Brunn-Minkowski Inequalities for the L_p and L_p Radial Blaschke-Minkowski Homomorphisms

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***Abstract*—The Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms were defined by Schuster. Recently, Wang *et al.* extended these concepts to L_p versions. In this paper, we establish Brunn-Minkowski type inequalities for the L_p and L_p radial Blaschke-Minkowski homomorphisms of dual quermassintegrals.**

***Index Terms*—L_p Blaschke-Minkowski homomorphism, L_p radial Blaschke-Minkowski homomorphism, Brunn-Minkowski inequality, dual quermassintegral.**

I. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbf{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbf{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_{os}^n$, respectively. In addition, let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbf{R}^n$. Let S^{n-1} denote the unit sphere and $V(K)$ denote the n-dimensional volume of the body K. For the standard unit ball B, its volume $V(B)=\omega_n$.

The projection bodies and intersection bodies played critical roles in the solutions of the Shephard problems and Busemann-petty problems, respectively (see [10], [22]). Through the work of Ludwig (see [16], [17]), projection bodies and intersection bodies were characterized as continuous and $GL(n)$ contravariant valuations. Recently, based on the properties of the well-known projection and intersection operators, Schuster [23] introduced two special valuations: the Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms which can be stated as follows:

Definition 1.A. *A map $\Phi:\mathcal{K}^n\rightarrow\mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism, if it satisfies the following conditions*:

(1) Φ is continuous;

(2) For all $K,L\in\mathcal{K}^n$, $\Phi(K\#L)=\Phi K+\Phi L$, where $K\#L$ denotes the Blaschke sum of K and L, and $\Phi K+\Phi L$ denotes the Minkowski addition of ΦK and ΦL;

(3) For all $K\in\mathcal{K}^n$ and every $v\in SO(n)$, $\Phi(vK)=v\Phi(K)$, where $SO(n)$ denotes the group of rotations in n-dimensions.

Definition 1.B. *A map $\Psi:\mathcal{S}_o^n\rightarrow\mathcal{S}_o^n$ is called a radial Blaschke-Minkowski homomorphism, if it satisfies the following conditions*:

(1) Ψ is continuous with respect to the radial metric;

(2) For all $K,L\in\mathcal{S}_o^n$, $\Psi(K\widehat{+}L)=\Psi K\widetilde{+}\Psi L$, where $K\widehat{+}L$ denotes the radial Blaschke sum of K and L, and $\Psi K\widetilde{+}\Psi L$ denotes the radial Minkowski addition of ΨK and ΨL;

(3) For all $K\in\mathcal{S}_o^n$ and every $v\in SO(n)$, $\Psi(vK)=v\Psi(K)$, where $SO(n)$ denotes the group of rotations in n-dimensions.

Associated with the Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms, Zhao [32] established following Brunn-Minkowski type inequalities.

Theorem 1.A. *If $K,L\in\mathcal{S}_o^n$ and $i,j\in\mathbf{R}$, $s\in\mathbf{N}$ satisfy $i\le n-1\le j\le n$ $(i\neq j)$, $0\le s\le n-1$, then*

$$\left(\frac{\widetilde{W}_i(\Psi_s(K\widetilde{+}_sL))}{\widetilde{W}_j(\Psi_s(K\widetilde{+}_sL))}\right)^{\frac{1}{j-i}}\le\left(\frac{\widetilde{W}_i(\Psi_sK)}{\widetilde{W}_j(\Psi_sK)}\right)^{\frac{1}{j-i}}+\left(\frac{\widetilde{W}_i(\Psi_sL)}{\widetilde{W}_j(\Psi_sL)}\right)^{\frac{1}{j-i}},$$

with equality if and only if K and L are dilates. Here Ψ_s denotes the mixed radial Blaschke-Minkowski homomorphisms of order s, and $\widetilde{+}_s$ denotes the L_s radial sum.

Theorem 1.B. *If $K,L\in\mathcal{K}^n$ in $\mathbf{R}^n$ and $i,j\in\mathbf{R}$ satisfy $i\le n+1\le j\le n$ and $i\neq j$, then*

$$\left(\frac{\widetilde{W}_i(\Phi^*(K\#L))}{\widetilde{W}_j(\Phi^*(K\#L))}\right)^{\frac{1}{i-j}}\le\left(\frac{\widetilde{W}_i(\Phi^*K)}{\widetilde{W}_j(\Phi^*K)}\right)^{\frac{1}{i-j}}+\left(\frac{\widetilde{W}_i(\Phi^*L)}{\widetilde{W}_j(\Phi^*L)}\right)^{\frac{1}{i-j}},$$

with equality if and only if K and L are homothetic. Here Φ^*K denotes the polar body of ΦK.

More results for the Blaschke-Minkowski homomorphisms and the radial Blaschke-Minkowski homomorphisms, also see [1], [4], [5], [6], [7], [8], [12], [15], [27], [28], [29], [30], [31], [33], [34], [35], [36], [37].

In 2013, based on the properties of L_p projection bodies, Wang [24] extended the notion of Blaschke-Minkowski homomorphisms to L_p version. Here, according to the range of solutions of L_p Minkowski problem (see Theorem 9.2.3 of book [22]), we improve Wang's definition as follows:

Definition 1.C. *For $p\ge1$, a map $\Phi_p:\mathcal{K}_o^n\rightarrow\mathcal{K}_o^n$ is called a L_p Blaschke-Minkowski homomorphism, if it satisfies the following conditions*:

(1) Φ_p is continuous;

(2) For all $K,L\in\mathcal{K}_o^n$, $\Phi_p(K\#_pL)=\Phi_pK+_p\Phi_pL$, where $K\#_pL$ denotes the L_p Blaschke sum of K and L, and $\Phi_pK+_p\Phi_pL$ denotes L_p Minkowski addition of Φ_pK and Φ_pL;

(3) For all $K\in\mathcal{K}_o^n$ and every $v\in SO(n)$, $\Phi_p(vK)=v\Phi_p(K)$, where $SO(n)$ denotes the group of rotations in n-dimensions.

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Remark 1.1. In Definition 1.C, if replace "$\Phi_p : \mathcal{K}_o^n \to \mathcal{K}_o^n$" by "$\Phi_p : \mathcal{K}_{os}^n \to \mathcal{K}_{os}^n$", then Definition 1.C is just the Wang's work (see [24]).

In [26], Wang, Liu and He defined the L_p radial Blaschke-Minkowski homomorphisms based on the radial Blaschke-Minkowski homomorphisms.

Definition 1.D. *For $p > 0$, a map $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ is called a L_p radial Blaschke-Minkowski homomorphism, if it satisfies the following conditions*:

(1) Ψ_p is continuous with respect to radial metric;

(2) For all $K, L \in \mathcal{S}_o^n$, $\Psi_p(K\widehat{+}_pL) = \Psi_pK\widetilde{+}_p\Psi_pL$, where $K\widehat{+}_pL$ denotes the L_p radial Blaschke addition of K and L, and $\Psi_pK\widetilde{+}_p\Psi_pL$ denotes the L_p radial Minkowski addition of Ψ_pK and Ψ_pL;

(3) For all $K \in \mathcal{S}_o^n$ and every $v \in SO(n)$, $\Psi_p(vK) = v\Psi_p(K)$, where $SO(n)$ denotes the group of rotations in n-dimensions.

From Definition 1.D, we easily see that the L_p radial Blaschke-Minkowski homomorphism is a more general form of the L_p intersection operator. Regarding the studies of the L_p Blaschke-Minkowski homomorphisms and L_p radial Blaschke-Minkowski homomorphisms, many results have been obtained in these articles (see [2], [3], [14], [25], [39]).

The purpose of this paper is to establish Brunn-Minkowski type inequalities for the L_p Blaschke-Minkowski homomorphisms and the L_p radial Blaschke-Minkowski homomorphisms based on Theorem 1.A and Theorem 1.B, respectively. Our results can be stated as follows:

Theorem 1.1. *For $p > 0$, let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be a L_p radial Blaschke-Minkowski homomorphism, $K_1, K_2, L_1, L_2 \in \mathcal{S}_o^n$, $i, j \in \mathbf{R}$ and $i \neq j$. If $i \leq n - p \leq j \leq n$, then*

$$\left(\frac{\widetilde{W}_i(\Psi_p(K_1\widehat{+}_pK_2))}{\widetilde{W}_j(\Psi_p(L_1\widehat{+}_pL_2))}\right)^{\frac{p}{j-i}} \leq \left(\frac{\widetilde{W}_i(\Psi_pK_1)}{\widetilde{W}_j(\Psi_pL_1)}\right)^{\frac{p}{j-i}} + \left(\frac{\widetilde{W}_i(\Psi_pK_2)}{\widetilde{W}_j(\Psi_pL_2)}\right)^{\frac{p}{j-i}}; \qquad (1.1)$$

if $n - p \leq i \leq n \leq j$, then

$$\left(\frac{\widetilde{W}_i(\Psi_p(K_1\widehat{+}_pK_2))}{\widetilde{W}_j(\Psi_p(L_1\widehat{+}_pL_2))}\right)^{\frac{p}{j-i}} \geq \left(\frac{\widetilde{W}_i(\Psi_pK_1)}{\widetilde{W}_j(\Psi_pL_1)}\right)^{\frac{p}{j-i}} + \left(\frac{\widetilde{W}_i(\Psi_pK_2)}{\widetilde{W}_j(\Psi_pL_2)}\right)^{\frac{p}{j-i}}. \qquad (1.2)$$

In each case, equality holds if and only if K_1 and K_2 are dilates, L_1 and L_2 are dilates and with the same dilation coefficient.

Theorem 1.2. *For $p \geq 1$, let $\Phi_p : \mathcal{K}_o^n \to \mathcal{K}_o^n$ be a L_p Blaschke-Minkowski homomorphism, $K_1, K_2, L_1, L_2 \in \mathcal{K}_o^n$, $i, j \in \mathbf{R}$ and $i \neq j$. If $i \geq n + p \geq j \geq n$, then*

$$\left(\frac{\widetilde{W}_i(\Phi_p^*(K_1\#_pK_2))}{\widetilde{W}_j(\Phi_p^*(L_1\#_pL_2))}\right)^{\frac{p}{i-j}} \leq \left(\frac{\widetilde{W}_i(\Phi_p^*K_1)}{\widetilde{W}_j(\Phi_p^*L_1)}\right)^{\frac{p}{i-j}} + \left(\frac{\widetilde{W}_i(\Phi_p^*K_2)}{\widetilde{W}_j(\Phi_p^*L_2)}\right)^{\frac{p}{i-j}}; \qquad (1.3)$$

if $n + p \geq i \geq n \geq j$, then

$$\left(\frac{\widetilde{W}_i(\Phi_p^*(K_1\#_pK_2))}{\widetilde{W}_j(\Phi_p^*(L_1\#_pL_2))}\right)^{\frac{p}{i-j}} \geq \left(\frac{\widetilde{W}_i(\Phi_p^*K_1)}{\widetilde{W}_j(\Phi_p^*L_1)}\right)^{\frac{p}{i-j}} + \left(\frac{\widetilde{W}_i(\Phi_p^*K_2)}{\widetilde{W}_j(\Phi_p^*L_2)}\right)^{\frac{p}{i-j}}. \qquad (1.4)$$

In each case, with equality if and only if Φ_pK_1 and Φ_pK_2 are dilates, Φ_pL_1 and Φ_pL_2 are dilates and with the same dilation coefficient. Here Φ_p^*K is the polar body of Φ_pK.

In this paper, the proofs of Theorem 1.1 and Theorem 1.2 will be given in Section III.

II. Preliminaries

A. Support function, radial function and polar set

Suppose that $\mathbf{R}$ is the set of real number. If $K \in \mathcal{K}^n$, the support function of K, $h_K = h(K, \cdot) : \mathbf{R}^n \to \mathbf{R}$, is defined by (see [10])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbf{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

For a compact star shaped (about the origin) K in $\mathbf{R}^n$, the radial function of K, $\rho_K = \rho(K, \cdot) : \mathbf{R}^n \backslash \{0\} \to [0, +\infty)$, is defined by (see [10])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbf{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, then K will be called a star body (respect to the origin).

If $E \subset \mathbf{R}^n$ is a nonempty subset, the polar of set E, E^*, is a convex set whose definition is given by (see [10], [22])

$$E^* = \{x \in \mathbf{R}^n : x \cdot y \leq 1, \ y \in E\}.$$

For $K \in \mathcal{K}_o^n$, it is not difficult to obtain $(K^*)^* = K$.

From the definitions of support function, radial function and polar, for $K \in \mathcal{K}_o^n$, then (see [10])

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \qquad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \qquad (2.1)$$

B. L_p Minkowski combination and L_p radial combination

For $K, L \in \mathcal{K}_o^n$, $\lambda, \mu \geq 0$ (not both zero) and real $p \geq 1$, the L_p Minkowski combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by (see [10], [22])

$$h(\lambda \cdot K +_p \mu \cdot L, \ \cdot)^p = \lambda h(K, \ \cdot)^p + \mu h(L, \ \cdot)^p, \qquad (2.2)$$

where "$+_p$" denotes the L_p Minkowski addition and $\lambda \cdot K$ denotes the L_p Minkowski scalar multiplication.

Let $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero) and real $p \neq 0$, the L_p radial combination, $\lambda \circ K \widetilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is given by (see [9], [22])

$$\rho(\lambda \circ K \widetilde{+}_p \mu \circ L, \ \cdot)^p = \lambda \rho(K, \ \cdot)^p + \mu \rho(L, \ \cdot)^p, \qquad (2.3)$$

where "$\widetilde{+}_p$" denotes the L_p radial sum and $\lambda \circ K$ denotes the L_p radial scalar multiplication.

C. Dual quermassintegrals

Lutwak [18] gave the notion of dual quermassintegrals. For $K \in \mathcal{S}_o^n$ and real i, the dual quermassintegral, $\widetilde{W}_i(K)$, of K is given by

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} du. \qquad (2.4)$$

Especially,

$$\widetilde{W}_0(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n du = V(K),$$

and

$$\widetilde{W}_n(K) = \frac{1}{n}\int_{S^{n-1}} du = \frac{1}{n}S(B) = V(B) = \omega_n.$$

D. L_p Blaschke combination

For $K, L \in \mathcal{K}_o^n,\ n \neq p$ and $\lambda, \mu \geq 0$ (not both zero), the L_p Blaschke combination, $\lambda \odot K \#_p\ \mu \odot L$, of K and L is defined by (see [22])

$$S_p(\lambda \odot K \#_p \mu \odot L, \cdot) = \lambda S_p(K, \cdot) + \mu S_p(L, \cdot),$$

where "$\#_p$" denotes the L_p Blaschke addition and $\lambda \odot K$ denotes the L_p Blaschke scalar multiplication. Here $S_p(M, \cdot)$ denotes the L_p surface area measure of $M \in \mathcal{K}_o^n$.

E. L_p projection body and L_p intersection body

The notion of L_p projection body was introduced by Lutwak, Yang and Zhang [19] as follows: For each $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p projection body, $\Pi_p K$, of K is an origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}}\int_{S^{n-1}} |u \cdot v|^p dS_p(K, v),$$

for all $u \in S^{n-1}$, and $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$.

In 2006, Haberl and Ludwing [11] defined the L_p intersection body as follows: For $K \in \mathcal{S}_o^n$ and $0 < p < 1$, the L_p intersection body, $I_p K$, of K is an origin-symmetric star body whose radial function is defined by

$$\rho^p_{I_p K}(u) = \int_K |u \cdot x|^{-p} dx = \frac{1}{n-p}\int_{S^{n-1}} |u \cdot v|^{-p} \rho_K^{n-p}(v) dS(v),$$

for all $u \in S^{n-1}$.

On the further study of the projection bodies and intersection bodies, we may refer to [13] and [20].

III. Results and Proofs

In this section, we will prove Theorem 1.1 and Theorem 1.2. In order to complete the proof of Theorem 1.1, we require the following lemmas.

Lemma 3.1 ([21]). *(Dresher's inequality) Let functions $f_1, f_2, g_1, g_2 \geq 0,\ E$ is a bounded measurable subset in $\mathbf{R}^n$, if $s \geq 1 \geq t \geq 0$ and $s \neq t$, then*

$$\left(\frac{\int_E (f_1+f_2)^s dx}{\int_E (g_1+g_2)^t dx}\right)^{\frac{1}{s-t}} \leq \left(\frac{\int_E f_1^s dx}{\int_E g_1^t dx}\right)^{\frac{1}{s-t}} + \left(\frac{\int_E f_2^s dx}{\int_E g_2^t dx}\right)^{\frac{1}{s-t}}, \qquad (3.1)$$

equality holds if and only if $f_1/f_2 = g_1/g_2$.

Lemma 3.2 ([38]). *(Reverse Dresher's inequality) Let functions $f_1, f_2, g_1, g_2 \geq 0,\ E$ is a bounded measurable subset in $\mathbf{R}^n$, if $1 \geq s \geq 0 \geq t$ and $s \neq t$, then*

$$\left(\frac{\int_E (f_1+f_2)^s dx}{\int_E (g_1+g_2)^t dx}\right)^{\frac{1}{s-t}} \geq \left(\frac{\int_E f_1^s dx}{\int_E g_1^t dx}\right)^{\frac{1}{s-t}} + \left(\frac{\int_E f_2^s dx}{\int_E g_2^t dx}\right)^{\frac{1}{s-t}}, \qquad (3.2)$$

equality holds if and only if $f_1/f_2 = g_1/g_2$.

Lemma 3.3 ([26]). *A map $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ is a L_p ($p > 0$) radial Blaschke-Minkowski homomorphism if and only if there is a non-negative measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e}\ )$ such that*

$$\rho(\Psi_p K, \cdot)^p = \rho(K, \cdot)^{n-p} * \mu.$$

Proof of Theorem 1.1. Since $K_1, K_2 \in \mathcal{S}_o^n$, by (2.4), Definition 1.D and (2.3), we have

$$\begin{aligned}
&\widetilde{W}_i(\Psi_p(K_1 \widehat{+}_p K_2)) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(\Psi_p(K_1 \widehat{+}_p K_2), u)^{n-i} du \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(\Psi_p K_1 \widetilde{+}_p \Psi_p K_2, u)^{n-i} du \\
&= \frac{1}{n}\int_{S^{n-1}} (\rho(\Psi_p K_1, u)^p + \rho(\Psi_p K_2, u)^p)^{\frac{n-i}{p}} du. \qquad (3.3)
\end{aligned}$$

Similarly, for $L_1, L_2 \in \mathcal{S}_o^n$, we get

$$\begin{aligned}
&\widetilde{W}_j(\Psi_p(L_1 \widehat{+}_p L_2)) \\
&= \frac{1}{n}\int_{S^{n-1}} (\rho(\Psi_p L_1, u)^p + \rho(\Psi_p L_2, u)^p)^{\frac{n-j}{p}} du. \qquad (3.4)
\end{aligned}$$

Since $i \leq n-p \leq j \leq n$, then $\frac{n-i}{p} \geq 1 \geq \frac{n-j}{p} \geq 0$, thus from (3.3), (3.4) and associated with (3.1), we obtain

$$\begin{aligned}
&\left(\frac{\widetilde{W}_i(\Psi_p(K_1 \widehat{+}_p K_2))}{\widetilde{W}_j(\Psi_p(L_1 \widehat{+}_p L_2))}\right)^{\frac{p}{j-i}} \\
&= \left(\frac{\int_{S^{n-1}} (\rho(\Psi_p K_1, u)^p + \rho(\Psi_p K_2, u)^p)^{\frac{n-i}{p}} du}{\int_{S^{n-1}} (\rho(\Psi_p L_1, u)^p + \rho(\Psi_p L_2, u)^p)^{\frac{n-j}{p}} du}\right)^{\frac{p}{j-i}} \\
&\leq \left(\frac{\int_{S^{n-1}} (\rho(\Psi_p K_1, u)^p)^{\frac{n-i}{p}} du}{\int_{S^{n-1}} (\rho(\Psi_p L_1, u)^p)^{\frac{n-j}{p}} du}\right)^{\frac{p}{j-i}} + \left(\frac{\int_{S^{n-1}} (\rho(\Psi_p K_2, u)^p)^{\frac{n-i}{p}} du}{\int_{S^{n-1}} (\rho(\Psi_p L_2, u)^p)^{\frac{n-j}{p}} du}\right)^{\frac{p}{j-i}} \\
&= \left(\frac{\int_{S^{n-1}} \rho(\Psi_p K_1, u)^{n-i} du}{\int_{S^{n-1}} \rho(\Psi_p L_1, u)^{n-j} du}\right)^{\frac{p}{j-i}} + \left(\frac{\int_{S^{n-1}} \rho(\Psi_p K_2, u)^{n-i} du}{\int_{S^{n-1}} \rho(\Psi_p L_2, u)^{n-j} du}\right)^{\frac{p}{j-i}} \\
&= \left(\frac{\widetilde{W}_i(\Psi_p K_1)}{\widetilde{W}_j(\Psi_p L_1)}\right)^{\frac{p}{j-i}} + \left(\frac{\widetilde{W}_i(\Psi_p K_2)}{\widetilde{W}_j(\Psi_p L_2)}\right)^{\frac{p}{j-i}}.
\end{aligned}$$

This yields inequality (1.1).

By the equality condition of (3.1), we know that equality holds in (1.1) if and only if $\frac{\rho(\Psi_p K_1,\ \cdot)}{\rho(\Psi_p K_2,\ \cdot)} = \frac{\rho(\Psi_p L_1,\ \cdot)}{\rho(\Psi_p L_2,\ \cdot)}$, and according to Lemma 3.3, we see that equality holds in (1.1) if and only if $\frac{\rho(K_1,\ \cdot)}{\rho(K_2,\ \cdot)} = \frac{\rho(L_1,\ \cdot)}{\rho(L_2,\ \cdot)}$, i.e., K_1 and K_2 are dilates, L_1 and L_2 are dilates and with the same dilation coefficient.

Similarly, for $n-p \leq i \leq n \leq j$, we can get desired inequality (1.2) from the inequalities (3.2), (3.3) and (3.4).

Taking $i = 0,\ j = n$ in Theorem 1.1, and notice that $\widetilde{W}_0(M) = V(M)$ and $\widetilde{W}_n(M) = \omega_n$ for any $M \in \mathcal{S}_o^n$, we have a following fact.

Corollary 3.1. *Let $\Psi_p : \mathcal{S}_o^n \to \mathcal{S}_o^n$ be a L_p radial Blaschke-Minkowski homomorphism, $K_1, K_2 \in \mathcal{S}_o^n$. If $0 < p < n$, then*

$$V(\Psi_p(K_1 \widehat{+}_p K_2))^{\frac{p}{n}} \le V(\Psi_p K_1)^{\frac{p}{n}} + V(\Psi_p K_2)^{\frac{p}{n}};$$

if $p > n$, then

$$V(\Psi_p(K_1 \widehat{+}_p K_2))^{\frac{p}{n}} \ge V(\Psi_p K_1)^{\frac{p}{n}} + V(\Psi_p K_2)^{\frac{p}{n}}.$$

In each case, equality holds if and only if K_1 and K_2 are dilates.

As a special example of the L_p radial Blaschke-Minkowski homomorphisms, the L_p intersection body have the following result by Theorem 1.1.

Corollary 3.2. *For $0 < p < 1$, $K_1, K_2, L_1, L_2 \in \mathcal{S}_o^n$, $i, j \in \mathbf{R}$ and $i \neq j$. If $i \le n - p \le j \le n$, then*

$$\left(\frac{\widetilde{W}_i(I_p(K_1 \widehat{+}_p K_2))}{\widetilde{W}_j(I_p(L_1 \widehat{+}_p L_2))}\right)^{\frac{p}{j-i}} \le \left(\frac{\widetilde{W}_i(I_p K_1)}{\widetilde{W}_j(I_p L_1)}\right)^{\frac{p}{j-i}} + \left(\frac{\widetilde{W}_i(I_p K_2)}{\widetilde{W}_j(I_p L_2)}\right)^{\frac{p}{j-i}};$$

if $n - p \le i \le n \le j$, then

$$\left(\frac{\widetilde{W}_i(I_p(K_1 \widehat{+}_p K_2))}{\widetilde{W}_j(I_p(L_1 \widehat{+}_p L_2))}\right)^{\frac{p}{j-i}} \ge \left(\frac{\widetilde{W}_i(I_p K_1)}{\widetilde{W}_j(I_p L_1)}\right)^{\frac{p}{j-i}} + \left(\frac{\widetilde{W}_i(I_p K_2)}{\widetilde{W}_j(I_p L_2)}\right)^{\frac{p}{j-i}}.$$

In each case, equality holds if and only if K_1 and K_2 are dilates, L_1 and L_2 are dilates and with the same dilation coefficient.

Proof of Theorem 1.2. For $K_1, K_2 \in \mathcal{K}_o^n$, from (2.4), Definition 1.C, (2.1) and (2.2), we obtain

$$\begin{aligned}
&\widetilde{W}_i(\Phi_p^*(K_1 \#_p K_2)) \\
&= \frac{1}{n}\int_{S^{n-1}} \rho(\Phi_p^*(K_1 \#_p K_2), u)^{n-i} du \\
&= \frac{1}{n}\int_{S^{n-1}} h(\Phi_p(K_1 \#_p K_2), u)^{-(n-i)} du \\
&= \frac{1}{n}\int_{S^{n-1}} h(\Phi_p K_1 +_p \Phi_p K_2, u)^{-(n-i)} du \\
&= \frac{1}{n}\int_{S^{n-1}} (h(\Phi_p K_1, u)^p + h(\Phi_p K_2, u)^p)^{\frac{-(n-i)}{p}} du. \qquad (3.5)
\end{aligned}$$

Similarly, for $L_1, L_2 \in \mathcal{K}_o^n$, we have

$$\begin{aligned}
&\widetilde{W}_j(\Phi_p^*(L_1 \#_p L_2)) \\
&= \frac{1}{n}\int_{S^{n-1}} (h(\Phi_p L_1, u)^p + h(\Phi_p L_2, u)^p)^{\frac{-(n-j)}{p}} du. \qquad (3.6)
\end{aligned}$$

For $i \ge n + p \ge j \ge n$, then $\frac{-(n-i)}{p} \ge 1 \ge \frac{-(n-j)}{p} \ge 0$, thus from (3.5), (3.6) and combined with (3.1), we know that

$$\begin{aligned}
&\left(\frac{\widetilde{W}_i(\Phi_p^*(K_1 \#_p K_2))}{\widetilde{W}_j(\Phi_p^*(L_1 \#_p L_2))}\right)^{\frac{p}{i-j}} \\
&= \left(\frac{\int_{S^{n-1}} (h(\Phi_p K_1, u)^p + h(\Phi_p K_2, u)^p)^{\frac{-(n-i)}{p}} du}{\int_{S^{n-1}} (h(\Phi_p L_1, u)^p + h(\Phi_p L_2, u)^p)^{\frac{-(n-j)}{p}} du}\right)^{\frac{p}{i-j}} \\
&\le \left(\frac{\int_{S^{n-1}} (h(\Phi_p K_1, u)^p)^{\frac{-(n-i)}{p}} du}{\int_{S^{n-1}} (h(\Phi_p L_1, u)^p)^{\frac{-(n-j)}{p}} du}\right)^{\frac{p}{i-j}} + \left(\frac{\int_{S^{n-1}} (h(\Phi_p K_2, u)^p)^{\frac{-(n-i)}{p}} du}{\int_{S^{n-1}} (h(\Phi_p L_2, u)^p)^{\frac{-(n-j)}{p}} du}\right)^{\frac{p}{i-j}} \\
&= \left(\frac{\int_{S^{n-1}} h(\Phi_p K_1, u)^{-(n-i)} du}{\int_{S^{n-1}} h(\Phi_p L_1, u)^{-(n-j)} du}\right)^{\frac{p}{i-j}} + \left(\frac{\int_{S^{n-1}} h(\Phi_p K_2, u)^{-(n-i)} du}{\int_{S^{n-1}} h(\Phi_p L_2, u)^{-(n-j)} du}\right)^{\frac{p}{i-j}} \\
&= \left(\frac{\int_{S^{n-1}} \rho(\Phi_p^* K_1, u)^{n-i} du}{\int_{S^{n-1}} \rho(\Phi_p^* L_1, u)^{n-j} du}\right)^{\frac{p}{i-j}} + \left(\frac{\int_{S^{n-1}} \rho(\Phi_p^* K_2, u)^{n-i} du}{\int_{S^{n-1}} \rho(\Phi_p^* L_2, u)^{n-j} du}\right)^{\frac{p}{i-j}} \\
&= \left(\frac{\widetilde{W}_i(\Phi_p^* K_1)}{\widetilde{W}_j(\Phi_p^* L_1)}\right)^{\frac{p}{i-j}} + \left(\frac{\widetilde{W}_i(\Phi_p^* K_2)}{\widetilde{W}_j(\Phi_p^* L_2)}\right)^{\frac{p}{i-j}}.
\end{aligned}$$

This is just the inequality (1.3).

By the equality condition of (3.1), equality holds in (1.3) if and only if $\frac{h(\Phi_p K_1, \cdot)}{h(\Phi_p K_2, \cdot)} = \frac{h(\Phi_p L_1, \cdot)}{h(\Phi_p L_2, \cdot)}$, i.e., $\Phi_p K_1$ and $\Phi_p K_2$ are dilates, $\Phi_p L_1$ and $\Phi_p L_2$ are dilates and with the same dilation coefficient.

Similar to the above method, if $n + p \ge i \ge n \ge j$, we can prove the inequality (1.4) by (3.2), (3.5) and (3.6).

In particular, if $i = n$ and $j = 0$ in (1.4), we obtain a result as follows.

Corollary 3.3. *Let $\Phi_p : \mathcal{K}_o^n \to \mathcal{K}_o^n$ be a L_p Blaschke-Minkowski homomorphism, $K_1, K_2 \in \mathcal{K}_o^n$, if $1 \le p \neq n$, then*

$$V(\Phi_p^*(K_1 \#_p K_2))^{-\frac{p}{n}} \ge V(\Phi_p^* K_1)^{-\frac{p}{n}} + V(\Phi_p^* K_2)^{-\frac{p}{n}},$$

with equality if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.

Since the L_p projection body is a special example of the L_p Blaschke-Minkowski homomorphisms, therefore, we can obtain the following fact from Theorem 1.2.

Corollary 3.4. *For $p \ge 1$, $K_1, K_2, L_1, L_2 \in \mathcal{K}_o^n$, $i, j \in \mathbf{R}$ and $i \neq j$. If $i \ge n + p \ge j \ge n$, then*

$$\left(\frac{\widetilde{W}_i(\Pi_p^*(K_1 \#_p K_2))}{\widetilde{W}_j(\Pi_p^*(L_1 \#_p L_2))}\right)^{\frac{p}{i-j}} \le \left(\frac{\widetilde{W}_i(\Pi_p^* K_1)}{\widetilde{W}_j(\Pi_p^* L_1)}\right)^{\frac{p}{i-j}} + \left(\frac{\widetilde{W}_i(\Pi_p^* K_2)}{\widetilde{W}_j(\Pi_p^* L_2)}\right)^{\frac{p}{i-j}};$$

if $n + p \ge i \ge n \ge j$, then

$$\left(\frac{\widetilde{W}_i(\Pi_p^*(K_1 \#_p K_2))}{\widetilde{W}_j(\Pi_p^*(L_1 \#_p L_2))}\right)^{\frac{p}{i-j}} \ge \left(\frac{\widetilde{W}_i(\Pi_p^* K_1)}{\widetilde{W}_j(\Pi_p^* L_1)}\right)^{\frac{p}{i-j}} + \left(\frac{\widetilde{W}_i(\Pi_p^* K_2)}{\widetilde{W}_j(\Pi_p^* L_2)}\right)^{\frac{p}{i-j}}.$$

In each case, equality holds if and only if $\Pi_p K_1$ and $\Pi_p K_2$ are dilates, $\Pi_p L_1$ and $\Pi_p L_2$ are dilates and with the same dilation coefficient. Here $\Pi_p^* K$ denotes the polar body of $\Pi_p K$.

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