique. Precisely in this paper, the finite Hankel-type generalized transformation is extended to a class of generalized functions, following the method investigated by [5]. $D(I)$ is the space of infinitely differentiable functions with compact support on . Its dual $D'(I)$ is the space of Hankel distributions. $E(I)$ is the space of all infinitely differentiable functions on I and $E'(I)$ is the space of distributions with compact support.
Finally, we will use the operators

$$\Delta_{\alpha,\beta,\nu,\mu} = x^{-\mu\nu+\alpha+1-2\nu} D x^{2\mu\nu+1} D x^{-\mu\nu-\alpha}.$$

It is expressed as

$$\Delta_{\alpha,\beta,\nu,\mu} = x^{2-2\nu}D^2 + (1-2\alpha)\,x^{1-2\nu}D - x^{-2\nu}\left\{(\mu\nu)^2 - \alpha^2\right\} \quad \text{(I.6)}$$

and $D = \frac{d}{dx}$.
The operational formula is easily computable

$$\Delta_{\alpha,\beta,\nu,\mu}\left[\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x)\right] = -(\lambda_m)^{2\nu}(\beta\nu)^2\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x). \quad \text{(I.7)}$$

In the study the finite generalized Hankel transformation is defined by:
$(\hbar_{\alpha,\beta,\nu,\mu}f)(n) = F_{\alpha,\beta,\nu,\mu}(n)$

$$= \beta^2\nu^2 \int_0^a x^{2\nu-2\alpha-1}(xy)^\alpha \mathscr{J}_\mu\left(\beta(xy)^\nu\right) f(x)dx. \quad \text{(I.8)}$$

For each real number μ, a function $\varphi(x)$ is in H_μ if and only if it is defined on $0 < x < 1$, it is complex-valued and smooth, and for each pair of nonnegative integers m and k,

$$\gamma^\mu_{m,k}(\varphi) \triangleq \sup_{0<x<1}\left|x^m\left(x^{-1}D\right)^k\left[x^{-\mu-1/2}\varphi(x)\right]\right| \quad \text{(I.9)}$$

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V. R. Lakshmi Gorty is presently a Professor in Mathematics, Basic Sciences and Humanities, SVKM's NMIMS, MPSTME, Vile Parle (W), Mumbai-400056, India.
e-mail: vr.lakshmigorty@nmims.edu.

exists (i.e., is finite). H_μ is a linear space. Also, $\gamma^{\mu}_{m,k}$ each is a seminorm on H_μ. The topology of H_μ is that generated by $\left\{\gamma^{\mu}_{m,k}\right\}^{\infty}_{m,k=0}$. The Hankel transformation h_μ, is an automorphism on H_μ whenever $\mu \geq -\frac{1}{2}$.
The generalized functions in the dual H'_μ of H_μ act like distributions of slow growth as $x \to \infty$. Moreover, H'_μ is the domain of H_μ the generalized Hankel transformation h_μ, it follows that h_μ is an automorphism on H'_μ . This procedure is reminiscent of Schwartz's method of extending the Fourier transformation to distributions of slow growth.
For a real number μ and a positive real number a, they constructed a testing function space $J_{a,\,\mu}$ as follows:
Let $J_{a,\,\mu}$ be a testing function space containing all $\varphi(x)$ which are defined and smooth on $I = (0, 1)$ and for which $\tau^{\mu,\,a}_{k}(\varphi)$

$$= \sup_{x\in I}\left|e^{-ax}x^{-\mu-1/2}\left(x^{-\mu-1/2}Dx^{2\mu+1}Dx^{-\mu-1/2}\right)^{k}(\varphi)\right| \tag{I.10}$$

$< \infty$ for $k = 0, 1, 2, \cdots$. $\mathscr{J}_{a,\,\mu}$ is assigned the topology generated by the countable multinorm $\{\tau^{\mu,\,a}_{k}\}^{\infty}_{k=0}$ $\cdot \mathscr{J}_{a,\mu}$ contains the kernel $xy^{\nu}\mathscr{J}_{\mu}(xy)$ as a function on
$0 < x < 1$ for each fixed complex y in the strip

$$\Omega = \{y : |\mathrm{Im}\ y| < a,\ y \neq 0 \text{ or a negative number}\}$$

. Also defined a new testing function space $M_{a,\,\mu}$ between H_μ and $\mathscr{J}_{a,\,\mu}$ namely $H_\mu \subset M_{a,\,\mu} \subset \mathscr{J}_{a,\,\mu}$, whereby $M_{a,\,\mu}$ still contains the kernel function. $\mathscr{J}'_{a,\,\mu} \subset M'_{a,\,\mu}$ is thus extended to a larger class of generalized functions. Many properties of $M_{a,\,\mu}$ and the countable union $M_\mu = \bigcup^{\infty}_{i=1} M_{a_i,\,\mu}$ are also established.
An inversion theorem and a representation for $M'_{a,\,\mu}$ are the main results. The Hankel transformation h_μ is now defined on the dual space $\mathscr{J}'_{a,\,\mu}$. The behaviours of $\mathscr{J}_\mu$ near the origin and the infinity are the following ones:

$$\mathscr{J}_\mu(x) = O\left(x^{\mu}\right) \tag{I.11}$$

as $x \to 0+$.

$$\mathscr{J}_\mu(x) \approx \left(\frac{2}{\pi x}\right)^{1/2}\left[\cos\left(x-\frac{1}{2}\mu\pi-\frac{1}{4}\pi\right)\sum^{\infty}_{m=0}\frac{(-1)^m(\mu,2m)}{(2x)^{2m}}\right] - \left(\frac{2}{\pi x}\right)^{1/2}\left[\sin\left(x-\frac{1}{2}\mu\pi-\frac{1}{4}\pi\right)\frac{(-1)^m(\mu,2m+1)}{(2x)^{2m+1}}\right] \tag{I.12}$$

where (μ, k) is understood as in [9].
The main differentiation formulas for J_μ are

$$\frac{d}{dx}\left\{(\lambda_m x)^{(\mu\nu+\nu)}\mathscr{J}_{\mu+1}\left(\beta(\lambda_m x)^{\nu}\right)\right\} = \nu\beta(\lambda_m x)^{(\mu\nu+2\nu)}x^{-1}\mathscr{J}_{\mu}\left(\beta(\lambda_m x)^{\nu}\right). \tag{I.13}$$

$$\frac{d}{dx}\left\{(\lambda_m x)^{(-\mu\nu)}\mathscr{J}_{\mu}\left(\beta(\lambda_m x)^{\nu}\right)\right\} = -\nu\beta(\lambda_m x)^{(-\mu\nu+\nu)}x^{-1}\mathscr{J}_{\mu+1}\left(\beta(\lambda_m x)^{\nu}\right). \tag{I.14}$$

for $x, y > 0$.

II. The testing function space $V_{\alpha,\beta,\nu,\mu}(I)$

For each pair of real numbers α, β, ν and μ, with $\mu \geq -\frac{1}{2}$ and $V_{\alpha,\beta,\nu,\mu}(I)$ is the space of all infinitely differentiable complex-valued functions $\phi(x)$ defined on I, such that

$$\gamma^{\theta,\alpha,\beta,\nu,\mu}_{k} = \sup\left|x^{\theta}\Delta^{k}{}_{\alpha,\beta,\nu,\mu}x^{-(-\mu\nu+\alpha+1-2\nu)}\phi(x)\right| < \infty \tag{II.1}$$

for each $k = 0, 1, 2, \cdots$.
The topology of the linear space $V_{\alpha,\beta,\nu,\mu}(I)$ is generated by the collection of seminorms$\left(\gamma^{\alpha,\beta,\nu,\mu}_{k}\right)$, $k = 0, 1, 2, \cdots$. Thus $V_{\alpha,\beta,\nu,\mu}(I)$ is a countably multinormed space.

Theorem 2.1: $V_{\alpha,\beta,\nu,\mu}(I)$ is complete and therefore a Fréchet space.

Proof. It can be proved with an argument similar to the one used by [4], [10]. Let J denote an arbitrary compact of I. Let x_1 be any fixed point of I and let D^{-1} be the integration operator $D^{-1} = \int_{x_1}^{x}\cdots dt$. Let (ϕ_n) be a Cauchy sequence in $V_{\alpha,\beta,\nu,\mu}(I)$. For each non-negative integer k, it follows from (II.1) that $\Delta^{k}_{\alpha,\beta,\nu,\mu}x^{-(-\mu\nu+\alpha+1-2\nu)}\phi_n(x)$ converges uniformly on J as $n \to \infty$.
If $k = 0$, then x $\phi_n(x)$ converges uniformly on J.
If $k = 1$, we have

$$\begin{aligned}
&x^{-(-\mu\nu+\alpha+1-2\nu)}\Delta_{\alpha,\beta,\nu,\mu}\phi_n(x)\\
&= x^{-(-\mu\nu+\alpha+1-2\nu)}\\
&\quad\times\left(x^{-\mu\nu+\alpha+1-2\nu}D_x x^{2\mu\nu+1}D_x x^{-\mu\nu-\alpha}\phi_n(x)\right)\\
&= x^{-(-\mu\nu+\alpha+1-2\nu)}x^{2-2\nu}D_x{}^2\phi_n(x)\\
&+ x^{-(-\mu\nu+\alpha+1-2\nu)}(1-2\alpha)x^{1-2\nu}D_x\phi_n(x)\\
&- x^{-(-\mu\nu+\alpha+1-2\nu)}x^{-2\nu}\left\{(\mu\nu)^2-\alpha^2\right\}\phi_n(x)
\end{aligned}$$

$$x^{-(-\mu\nu+\alpha+1-2\nu)}\Delta_{\alpha,\beta,\nu,\mu}\phi_n(x) = -x^{-(-\mu\nu+\alpha+1-2\nu)}(\lambda_n)^{2\nu}(\beta\nu)^2 J_{\alpha,\beta,\nu,\mu}(\lambda_n x) \tag{II.2}$$

$$\begin{aligned}
&D^{-1}x^{-(-\mu\nu+\alpha+1-2\nu)}\Delta_{\alpha,\beta,\nu,\mu}\phi_n(x)\\
&= D^{-1}x^{-(-\mu\nu+\alpha+1-2\nu)}\\
&\quad\times\left(x^{-\mu\nu+\alpha+1-2\nu}D_x x^{2\mu\nu+1}D_x x^{-\mu\nu-\alpha}\phi_n(x)\right)\\
&= x^{2\mu\nu+1}D_x x^{-\mu\nu-\alpha}\phi_n(x) - x_1^{2\mu\nu+1}D_{x_1}x_1^{-\mu\nu-\alpha}\phi_n(x_1)
\end{aligned} \tag{II.3}$$

And

$$\begin{aligned}
&x^{-(-\mu\nu+\alpha+1-2\nu)}D^{-1}x^{-(2\mu\nu+1)}D^{-1}x^{-(-\mu\nu+\alpha+1-2\nu)}\\
&\quad\times\Delta_{\alpha,\beta,\nu,\mu}\phi_n(x)\\
&= x^{-(-\mu\nu+\alpha+1-2\nu)}\int_{x_1}^{x}x^{-(2\mu\nu+1)}x^{2\mu\nu+1}D_x x^{-\mu\nu-\alpha}\phi_n(x)\,dx\\
&- x^{-(-\mu\nu+\alpha+1-2\nu)}\int_{x_1}^{x}x_1^{2\mu\nu+1}D_{x_1}x_1^{-\mu\nu-\alpha}\phi_n(x_1)\,dx
\end{aligned}$$

$$= x^{-(-\mu\nu+\alpha+1-2\nu)} \int_{x_1}^{x} \left[D_x x^{-\mu\nu-\alpha}\phi_n(x)\right] dx$$
$$- x^{-(-\mu\nu+\alpha+1-2\nu)} \int_{x_1}^{x} \left(\frac{x_1}{x}\right)^{2\mu\nu+1} D_{x_1} x_1^{-\mu\nu-\alpha}\phi_n(x_1)\, dx$$
$$= x^{-(-\mu\nu+\alpha+1-2\nu)} \left[x^{-\mu\nu-\alpha}\phi_n(x) - x_1^{-\mu\nu-\alpha}\phi_n(x_1)\right]$$
$$- x^{-(-\mu\nu+\alpha+1-2\nu)} \int_{x_1}^{x} \left(\frac{x_1}{x}\right)^{2\mu\nu+1} D_{x_1} x_1^{-\mu\nu-\alpha}\phi_n(x_1) dx$$
$$= \left[\phi_n(x) - \left(\tfrac{x}{x_1}\right)^{2\mu\nu-1+2\nu}\phi_n(x_1)\right]$$
$$-x^{-(-\mu\nu+\alpha+1-2\nu)}(x_1)^{2\mu\nu+1}\left[\tfrac{x^{-2\mu\nu}}{2\mu\nu} - \tfrac{x_1{}^{-2\mu\nu}}{2\mu\nu}\right] \times D_{x_1} x_1^{-\mu\nu-\alpha}\phi_n(x_1)$$
$$= \phi_n(x) - \left(\tfrac{x}{x_1}\right)^{2\mu\nu-1+2\nu}\phi_n(x_1)$$
$$-\phi_{x_1}(x)\left\{x^{-(-\mu\nu+\alpha+1-2\nu)}(x_1)^{2\mu\nu+1}\right\} \times D_{x_1} x_1^{-\mu\nu-\alpha}\phi_n(x_1) \tag{II.4}$$

where we set

$$\phi_{x_1}(x) = (2\mu\nu)^{-1}\left[x^{-2\mu\nu} - x_1{}^{-2\mu\nu}\right]$$

when $\mu \neq 0$ and $\phi_{x_1}(x) = \log\frac{x}{x_1}$ when $\mu = 0$.
The left-hand sides of (II.2), (II.3) and (II.4) converges uniformly on J as $n \to \infty$. Then, it follows from these expressions that $D\phi_n(x)$ and $D^2\phi_n(x)$ also converge, uniformly on J.
A simple inductive process shows that $D^k\phi_n(x)$ converges uniformly on J as $n \to \infty$, for each non-negative integer k. Thus there exists an infinitely differentiable function $\phi(x)$ defined on I such that $D^k\phi_n(x) \to D^k\phi(x)$ as $n \to \infty$. Finally, it is obvious that $\phi \in V_{\alpha,\beta,\nu,\mu}(I)$ and $\phi(x)$ is the limit of the sequence $\phi_n(x)$ in this space.
${V'}_{\alpha,\beta,\nu,\mu}(I)$ is the dual space of $V_{\alpha,\beta,\nu,\mu}(I)$. We assign to ${V'}_{\alpha,\beta,\nu,\mu}(I)$ the usual weak convergence. Therefore, ${V'}_{\alpha,\beta,\nu,\mu}(I)$ is a complete space. We now list some properties of these spaces:

(i) $D(I) \subset V_{\alpha,\beta,\nu,\mu}(I)$ and the topology of $D(I)$ is stronger than that induced of it by $V_{\alpha,\beta,\nu,\mu}(I)$. Hence, the restriction of any $f \in {V'}_{\alpha,\beta,\nu,\mu}(I)$ to $D(I)$ is in $D'(I)$. This can be inferred if we note that
$$\Delta^k_{\alpha,\beta,\nu,\mu,x}\phi(x) = \sum_{j=1}^{2k} a_j x^{j-2k} D^k\phi(x) \tag{II.5}$$
where the constants a_j depend only on $V_{\alpha,\beta,\nu,\mu}(I)$.

(ii) $V_{\alpha,\beta,\nu,\mu}(I) \subset E(I)$ and $E'(I)$ is a subspace of ${V'}_{\alpha,\beta,\nu,\mu}(I)$.

(iii) For each $f \in {V'}_{\alpha,\beta,\nu,\mu}(I)$ there exists a positive constant C and a nonnegative integer a such that [4]
$$|\langle f,\phi\rangle| \leq C \max_{0\leq k\leq\infty} \gamma_k^{\alpha,\beta,\nu,\mu}(\phi) \tag{II.6}$$
for every $\phi \in V_{\alpha,\beta,\nu,\mu}(I)$.

(iv) Let $f(x)$ be a function defined on I such that $\int_0^1 x^{2\nu-2\alpha-1}|f(x)|\,dx$ exists for $\theta \geq \mu + \frac{1}{2}$. Then $f(x)$ generates a regular generalized function in ${V'}_{\alpha,\beta,\nu,\mu}(I)$ by
$$|\langle f,\phi\rangle| = \int_0^1 f(x)\phi(x)\,dx, \ \ \phi \in V_{\alpha,\beta,\nu,\mu}(I). \tag{II.7}$$

(v) For each $m = 1, 2, \ldots$ and $\mu \geq -\frac{1}{2}$ the function $x^{2\nu-2\alpha-1}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x)$, $0 < x < 1$ is a member of $V_{\alpha,\beta,\nu,\mu}(I)$. To see this, from (1.3) it follows:
$$\gamma_k^{\theta,\alpha,\beta,\nu,\mu}\left[x^{(-\mu\nu+\alpha+1-2\nu)}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x)\right]$$
$$= \sup_{\mathrm{I}}\left|x^\theta(\lambda_n)^{2\nu k}(\beta\nu)^2\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m x)\right| < \infty \tag{II.8}$$
for each $k = 0, 1, 2, \ldots$.

III. The testing function spaces and $M_{a,\,\mu}$ and M_μ

Let $a \in I$ and $\mu \in R$. Define $M_{a,\,\mu}$ as the space of testing functions $\varphi(x)$ which are defined and smooth on $0 < x < 1$ for which
$$\gamma^{a,\,\mu\nu+\alpha-1/2}_{m,\,k}(\varphi)$$
$$= \sup_{x\in I}\left|e^{-ax}x^m\left(x^{1-2\nu}D\right)^k\left[x^{-\mu\nu-\alpha}\varphi(x)\right]\right|$$
for $m, k = 0, 1, 2, ...$
$$= \sup_{x\in I}\left|e^{-ax}x^{m-\mu\nu-\alpha}\left[A_k\right]\right|$$
where
$$A_k = a_{k0}\frac{\varphi(x)}{x^{(1-2\nu)k}} + a_{k1}\frac{\varphi^{(1)}(x)}{x^{(1-2\nu)k-1}} + \cdots + a_{kk}\frac{\varphi^{(k)}(x)}{x^k}$$
$$\leq M\sup_{x\in I}\left|a_{k0}\tfrac{\varphi(x)}{x^{(1-2\nu)k}}\right| + M\sup_{x\in I}\left|a_{k1}\tfrac{\varphi^{(1)}(x)}{x^{(1-2\nu)k-1}}\right| + + ... + M\sup_{x\in I}\left|a_{kk}\tfrac{\varphi^{(k)}(x)}{x^k}\right|$$
$< \infty$.
Thus
$$\gamma^{a,\,\mu\nu+\alpha-1/2}_{m,\,k}(\varphi) < \infty. \tag{III.1}$$

Assigned to $M_{a,\,\mu}$ the topology generated by the countable multinorm $\left\{\gamma^{a,\,\mu\nu+\alpha-1/2}_{m,\,k}\right\}$. $M_{a,\,\mu}$ is a Hausdorff space since $\gamma^{a,\,\mu\nu+\alpha-1/2}_{m,\,k}$ is a norm.
Let $\mu \geq -\frac{1}{2}$. For a fixed complex number y belonging to the strip
$$\Omega = \{y : |\mathrm{Im}\ y| < a,\ y \neq 0 \text{ or a negative number}\} \tag{III.2}$$
$$\frac{\partial^m}{\partial y^m}\left(\nu\beta y^{-1-2\alpha+2\nu}(xy)^\alpha J_\mu\left(\beta(xy)^\nu\right)\right) \in M_{a,\,\mu}. \tag{III.3}$$
Indeed, it is easily verified that (see [[5], [9])
$$\frac{\partial^m}{\partial y^m}\left(\nu\beta y^{-1-2\alpha+2\nu}(xy)^\alpha J_\mu\left(\beta(xy)^\nu\right)\right)$$
$$= \sum_{j=0}^{m} a_j(\mu)\, y^{-1-\alpha+2\nu}y^{j\nu-m}x^{\alpha+j\nu}J_{\mu-j}\left(\beta(xy)^\nu\right) \tag{III.4}$$
where $a_j(\mu)$ are constants depending on μ only.
Considering

$$\left(x^{1-2\nu}D\right)^{k}\left[x^{-\mu\nu-\alpha}(xy)^{\alpha+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$= \quad y^{\alpha+\mu\nu}\left(x^{1-2\nu}D\right)^{k}\left[(xy)^{-\mu\nu+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$= y^{\alpha+\mu\nu}\left(x^{1-2\nu}D\right)^{k-1}\left(x^{1-2\nu}\right)$$
$$\times D\left[(xy)^{-\mu\nu+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$= y^{\alpha+\mu\nu}\left(x^{1-2\nu}D\right)^{k-1}\left(x^{1-2\nu}\right)$$
$$\times\left[-\left(\nu\beta\right)(xy)^{-\mu\nu+j\nu+\nu}x^{-1}\mathscr{J}_{\mu-j+1}\left(\beta(xy)^{\nu}\right)\right]$$
$$= y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}D\right)^{k-2}\left(x^{1-2\nu}\right)(-1)\left(\nu\beta\right)$$
$$\times D\left[(xy)^{-\mu\nu+j\nu-\nu}\mathscr{J}_{\mu-j+1}\left(\beta(xy)^{\nu}\right)\right]$$
$$y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}D\right)^{k-2}\left(x^{1-2\nu}\right)$$
$$\times(-1)^{2}(\nu\beta)^{2}\left[(xy)^{-\mu\nu+j\nu}x^{-1}\mathscr{J}_{\mu-j+2}\left(\beta(xy)^{\nu}\right)\right]$$
$$= y^{\alpha+\mu\nu+2k\nu}(-1)^{k}(\nu\beta)^{k}$$
$$\times\left[(xy)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right]$$
$$= (-1)^{k}(\nu\beta)^{k}y^{\alpha+k\nu}\left[(x)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right]$$

Thus

$$\left(x^{1-2\nu}D\right)^{k}\left[x^{-\mu\nu-\alpha}(xy)^{\alpha+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$= (-1)^{k}(\nu\beta)^{k}y^{\alpha+k\nu}\left[(x)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right] \tag{III.5}$$

and

$$\left[(xy)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right]$$
$$\sim\frac{1}{2^{\mu\nu-j\nu+k\nu}\Gamma\left(\mu\nu-j\nu+k\nu+1\right)}$$

as $x\to 0^{+}$.

And

$$\left[(xy)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right]$$
$$= O\left[(xy)^{-(\mu\nu-j\nu+k\nu)-1/2}e^{x|\text{Im } y|}\right]$$

as $x\to\infty$.

Then it follows that

$$\gamma_{m,\,k\nu}^{a,\,\mu\nu}\left(y^{-1-\alpha+2\nu}x^{\alpha+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right)<\infty. \tag{III.6}$$

Therefore

$$\gamma_{m,\,k}^{a,\,\mu}\left[\frac{\partial^{m}}{\partial y^{m}}\left\{\nu\beta y^{-1-2\alpha+2\nu}(xy)^{\alpha}\mathscr{J}_{\mu}\left(\beta(xy)^{\nu}\right)\right\}\right]$$
$$\leq\sum_{j=0}^{m}\left|a_{j}\left(\mu\right)\right||y|^{\nu j-m}\gamma_{m,k}^{a,\mu}\left(y^{-1-\alpha+2\nu}x^{\alpha+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right) \tag{III.7}$$

$<\infty$ for a fixed $y\in\Omega$.

IV. Finite generalized Hankel transformation and continuous linear mapping

The differential operator $N_{\mu}\triangleq x^{(2\mu\nu+1)/2}Dx^{-\mu\nu-\alpha}$ is continuous from $M_{a,\,\mu}$ into $M_{a,\,\mu+1}$.

Theorem 4.1: The operation $\varphi\to N_{\mu}\varphi$ is a continuous linear mapping of $M_{a,\,\mu}$ into $M_{a,\,\mu+1}$.

Proof. For $\varphi\in M_{a,\,\mu}$,

$$\gamma_{m,\,k}^{a,\,\mu\nu+\alpha+1/2}\left(N_{\mu}\varphi\right)$$
$$=\sup_{x\in I}\left|e^{-ax}x^{m}\left(x^{1-2\nu}D\right)^{k}x^{-\mu\nu-2\nu+1/2}x^{\mu\nu+1/2}Dx^{-\mu\nu-\alpha}\varphi\left(x\right)\right|$$
$$=\sup_{x\in I}\left|e^{-ax}x^{m}\left(x^{1-2\nu}D\right)^{k}\left(x^{1-2\nu}D\right)x^{-\mu\nu-\alpha}\varphi\left(x\right)\right|$$
$$=\sup_{x\in I}\left|e^{-ax}x^{m}\left(x^{1-2\nu}D\right)^{k+1}x^{-\mu\nu-\alpha}\varphi\left(x\right)\right|.$$

$$\gamma_{m,\,k}^{a,\,\mu\nu+\alpha+1/2}\left(N_{\mu}\varphi\right)=\gamma_{m,\,k+1}^{a,\mu\nu+\alpha-1/2}\left(\varphi\right).$$

Indeed

$$\gamma_{m,\,k}^{a,\,\mu\nu+\alpha+1/2}\left(N_{\mu}\varphi\right)=\gamma_{m,\,k+1}^{a,\mu\nu+\alpha-1/2}\left(\varphi\right). \tag{IV.1}$$

Thus, $\varphi\in M_{a,\,\mu}$ implies that $N_{\mu}\varphi\in M_{a,\,\mu+1}$.

Also

$$N_{\mu}\left(\alpha_{1}\varphi_{1}+\alpha_{2}\varphi_{2}\right)$$
$$=x^{(2\mu\nu+1)/2}Dx^{-\mu\nu-\alpha}\left(\alpha_{1}\varphi_{1}+\alpha_{2}\varphi_{2}\right)$$
$$=\alpha_{1}x^{(2\mu\nu+1)/2}Dx^{-\mu\nu-\alpha}\varphi_{1}+\alpha_{2}x^{(2\mu\nu+1)/2}Dx^{-\mu\nu-\alpha}\varphi_{2}.$$

$$N_{\mu}\left(\alpha_{1}\varphi_{1}+\alpha_{2}\varphi_{2}\right)=\alpha_{1}N_{\mu}\varphi_{1}+\alpha_{2}N_{\mu}\varphi_{2}. \tag{IV.2}$$

Thus N_{μ} is linear. Assume $\{\varphi_{n}\}$ converges in the sense of $M_{a,\,\mu}$.

Then

$$\gamma_{m,\,k+1}^{a,\mu\nu+\alpha-1/2}\left(\varphi_{n}-\varphi_{0}\right)\to 0 \tag{IV.3}$$

as $n\to\infty$.

But

$$\gamma_{m,\,k+1}^{a,\mu\nu+\alpha-1/2}\left(\varphi_{n}-\varphi_{0}\right)$$
$$=\gamma_{m,\,k+1}^{a,\mu\nu+\alpha-1/2}\left(N_{\mu}\varphi_{n}-N_{\mu}\varphi\right)\to 0\left(\varphi\right) \tag{IV.4}$$

as $n\to\infty$. Thus N_{μ} is a continuous linear mapping of $M_{a,\,\mu}$ into $M_{a,\,\mu+1}$ as in [12].

Note: It is impossible for us to define N_{μ}^{-1} on $M_{a,\,\mu+1}$.

Theorem 4.2: The differential operator $\overline{M}_{\mu}\triangleq x^{-\mu\nu-\alpha}Dx^{(2\mu\nu+1)/2}$ is continuous from $M_{a,\,\mu+1}$ into $M_{a,\,\mu}$.

Proof. Indeed,

$$\gamma_{m,k}^{\mu\nu+2\alpha-1}\left(\overline{M}_{\mu}\varphi\right)$$
$$\leq t^{m}\int_{0}^{1}\left\|x^{-\mu\nu-\alpha}x^{-\mu\nu-\alpha}Dx^{(\mu\nu+1/2)}\varphi\left(x\right)\right\|dx$$
$$\leq\int_{0}^{1}\left\|x^{m}x^{-2\mu\nu-2\alpha}Dx^{(\mu\nu+1/2)}\varphi\left(x\right)\right\|dx$$
$$\leq\int_{0}^{1}\left\|x^{m}x^{-2\mu\nu-2\alpha}Dx^{(2\mu\nu+2\alpha+1)}x^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)\right\|dx$$
$$\leq\int_{0}^{1}\left\|x^{m}\left(2\mu\nu+2\alpha+1+2\nu k\right)x^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)\right\|dx$$
$$+\int_{0}^{1}\left\|x^{m+2\nu}x^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)\right\|dx$$
$$\leq\left(2\mu\nu+2\alpha+1+2\nu k\right)\gamma_{m,k}^{\mu\nu+2\alpha}+\gamma_{m+2\nu,1}^{\mu\nu+2\alpha}.$$

This implies

$$\gamma_{m,k}^{\mu\nu+2\alpha-1}\left(\overline{M}_{\mu}\varphi\right)\leq\left(2\mu\nu+2\alpha+1+2\nu k\right)\gamma_{m,k}^{a,\,\mu\nu+2\alpha}+\gamma_{m+2\nu,\,k+1}^{a,\,\mu\nu+2\alpha}.\quad\text{(IV.5)}$$

Thus $\overline{M}_{\mu}\varphi\in M_{a,\,\mu}$ for every $\varphi\in M_{a,\,\mu+1}$. But if $\{\varphi_n\}$ converges in the sense of $M_{a,\,\mu+1}$, then $\overline{M}_{\mu}\varphi_n\in M_{a,\,\mu}$ for every n and

$$\gamma_{m,k}^{a,\,\mu\nu+2\alpha-1}\left(\overline{M}_{\mu}\varphi_n-\overline{M}_{\mu}\varphi_0\right)\leq\left(2\mu\nu+2\alpha+1+2\nu k\right)\gamma_{m,k}^{a,\,\mu\nu+2\alpha}\left(\varphi_n-\varphi_0\right)$$
$$+\gamma_{m+2\nu,\,k+1}^{a,\,\mu\nu+2\alpha}\left(\varphi_n-\varphi_0\right).\quad\text{(IV.6)}$$

$\left\{\overline{M}_{\mu}\varphi_n\right\}$ converges in the sense of $M_{a,\,\mu}$ and $\overline{M}_{\mu}$ is continuous.

V. Some operational formulae

Lemma 1: For each real number μ, a function $\varphi\left(x\right)$ is in H_{μ}, k is a positive integer and $k\geq-\mu-\frac{1}{2}$. Then for every $\varphi\in H_{\mu}\left(A\right)$,

$$h_{\mu+1,k}\left(N_{\mu}\varphi\right)=-xh_{\mu,k}\left(\varphi\right).\quad\text{(V.1)}$$

Proof. By definition

$$x^{-k+\nu k+\nu-1-\mu\nu+1/2}N_{\mu\nu-(1-2\nu)(k-1)}...N_{\mu\nu-(1-2\nu)}$$
$$\times N_{\mu\nu-(1-2\nu)(k-k+1)}N_{\mu\nu-(1-2\nu+1)}\varphi\left(x\right)$$
$$=D^{k+1}x^{-\mu\nu-\alpha}\varphi\left(x\right).$$

$$\gamma_{m,k}^{a,\,\mu\nu+2\alpha-1}\left(\overline{M}_{\mu}\varphi\right)$$
$$=\sup_{x\in I}\left\|e^{-ax}x^{m}\left(x^{1-2\nu}D\right)^{k}\left[x^{-2\mu\nu-2\alpha}Dx^{(\mu\nu+1/2)}\varphi\left(x\right)\right]\right\|$$
$$=\sup_{x\in I}\left\|\psi(x)\right\|$$

where $\psi(x)$ is given by

$$=e^{-ax}x^{m}\left(x^{1-2\nu}D\right)^{k}$$
$$\times x^{-2\mu\nu-2\alpha}\left(2\mu\nu+2\alpha+1\right)x^{(2\mu\nu+2\alpha)}x^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)$$
$$+xDx^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)$$

which is given as

$$=\sup_{x\in I}\left\|\psi_1(x)\right\|$$

where $\psi_1(x)$ is given by

$$=e^{-ax}x^{m}\left(x^{1-2\nu}D\right)^{k}\left(2\mu\nu+2\alpha+1\right)x^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)$$
$$+\left(x^{2\nu}\right)x^{1-2\nu}Dx^{-\mu\nu-2\alpha-1/2}\varphi\left(x\right)$$
$$\leq\left(2\mu\nu+2\alpha+1+2\nu k\right)\gamma_{m,\,k}^{a,\,\mu\nu+2\alpha}+\gamma_{m+2\nu,\,k+1}^{a,\,\mu\nu+2\alpha}.$$

This implies

$$\gamma_{m,\,k}^{a,\,\mu\nu+2\alpha-1}\left(\overline{M}_{\mu}\varphi\right)$$
$$\leq\left(2\mu\nu+2\alpha+1+2\nu k\right)\gamma_{m,\,k}^{a,\,\mu\nu+2\alpha}+\gamma_{m+2\nu,\,k+1}^{a,\,\mu\nu+2\alpha}\quad\text{(V.2)}$$

Also noted that

$$\left(x^{1-2\nu}D\right)^{k}\left[x^{-\mu\nu-\alpha}(xy)^{\alpha+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu}\left(x^{1-2\nu}D\right)^{k}\left[(xy)^{-\mu\nu+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu}\left(x^{1-2\nu}D\right)^{k-1}\left(x^{1-2\nu}\right)$$
$$\times D\left[(xy)^{-\mu\nu+j\nu}\mathscr{J}_{\mu-j}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu}\left(x^{1-2\nu}D\right)^{k-1}\left(x^{1-2\nu}\right)$$
$$\times\left[-\left(\nu\beta\right)(xy)^{-\mu\nu+j\nu+\nu}x^{-1}\mathscr{J}_{\mu-j+1}\left(\beta(xy)^{\nu}\right)\right].$$
$$=y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}D\right)^{k-1}\left(x^{1-1}\right)\left(-1\right)\left(\nu\beta\right)$$
$$\times\left[(xy)^{-\mu\nu+j\nu-\nu}\mathscr{J}_{\mu-j+1}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}D\right)^{k-2}\left(x^{1-2\nu}D\right)\left(-1\right)\left(\nu\beta\right)$$
$$\times\left[(xy)^{-\mu\nu+j\nu-\nu}\mathscr{J}_{\mu-j+1}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}D\right)^{k-2}\left(x^{1-2\nu}\right)\left(-1\right)\left(\nu\beta\right)$$
$$\times D\left[(xy)^{-\mu\nu+j\nu-\nu}\mathscr{J}_{\mu-j+1}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}D\right)^{k-2}\left(x^{1-2\nu}\right)(-1)^{2}(\nu\beta)^{2}$$
$$\times\left[(xy)^{-\mu\nu+j\nu}x^{-1}\mathscr{J}_{\mu-j+2}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+2\nu}\left(x^{1-2\nu}\times D\right)^{k-3}\left(x^{1-2\nu}\right)(-1)^{2}(\nu\beta)^{2}$$
$$\times D\left[(xy)^{-\mu\nu+j\nu-2\nu}\mathscr{J}_{\mu-j+2}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+4\nu}\left(x^{1-2\nu}D\right)^{k-3}(-1)^{3}(\nu\beta)^{3}$$
$$\left[(xy)^{-\mu\nu+j\nu-3\nu}\mathscr{J}_{\mu-j+3}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+4\nu}\left(x^{1-2\nu}D\right)^{k-3}(-1)^{3}(\nu\beta)^{3}$$
$$\times\left[(xy)^{-\mu\nu+j\nu-3\nu}\mathscr{J}_{\mu-j+3}\left(\beta(xy)^{\nu}\right)\right]$$
$$=y^{\alpha+\mu\nu+2k\nu}(-1)^{k}(\nu\beta)^{k}\left[(xy)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right]$$
$$=(-1)^{k}(\nu\beta)^{k}y^{\alpha+k\nu}\left[(x)^{-\mu\nu+j\nu-k\nu}\mathscr{J}_{\mu-j+k}\left(\beta(xy)^{\nu}\right)\right].$$

Lemma 2: Let μ be a fixed real number and k be a positive integer $\geq-\frac{1}{2}$. Then for every $\varphi\in H_{\mu+1}\left(A\right)$, then

$$h_{\mu,k}\left(M_{\mu}\varphi\right)=x^{\nu}h_{\mu+1,k}\left(\varphi\right).\quad\text{(V.3)}$$

Proof. Using

$$N_{\mu\nu-(1-2\nu)(k-1)}...N_{\mu\nu-(1-2\nu)}N_{\mu\nu+2\nu}\varphi\left(x\right)$$

$$= x^{\mu\nu+\nu k+\nu-1/2}\left(x^{1-2\nu}D\right)^{k+1}x^{-\mu\nu-\alpha}\varphi(x).$$

(V.4)

$$h_{\mu,k}(M_\mu\varphi)$$
$$=(-1)^k(\nu\beta)^{2k}x^{2\nu-2\alpha-1-k\nu}$$
$$\int_0^1(2\mu\nu+2\nu k+2\nu)\,y^{2\mu\nu+2\nu k+2\nu-1}(xy)^\alpha J_{\mu+k}$$
$$\times(\beta(xy)^\nu)\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\,dy$$
$$+(-1)^k(\nu\beta)^{2k}x^{2\nu-2\alpha-1-k\nu}$$
$$\int_0^1\nu\beta y^{2\mu\nu+2\nu k+2\nu+1}(xy)^\alpha J_{\mu+k}$$
$$\times(\beta(xy)^\nu)\left(y^{1-2\nu}D\right)^{k+1}y^{-\mu\nu-\alpha-1}\varphi(y)\,dy.$$

Thus follows:

$$x^\nu h_{\mu+1,k}(\varphi)$$
$$=(-1)^k(\nu\beta)^k x^{\nu-k\nu}\int_0^1(\nu\beta)\,x^\nu y^{-\mu\nu-k\nu-1+\nu}y^{2\mu\nu+2\nu k+2\nu}$$
$$\times J_{\mu+k+1}(\beta(xy)^\nu)\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\,dy.$$

Also can be written as analogously in [13]

$$x^\nu h_{\mu+1,k}(\varphi)$$
$$=(-1)^k(\nu\beta)^k x^{-k\nu+\nu}$$
$$\times\int_0^1\left[y^{2\mu\nu+2\nu k+2\nu}\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\right]$$
$$\times(\nu\beta)\,x^{-1}(xy)^{-\mu\nu-k\nu+\nu}J_{\mu+k+1}(\beta(xy)^\nu)\,dy.$$

Note that

$$J_{\mu+k+1}(\beta(xy)^\nu)$$
$$=-(\nu\beta)^{-1}x^{\mu\nu+k\nu-\nu+1}y^{\mu\nu+k\nu-\nu}$$
$$\times D\left[y^{-\mu\nu-k\nu}\mathscr{J}_{\mu+k}(\beta(xy)^\nu)\right] \quad \text{(V.5)}$$

Thus

$$x^\nu h_{\mu+1,k}(\varphi)$$
$$=(-1)^{k+1}(\nu\beta)^k x^{\nu-k\nu}$$
$$\times\int_0^1\left[y^{2\mu\nu+2\nu k+2\nu}\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\right]$$
$$\times D\left[y^{-\mu\nu-k\nu}\mathscr{J}_{\mu+k}(\beta(xy)^\nu)\right]dy.$$
$$=(-1)^{k+1}(\nu\beta)^k x^{\nu-k\nu}y^{-\mu\nu-k\nu}y^{2\mu\nu+2\nu k+2\nu}\mathscr{J}_{\mu+k}(\beta(xy)^\nu)$$
$$\times\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\,|_0^1$$

$$-(-1)^{k+1}(\nu\beta)^k x^{\nu-k\nu}\int_0^1\left[y^{-\mu\nu-k\nu}\mathscr{J}_{\mu+k}(\beta(xy)^\nu)\right]$$
$$\times D\left[y^{2\mu\nu+2\nu k+2\nu}\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\right]dy.$$

Also

$$D\left[y^{2\mu\nu+2\nu k+2\nu}\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)\right]$$
$$=(2\mu\nu+2\nu k+2\nu)\,y^{2\mu\nu+2\nu k+2\nu-1}\left(y^{1-2\nu}D\right)^k y^{-\mu\nu-\alpha-1}\varphi(y)$$
$$+y^{2\mu\nu+2\nu k+2\nu+1}\left(y^{1-2\nu}D\right)^{k+1}y^{-\mu\nu-\alpha-1}\varphi(y).$$

(V.6)

Thus completes the proof.

VI. The finite generalized Hankel-Clifford transformation: continuous mapping in $S_{\alpha,\beta,\nu,\mu}$ and $L_{\alpha,\beta,\nu,\mu}$

In this section, a space onto $S_{\alpha,\beta,\nu,\mu}$ of functions and a space $L_{\alpha,\beta,\nu,\mu}$ of complex sequences is introduced. The finite generalized Hankel-Clifford transform $\hbar^*_{\alpha,\beta,\nu,\mu}$ on them is investigated. $S_{\alpha,\beta,\nu,\mu}$ is defined as the space of all complex valued functions $\phi(x)$ on $(0,1]$ such that $\phi(x)$ is infinitely differentiable and satisfies for every $k\in N$.

(i) ${\Delta^*}^k_{\alpha,\beta,\nu,\mu}\phi(1)=0$,

(ii) $x^{-\mu\nu+\alpha+1-2\nu}{\Delta^*}^k_{\alpha,\beta,\nu,\mu}\phi(x)\to0$ and $x^{\mu\nu-1-\alpha+2\nu}\frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}{\Delta^*}^k_{\alpha,\beta,\nu,\mu}\phi(x)\right)\to0$ as $x\to0+$ and

(iii) $x^{(2\nu+\mu\nu-\alpha-1)/2}{\Delta^*}^k_{\alpha,\beta}\phi(x)\in L(0,1)$.

$S_{\alpha,\beta,\nu,\mu}$ is endowed with the topology generated by the family of seminorms $\{\|\|_k\}_{k=0}^\infty$ and also is a Hausdorff topological linear space that verifies the first countability axiom, where $\|\phi\|_k=\int_0^1 x^{-\alpha/2}\left|{\Delta^*}^k_{\alpha,\beta,\nu,\mu}\phi(x)\right|dx$ for $\phi\in S_{\alpha,\beta,\nu,\mu}$ and $k\in N$. Moreover, the operator ${\Delta^*}^k_{\alpha,\beta,\nu,\mu}$ defines a continuous mapping from $S_{\alpha,\beta,\nu,\mu}$ into itself. $S'_{\alpha,\beta,\nu,\mu}$ is the dual space of $S_{\alpha,\beta,\nu,\mu}$ and it is equipped with the usual weak topology.

Proposition 6.1: If $f(x)$ is a function defined on $(0,1)$ such that $x^{(2\nu-2\alpha-1)/2}f(x)$ is bounded on $(0,1)$ then $f(x)$ generates a member of $S'_{\alpha,\beta,\nu,\mu}$ through the definition is given by

$$\langle f(x),\phi(x)\rangle=\int_0^1 f(x),\phi(x)\,dx\,,\ \phi\in S_{\alpha,\beta,\nu,\mu}. \quad \text{(VI.1)}$$

Proof. The result easily follows from the inequality

$$|\langle f(x),\phi(x)\rangle|\le\|\phi\|_0\sup_{0<x<1}\left|x^{(2\nu-2\alpha-1)/2}f(x)\right| \quad \text{(VI.2)}$$

for $\phi\in S_{\alpha,\beta,\nu,\mu}$.

The spaces $V_{\theta,\alpha,\beta}(I)$ defined by Malgonde [12] analogously defined as $V_{\theta,\alpha,\beta,\nu,\mu}(I)$ related to $S_{\alpha,\beta,\nu,\mu}$ as follows:

Proposition 6.2: Let $\mu\ge-\frac{1}{2}$ and $\theta>\frac{1}{4}-\frac{(2\nu-2\alpha-1)}{2}$. Then $S_{\alpha,\beta,\nu,\mu}\subset V_{\theta,\alpha,\beta,\nu,\mu}(I)$ and the topology of $S_{\alpha,\beta,\nu,\mu}$ is stronger than that induced on it by $V_{\theta,\alpha,\beta,\nu,\mu}(I)$.

Proof. Let $\phi\in S_{\alpha,\beta,\nu,\mu}$. In virtue of the conditions (i) and (ii), we can write

$$x^{\theta-(2\nu+\mu\nu-\alpha-1)}{\Delta^*}^k_{\alpha,\beta}\phi(x)$$
$$=x^{\theta+\mu}\int_1^x t^{-2\mu-(2\nu+\mu\nu-\alpha-1)}\int_0^t u^\mu{\Delta^*}^{k+1}_{\alpha,\beta,\nu,\mu}\phi(u)\,du\,dt \quad \text{(VI.3)}$$

for every $x\in(0,1)$ and $k\in N$.

$$\left|x^{\theta-(2\nu+\mu\nu-\alpha-1)}{\Delta^*}^k_{\alpha,\beta,\nu,\mu}\phi(x)\right|$$
$$\le x^{\theta-\frac{(2\nu+\mu\nu-\alpha-1)}{2}}\int_0^1 u^{-(2\nu+\mu\nu-\alpha-1)/2}\left|{\Delta^*}^{k+1}_{\alpha,\beta,\nu,\mu}\phi(u)\right|du$$

$$\leq \int_0^1 u^{-(2\nu+\mu\nu-\alpha-1)/2} \left| {\Delta^*}_{\alpha,\beta,\nu,\mu}^{k+1} \phi(u) \right| du \tag{VI.4}$$

for every $x \in (0,1)$ and $k \in N$.
Hence, for every $\phi \in S_{\alpha,\beta,\nu,\mu}$ and $k \in N$,

$$\sup_{0<x<1} \left| x^{\theta-(2\nu+\mu\nu-\alpha-1)} {\Delta^*}_{\alpha,\beta,\nu,\mu}^{k} \phi(x) \right| \leq \|\phi\|_{k+1} \tag{VI.5}$$

and $S_{\alpha,\beta,\nu,\mu}$ is contained in $V_{\theta,\alpha,\beta}(I)$ and the inclusion is continuous.
From Proposition 6.2, if $f \in V'_{\theta,\alpha,\beta}(I)$, then the restriction of f to $S_{\alpha,\beta,\nu,\mu}$ is a member of $S'_{\alpha,\beta,\nu,\mu}$.
Define $L_{\alpha,\beta,\nu,\mu}$ as the space of all complex sequences $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} a_n \lambda_n^{2k} = 0$, for every $k \in N$, where $\lambda_n, n = 0,1,2,\cdots$ represent the positive roots of the equation $\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x) = 0$, arranged in ascending order of magnitude. The topology of $L_{\alpha,\beta,\nu,\mu}$ is that generated by the family of norms $\left\{\gamma_{\alpha,\beta,\nu,\mu}^k\right\}_{k=0}^{\infty}$, where

$$\gamma_{\alpha,\beta,\nu,\mu}^k \left((a_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} |a_n| \lambda_n^{2k} \tag{VI.6}$$

for every $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta,\nu,\mu}$ and $k \in N$.
Thus $L_{\alpha,\beta,\nu,\mu}$ is a Hausdorff topological linear space that satisfies the first countability axiom. $L'_{\alpha,\beta,\nu,\mu}$ denotes the dual space of $L_{\alpha,\beta,\nu,\mu}$ and it is endowed with the weak topology.
Proposition 6.3: Let $(b_n)_{n=0}^{\infty}$ be a complex sequence such that $|b_n| \leq M\lambda_n^l$ for every $n \in N$ and for some $l \in N$ and $M > 0$. Then the linear operator $(a_n)_{n=0}^{\infty} \to (a_n b_n)_{n=0}^{\infty}$ is a continuous mapping from $L_{\alpha,\beta,\nu,\mu}$ into itself. Moreover, the operator in $L'_{\alpha,\beta,\nu,\mu}, B \to (b_n)_{n=0}^{\infty} B$ where

$$\left\langle (b_n)_{n=0}^{\infty} B, (a_n)_{n=0}^{\infty} \right\rangle = \left\langle B, (a_n b_n)_{n=0}^{\infty} \right\rangle, \tag{VI.7}$$

for $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta,\nu,\mu}$. is a continuous mapping from $L'_{\alpha,\beta,\nu,\mu}$ into itself.
Proof. It is sufficient to see that
$\gamma_{\alpha,\beta,\nu,\mu}^k \left((a_n)_{n=0}^{\infty}\right)$

$$\leq M \sum_{n=0}^{\infty} |a_n| \lambda_n^{2k+l} \leq M_1 \, \gamma_{\alpha,\beta,\nu,\mu}^{k+l} \left((a_n)_{n=0}^{\infty}\right) \tag{VI.8}$$

for $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta,\nu,\mu}$ and $k \in N$. M_1 being a suitable positive constant. Thus $(b_n)_{n=0}^{\infty}$ generates a member of $L'_{\alpha,\beta,\nu,\mu}$ by

$$\left\langle (b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty} \right\rangle = \sum_{n=0}^{\infty} a_n b_n \tag{VI.9}$$

for $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta,\nu,\mu}$.
The fundamental theorem in the theory of a finite generalized Hankel transformation asserts that the conventional finite generalized Hankel transformation $\hbar^*_{\alpha,\beta,\nu,\mu}$ is an isomorphism from $S_{\alpha,\beta,\nu,\mu}$ onto $L_{\alpha,\beta,\nu,\mu}$. The proof of this fact is the next object:

Theorem 6.1: For $\mu \geq -\frac{1}{2}$, the finite generalized Hankel-Clifford transformation $\hbar^*_{\alpha,\beta,\nu,\mu}$ is an isomorphism from $S_{\alpha,\beta,\nu,\mu}$ onto $L_{\alpha,\beta,\nu,\mu}$.

Proof. Let $\phi \in S_{\alpha,\beta,\nu,\mu}$. As it is known, $\hbar^*_{\alpha,\beta,\nu,\mu}\phi = (a_n)_{n=0}^{\infty}$

$$a_n = \frac{2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)} \int_0^1 \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)\phi(x)dx, \tag{VI.10}$$

for $n \in N$.
In virtue of the operational rule, for every $n \in N$ it is obtained as:
$\Delta_{\alpha,\beta,\nu,\mu} = x^{2-2\nu}D^2 + (1-2\alpha)x^{1-2\nu}D - x^{-2\nu}\left\{(\mu\nu)^2 - \alpha^2\right\}$
is not self adjoint.
Together with $\Delta_{\alpha,\beta,\nu,\mu}$, the operator $\Delta^*_{\alpha,\beta,\nu,\mu}$ is defined as

$$\Delta^*_{\alpha,\beta,\nu,\mu} = x^{-\alpha-\nu\mu} D x^{2\mu\nu+1} D x^{-\mu\nu+\alpha+1-2\nu}. \tag{VI.11}$$

Therefore

$$\Delta^*_{\alpha,\beta,\nu,\mu} = x^{2-2\nu}D^2 + (4\nu-2\alpha-3)x^{1-2\nu}D - x^{-2\nu}\left\{(\mu\nu)^2 - (\alpha+1-2\nu)^2\right\} \tag{VI.12}$$

$$\begin{aligned}
\lambda_n^2 a_n &= \frac{\lambda_n^2 2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)} \int_0^1 \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)\phi(x)dx \\
&= \frac{\lambda_n 2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)} \\
&\int_0^1 \frac{d}{dx}\left(x^{-(-\mu\nu+\alpha+1-2\nu)} \mathscr{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n x)\right) \\
&\quad \times x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\,dx \\
&= \frac{\lambda_n 2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)} \left\{ \mathscr{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n x)\, \phi(x)\big|_0^1 \right\} \\
&- \frac{\lambda_n 2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)} \int_0^1 x^{-(-\mu\nu+\alpha+1-2\nu)} \mathscr{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n x) \\
&\quad \times \frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\right)dx
\end{aligned}$$

for $n \in N$.
Moreover, according to (VI.4) since $\mathscr{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n x)\,\phi(x)\big|_0^1 = 0$ and $\phi(1) = 0$ and

$$\lim_{x\to 0+} x^{-\mu\nu+\alpha+1-2\nu}\phi(x) = 0. \tag{VI.13}$$

Hence
$\lambda_n^2 a_n$

$$= -\frac{\lambda_n 2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)} \int_0^1 x^{-(-\mu\nu+\alpha+1-2\nu)} \mathscr{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n x) \times \frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\right)dx. \tag{VI.14}$$

Now, by invoking (VI.13), one has

$$\begin{aligned}
&= -\int_0^1 \tfrac{d}{dx}\left(x^{-\alpha-\nu\mu+2\nu-2} \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)\right) x^{2\mu\nu+1} \\
&\times \tfrac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\right)dx \\
&= -\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x) x^{-\alpha-\nu\mu+2\nu-2} x^{2\mu\nu+1} \tfrac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\right)\Big|_0^1 \\
&+ \int_0^1 \Delta^*_{\alpha,\beta,\nu,\mu}\phi(x)\, \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)dx.
\end{aligned}$$

Also in this case by (VI.13), the limit terms are equal to zero because
$\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n)=0, \phi\in C^{\infty}((0,1])$

$$\lim_{x\to 0+} x^{-\alpha+\nu\mu+2\nu-1}\frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\right)=0. \tag{VI.15}$$

Therefore
$$\lambda_n\int_0^1 x^{-(-\mu\nu+\alpha+1-2\nu)}\mathscr{J}_{\alpha,\beta,\nu,\mu+1}(\lambda_n x) \times\frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}\phi(x)\right)dx$$
$$=\int_0^1 \Delta^{*}_{\alpha,\beta,\nu,\mu}\phi(x)\,\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)dx. \tag{VI.16}$$

By combining (VI.15) and (VI.16), it is obtained as
$$\lambda_n^2 a_n = -\frac{2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)}\int_0^1 \Delta^{*}_{\alpha,\beta,\nu,\mu}\phi(x)\,\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)dx \tag{VI.17}$$
for every $n\in N$.

An inductive procedure allows to establish that
$$\lambda_n^{2k}a_n = (-1)^k\frac{2\nu}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)}\int_0^1 {\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x)\,\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)dx \tag{VI.18}$$
for every $n,k\in N$.

From (VI.18), according to Riemann-Lebesgue Lemma [13], one follows to
$$\lambda_n^{2k}a_n\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)\to 0 \tag{VI.19}$$
as $n\to\infty$.

Moreover by (VI.17), there exists a positive constant M such that
$$\lambda_n^{2k}|a_n|\le M\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)\lambda_n^{2k+1}|a_n| \tag{VI.20}$$
and then $\lambda_n^{2k}a_n\to 0$ as $n\to\infty$ for every $k\in N$.

Also, for certain $M_i>0, i=1,2$. $\sum\limits_{n=0}^{\infty}\lambda_n^{2k}|a_n|$
$$=\sum_{n=0}^{\infty}\frac{1}{\mathscr{J}^2_{\alpha,\beta,\nu,\mu+1}(\lambda_n)\lambda_n^{4(2\nu+\mu\nu-\alpha-1)}} \times\left|\int_0^1 {\Delta^{*}}^{k+2}_{\alpha,\beta,\nu,\mu}\phi(x)\,\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)dx\right|$$
$$\le M_1\sum_{n=0}^{\infty}\lambda_n^{-5(2\nu+\mu\nu-\alpha-1)/2} \times\int_0^1\left|(\lambda_n x)^{(2\nu+\mu\nu-\alpha-1)/2}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)\right| \times\left|x^{-(2\nu+\mu\nu-\alpha-1)/2}{\Delta^{*}}^{k+2}_{\alpha,\beta,\nu,\mu}\phi(x)\right|dx$$
$$\le M_2\sum_{n=0}^{\infty}\lambda_n^{-2(2\nu+\mu\nu-\alpha-1)} \times\int_0^1 x^{-(2\nu+\mu\nu-\alpha-1)/2}\left|{\Delta^{*}}^{k+2}_{\alpha,\beta,\nu,\mu}\phi(x)\right|dx.$$

$$\sum_{n=0}^{\infty}\lambda_n^{2k}|a_n| \le M_2\sum_{n=0}^{\infty}\lambda_n^{-2(2\nu+\mu\nu-\alpha-1)}\int_0^1 x^{-(2\nu+\mu\nu-\alpha-1)/2}\left|{\Delta^{*}}^{k+2}_{\alpha,\beta,\nu,\mu}\phi(x)\right|dx. \tag{VI.21}$$

Hence, since $\sum\limits_{n=0}^{\infty}\lambda_n^{-2(2\nu+\mu\nu-\alpha-1)}<\infty$, the relation follows as:
$$\gamma^k_{\alpha,\beta,\nu,\mu}\left((a_n)_{n=0}^{\infty}\right)\le M_3\|\phi\|_{k+2} \tag{VI.22}$$
for every $k\in N$ and $\phi\in S_{\alpha,\beta,\nu,\mu}$ and for some $M_3>0$. This inequality proves that the linear mapping $\hbar^{*}_{\alpha,\beta,\nu,\mu}$ is continuous from $S_{\alpha,\beta,\nu,\mu}$ into $L_{\alpha,\beta,\nu,\mu}$.

Let now $(a_n)_{n=0}^{\infty}\in L_{\alpha,\beta,\nu,\mu}$ and define
$$\tau_{\alpha,\beta,\nu,\mu}\left((a_n)_{n=0}^{\infty}\right)(x)=\phi(x)=\sum_{n=0}^{\infty}a_n x^{\alpha}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)$$
for $x\in(0,1]$.

By (VI.13) and (VI.14), given as:
$$\sum_{n=0}^{\infty}\left|a_n x^{(2\nu+\mu\nu-\alpha-1)}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x)\right| \le Mx^{(2\nu+\mu\nu-\alpha-1)/2}\sum_{n=0}^{\infty}|a_n|, x>0 \tag{VI.23}$$
for a suitable $M>0$. Therefore $\phi\in C(0,\infty)$. In a similar way $\phi\in C^{\infty}(0,\infty)$ can be proved as in [6].

From (VI.18), it follows:
$${\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x) =\sum_{n=0}^{\infty}(-1)^k\lambda_n^{2k}a_n x^{(2\nu+\mu\nu-\alpha-1)}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n x) \tag{VI.24}$$
for $x>0$ and $k\in N$. Then ${\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(1)=0$ for each $k\in N$.

Also
$$\left|x^{(2\nu+\mu\nu-\alpha-1)}{\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x)\right| \le M_1\,x^{(2\nu+\mu\nu-\alpha-1)/2}\sum_{n=0}^{\infty}|a_n|\lambda_n^{2k}. \tag{VI.25}$$

And from (VI.13), (VI.14) and (VI.15) it follows as:
$$\left|x^{2\nu+\mu\nu-\alpha-1}\frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}{\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x)\right)\right| \le M_2\,x^{2\nu+\mu\nu-\alpha-1+1}\sum_{n=0}^{\infty}|a_n|\lambda_n^{2k-(-\mu\nu+\alpha+1-2\nu)+1}. \tag{VI.26}$$

Hence
$$\lim_{x\to 0+}x^{-\mu\nu+\alpha+1-2\nu}{\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x) =\lim_{x\to 0+}x^{2\nu+\mu\nu-\alpha-1}\frac{d}{dx}\left(x^{-\mu\nu+\alpha+1-2\nu}{\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x)\right)=0. \tag{VI.27}$$

On the other hand, since the series defining ${\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x)$ is uniformly convergent in $x\in(0,1)$, there exists a positive constant M_3 such that
$$\int_0^1 x^{(2\nu+\mu\nu-\alpha-1)/2}\left|{\Delta^{*}}^{k}_{\alpha,\beta,\nu,\mu}\phi(x)\right|dx\le M_3\sum_{n=0}^{\infty}\lambda_n^{2k}|a_n| \tag{VI.28}$$
for each $k\in N$. Therefore $\tau_{\alpha,\beta,\nu,\mu}$ is a continuous mapping from $L_{\alpha,\beta,\nu,\mu}$ into $S_{\alpha,\beta,\nu,\mu}$.

VII. Applications

Recently in [14], a fingerprint recognition method based on Gabor wavelet transform and discrete cosine transform (DCT) was proposed useful development in computer engineering. In this section, applications to Mathematical Physics have been demonstrated with wider use in Mechanical Engineering field.

(A) Dirichlet problem in cylindrical coordinates. The generalized finite Hankel-Type Integral Transformation of the first type when applied with generalized Dirichlet's problem:

Find the conventional solution $v(r, z)$ of the equation

$$r^{2-2\nu}\frac{\partial^2 v}{\partial r^2}+(1-2\alpha)r^{1-2\nu}\frac{\partial v}{\partial r}-r^{-2\nu}\left\{(\mu\nu)^2-\alpha^2\right\}v+\frac{\partial^2 v}{\partial z^2} = 0, \quad \text{(VII.1)}$$

$0 < r < 1, 0 < z < \infty$ satisfying boundary conditions:

i) As $z \to \infty+, v(r, z)$ converges to zero in the sense of $D'(I)$
ii) As $z \to 0+, v(r, z)$ converges in the sense of $D'(I)$ to $f(r) \in V'_{\alpha,\beta,\nu,\mu}(I)$
iii) As $r \to 1-, v(r, z)$ converges to 0 on $c \le z < \infty$ for each $c < 0$
iv) As $r \to 0+, v(r, z) = O(1)$ on $c \le z < \infty$

Let us denote $V(n, z) = \hbar_{\alpha,\beta,\nu,\mu}(v(r, z))$.
According to (VII.1) becomes

$$\Delta_{\alpha,\beta,\nu,\mu}v + \frac{\partial^2 v}{\partial z^2} = 0. \quad \text{(VII.2)}$$

By applying $\hbar_{\alpha,\beta,\nu,\mu}$ to (VII.2)

$$\hbar_{\alpha,\beta,\nu,\mu}\left(\Delta^k_{\alpha,\beta,\nu,\mu}\right) = (-1)^k(\nu\beta{\lambda_n}^{\nu})^{2k}\left(\hbar_{\alpha,\beta,\nu,\mu}\right)[f(x)].$$

Equation (VII.2) becomes

$$(-1)\left(\nu\beta\lambda_n^{\nu}\right)^2\left(\hbar_{\alpha,\beta,\nu,\mu}\right)[v(r, z)] + \frac{\partial^2 \hbar_{\alpha,\beta,\nu,\mu}[v(r, z)]}{\partial z^2} = 0.$$

$$(-1)\left(\nu\beta\lambda_n^{\nu}\right)^2 V(n, z) + \frac{\partial^2}{\partial z^2}V(n, z) = 0. \quad \text{(VII.3)}$$

Considering $\left(\hbar^*_{\alpha,\beta,\nu,\mu}f\right) = F^*_{\alpha,\beta,\nu,\mu}(n)$ the solution becomes

$$V(n, z) = F_{\alpha,\beta,\nu,\mu}(n)e^{-\nu\beta z\lambda_n}$$

because of the boundary conditions (i) and (ii).
Also the inversion formulae can be obtained

$$v(r, z) = (\hbar^{-1}_{\alpha,\beta,\nu,\mu}F_{\alpha,\beta,\nu,\mu})(n) = f(r)$$

$$= \sum_{n=1}^{\infty}\frac{2\nu F(n)e^{-\nu\beta\lambda_n z}\mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_n r)}{(\lambda_n)^2\,{\mathscr{J}^2}_{\alpha,\beta,\nu,\mu}(\lambda_n)} \quad \text{(VII.4)}$$

Recall that $F(m) = O\left(\lambda_m^{2s-\frac{(-\mu\nu+\alpha-2\nu)}{2}+\frac{1}{2}}\right)$as $m \to \infty$ for some nonnegative integer 's'. On the other hand, $\lambda_m \approx \left(m + \frac{(-\mu\nu+\alpha-2\nu)}{4} + \frac{1}{4}\right)$ is uniformly bounded on $0 < r < 1$ (for each $m = 1, 2, \dots$) and $e^{-\nu\beta z\lambda_m} = O(e^{-c\pi m})$ uniformly on $c \le z < \infty$. These facts imply that (VII.4) and the series obtained by applying the operators $\Delta_{\alpha,\beta,\nu,\mu,r}$ and $\frac{\partial^2}{\partial z^2}$ under the summation sign in (VII.4) converge uniformly on $0 < r < 1, 0 < z < \infty$.

Thus applying

$$\Delta_{\alpha,\beta,\nu,\mu}v + \frac{\partial^2 v}{\partial z^2} = 0 \quad \text{(VII.5)}$$

term by term, we conclude that (VII.4) is a solution of the differential equation (VII.1).

The uniform convergence of (VII.4) allows to take limits when $r \to 0+$ and $r \to 1-$ under the summation sign. Therefore, the boundary conditions (iii) and (iv) are directly verified.

Finally, note that (VII.4) defines a continuous function on $0 < r < 1, 0 < z < \infty$.

Consequently, it generates a regular member in $D'(I)$. This last result, the uniform convergence of the series and the inversion theorem ensure the fulfillment of the boundary conditions (i) and (ii).

Remarks: When $\mu = 0$, (VII.5) is the Laplace's equation and the problem described here coincides exactly with the problem investigated by Zemanian in [4], although it is considered in the space $A'$and not in our space $V'_{\alpha,\beta,\nu,\mu}(I)$.

Moreover, note that the use of the generalized finite Hankel-Type Integral Transformation makes unnecessary the change of variables $u(r, z) = \sqrt{r}\,v(r, z)$, which must be done in [8] previous to the employment of the finite Hankel transformations. On the contrary, the transformation $[\hbar_{\alpha,\beta,\nu,\mu}$ allows to obtain directly the solution $v(r, z)$ to (VII.4) in agreement with solution given by Zemanian [4], in view of that $\mathscr{J}_{\alpha,\beta,\nu,0}(z) = j_0(z)$ and $v(r, z) = r^{-\frac{1}{2}}u(r, z)$.

(B) Find a function $u(r, t)$ on the domain $\{(r, t) : 0 < r < 1, t > 0\}$ that satisfies the differential equation:

$$r^{2-2\nu}\frac{\partial^2 u}{\partial r^2}+(1-2\alpha)r^{1-2\nu}\frac{\partial u}{\partial r}-r^{-2\nu}\left\{(\mu\nu)^2-\alpha^2\right\}u+\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{(VII.6)}$$

$\mu \ge 0$ satisfying boundary conditions:

v) As $t \to \infty+, u(r, t)$ converges to zero in the sense of $D'(I)$
vi) As $r \to 0+, u(r, t)$ converges in the sense of $D'(I)$ to $g(r) \in V'_{\alpha,\beta,\nu,\mu}(I)$
vii) As $r \to 1-, u(r, t)$ converges to zero on $c \le t < \infty$ for each $c < 0$
viii) As $r \to 0+, u(r, t) = O(1)$ on $c \le t < \infty$.

Let us denote $U(n, t) = \hbar_{\alpha,\beta,\nu,\mu}(u(r, t))$.
According to (VII.2), (VII.6) becomes

$$\Delta_{\alpha,\beta,\nu,\mu}u + \frac{\partial^2 u}{\partial t^2} = 0.$$

By applying $\hbar_{\alpha,\beta,\nu,\mu}$ to (VII.3)

$$\hbar_{\alpha,\beta,\nu,\mu}\left(\Delta^k_{\alpha,\beta,\nu,\mu}\right) = (-1)^k(\nu\beta{\lambda_n}^{\nu})^{2k}\left(\hbar_{\alpha,\beta,\nu,\mu}\right)[f(x)].$$

Equation (VII.6) becomes

$$(-1)\left(\nu\beta\lambda_n^{\nu}\right)^2\left(\hbar_{\alpha,\beta,\nu,\mu}\right)[u(r, t)] + \frac{\partial^2 \hbar_{\alpha,\beta,\nu,\mu}[u(r, t)]}{\partial t^2} = 0.$$

$$(-1)\left(\nu\beta\lambda_n^{\nu}\right)^2 U(n, t) + \frac{\partial^2}{\partial t^2}U(n, t) = 0.$$

The solution becomes:

$$U(n, t) = F_{\alpha,\beta,\nu,\mu}(n)e^{-\nu\beta t\lambda_n} \quad \text{(VII.7)}$$

because of the boundary conditions (i) and (ii).
Also the inversion formulae can be obtained

$$u(r,t) = \sum_{n=1}^{\infty} \frac{2\nu G(m) e^{-\nu\beta\lambda_m t} \mathscr{J}_{\alpha,\beta,\nu,\mu}(\lambda_m r)}{(\lambda_m)^{2\nu} \mathscr{J}^2{}_{\alpha,\beta,\nu,\mu}(\lambda_m)}. \quad \text{(VII.8)}$$

It can be proved that (VII.8) truly is a solution to problem (VII.6) in the same way as in the first example.
This section is concluded with the note that many other problems in Mathematical Physics have the same form. Given a partial differential equation involving the n-dimensional Laplacian operator

$$\Delta_{\alpha,\beta,\nu,\mu} u = D^2_{x_1} u + D^2_{x_2} u + ... + D^2_{x_n} u,$$

finding solutions depending only on $r = (x_1^2 + x_2^2 + x_3^2 + ... + x_{n-1}^2)^{\frac{1}{2}}$ and $z = x_n$ it follows that $u(r,z)$ must satisfy equations analogous to (VII.4) and (VII.8) with $\mu = (n-3)/2; n \geq 3$ [6]. The finite Hankel-type integral transformation provides an elegant and straightforward method to solve both equations for any value of $\mu \geq 0$ (i.e., for each $n \geq 3$).

Compliance with Ethical Standards

Disclosure of potential conflicts of interest: I declare as an author of this study that there is no conflict of interest from whomsoever concerned.
Research involving Human Participants and/or Animals: This article does not contain any studies with human participants or animals performed by the author.

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