

# Adomian Polynomial and Elzaki Transform Method for Solving Sine-Gordon Equations

Olufemi Elijah Ige, Razaq Adekola Oderinu, and Tarig M. Elzaki

**Abstract**—Elzaki transform is combined with Adomian polynomial to obtain an approximate analytical solutions of nonlinear Sine Gordon equations. The necessity of Adomian polynomial is to linearise the nonlinear function(s) that is present in any given differential equation(s) because Elzaki transform, like other integral transforms, cannot be used to solve nonlinear differential equation independently. The approximate analytical solutions are presented in series form. In order to investigate the performance of the method, two single nonlinear Sine Gordon equations and one coupled Sine Gordon equation were considered in this paper. The method is very powerful because one of the problems considered converges to the exact solution and this shows the effectiveness of this method in solving nonlinear Sine Gordon equations.

**Index Terms**—Elzaki transform method, Adomian polynomial, Sine Gordon equations.

## I. INTRODUCTION

THE Sine Gordon equation is one of the most important equations in partial differential equations because of its applications in applied mathematics. This equation plays a major role in the propagation of fluxons in Josephson junctions between two superconductors [1], [2], [3]. It is also useful in many scientific fields such as the rigid pendulum motion attached to a stretched wire [4]. Furthermore, it is applicable in nonlinear optics [5], solid state physics, and stability of fluid motions [6].

Generally, Sine Gordon equation is of the form:

$$u_{tt} - u_{xx} + \sin u = 0, \quad (1)$$

with the initial conditions

$$u(x, 0) = a(x), \quad u_t(x, 0) = b(x), \quad (2)$$

where  $u$  is a function of  $x$  and  $t$ ,  $\sin u$  is the nonlinear term in this case,  $a(x)$  and  $b(x)$  are the known analytic function [6].

Several methods have been developed to obtain the approximate analytical solutions of Sine Gordon equations. Some of these methods for solving nonlinear differential equations are the: Exp function method [7], reduced differential transform method [6], the Homotopy Analysis Method, Adomian Decomposition Method [8], [9], Variation Iteration Method [10], [11], Homotopy Perturbation Method [12], [13], [14] and the variable separated ODE method [15].

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In this paper, we find the solutions of nonlinear Sine Gordon equations and coupled Sine Gordon equation by Elzaki transform method and Adomian Polynomial. This method gives the solutions in series form and most of the time, it yields exact solutions with few iterations.

This paper is structured as follows: Section 2 contains the basic definitions and the properties of the proposed method. Section 3 shows the theoretical approach of the proposed method on Sine Gordon equations. In Section 4, the Elzaki transform method and Adomian polynomial is applied to solve three problems in order to show its efficiency. Section 5 contains the discussion of results and the conclusion is presented in Section 6.

## II. PROPERTIES OF ELZAKI TRANSFORM

Elzaki transform [16], [17], [18], [19], [20], [21], [22] is defined for functions of exponential order. Consider the functions in the set  $A$  defined by

$$A = \left\{ f(t) : \exists M, c_1, c_2 > 0, |f(t)| < M e^{\frac{|t|}{c_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.$$

where  $c_1, c_2$  may either be finite or infinite, but  $M$  must be infinite.

According to Tarig [16], Elzaki transform is defined as:

$$E[f(t)] = u^2 \int_0^\infty f(ut) e^{-t} dt = T(u), \quad t \geq 0, u \in (c_1, c_2)$$

or

$$E[f(t)] = u \int_0^\infty f(t) e^{-\frac{t}{u}} dt = T(u), \quad t \geq 0, u \in (c_1, c_2) \quad (3)$$

where  $u$  in equation (3) is used to factor out  $t$  in the analysis of the function  $f$ .

Let  $T(u)$  be the Elzaki transform of  $f(t)$  such that

$$E[f(t)] = T(u).$$

Then:

- (i)  $E[f'(t)] = \frac{T(u)}{u} - u f(0)$ ,
- (ii)  $E[f''(t)] = \frac{T''(u)}{u^2} - f(0) - u f'(0)$ ,
- (iii)  $E[f^{(n)}(t)] = \frac{T^{(n)}(u)}{u^n} - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0)$ .

The equation  $E[f(t)] = T(u)$  means that  $T(u)$  is the Elzaki transform of  $f(t)$ , and  $f(t)$  is the inverse Elzaki transform of  $T(u)$ . That is,

$$f(t) = E^{-1}[T(u)].$$

In order to obtain the Elzaki transform of a partial derivative, integration by parts is used on the definition of the Elzaki transform and the resulting expressions are [23],

$$\begin{aligned} E \left[ \frac{\partial f(x, t)}{\partial t} \right] &= \frac{T(x, v)}{v} - v f(x, 0), \\ E \left[ \frac{\partial^2 f(x, t)}{\partial t^2} \right] &= \frac{T(x, v)}{v^2} - f(x, 0) - v \frac{\partial f(x, 0)}{\partial t}, \\ E \left[ \frac{\partial f(x, t)}{\partial x} \right] &= \frac{d}{dx} [T(x, v)], \\ E \left[ \frac{\partial^2 f(x, t)}{\partial x^2} \right] &= \frac{d^2}{dx^2} [T(x, v)]. \end{aligned}$$

### III. THEORETICAL APPROACH: ELZAKI TRANSFORM ON SINE GORDON EQUATION

The main focus of this work is to solve the nonlinear partial differential equations which is Sine Gordon equations using the combination of Elzaki transform method (ETM) and Adomian polynomial.

According to [24], [25], consider;

$$\frac{\partial^w u(x, t)}{\partial t^w} + Ru(x, t) + Nu(x, t) = f(x, t), \quad (4)$$

where  $w = 1, 2, 3$ .

The initial condition is given as

$$\frac{\partial^{w-1} u(x, t)}{\partial t^{w-1}} \Big|_{t=0} = g_{w-1}(x).$$

The partial derivative of the function  $u(x, t)$  of  $w^{th}$  order is the one given as  $\frac{\partial^w u(x, t)}{\partial t^w}$ ,  $R$  represents the linear differential operator,  $N$  indicates the nonlinear terms of differential equations, and  $f(x, t)$  is the non-homogeneous/source term.

By applying the Elzaki transform on equation (4), we get:

$$E \left[ \frac{\partial^w u(x, t)}{\partial t^w} \right] + E[Ru(x, t)] + E[Nu(x, t)] = E[f(x, t)], \quad (5)$$

where

$$E \left[ \frac{\partial^w u(x, t)}{\partial t^w} \right] = \frac{E[u(x, t)]}{v^w} - \sum_{k=0}^{w-1} v^{2-w+k} \frac{\partial^k u(x, 0)}{\partial t^k}. \quad (6)$$

Substituting equation (6) into equation (5) gives:

$$\begin{aligned} \frac{E[u(x, t)]}{v^w} - \sum_{k=0}^{w-1} v^{2-w+k} \frac{\partial^k u(x, 0)}{\partial t^k} + E[Ru(x, t)] \\ + E[Nu(x, t)] = E[f(x, t)]. \end{aligned}$$

This is the same as

$$\begin{aligned} \frac{E[u(x, t)]}{v^w} = E[f(x, t)] + \sum_{k=0}^{w-1} v^{2-w+k} \frac{\partial^k u(x, 0)}{\partial t^k} \\ - \{E[Ru(x, t)] + E[Nu(x, t)]\}. \end{aligned} \quad (7)$$

By simplifying equation (7), we have:

$$\begin{aligned} E[u(x, t)] = v^w E[f(x, t)] + \sum_{k=0}^{w-1} v^{2+k} \frac{\partial^k u(x, 0)}{\partial t^k} \\ - v^w \{E[Ru(x, t)] + E[Nu(x, t)]\}. \end{aligned} \quad (8)$$

Applying the inverse Elzaki transform to equation (8) gives

$$\begin{aligned} u(x, t) = E^{-1} \left[ v^w E[f(x, t)] + \sum_{k=0}^{w-1} v^{2+k} \frac{\partial^k u(x, 0)}{\partial t^k} \right] \\ - E^{-1} [v^w \{E[Ru(x, t)] + E[Nu(x, t)]\}]. \end{aligned}$$

We can rewrite this as

$$u(x, t) = F(x, t) - E^{-1} [v^w \{E[Ru(x, t)] + E[Nu(x, t)]\}], \quad (9)$$

where  $F(x, t)$  denotes the expression that arises from the given initial condition and the source terms after simplification. The solution will be in the form of infinite series as indicated below

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (10)$$

The nonlinear term is decomposed as

$$Nu(x, t) = \sum_{n=0}^{\infty} A_n, \quad (11)$$

where  $A_n$  is defined as the Adomian polynomials which can be calculated by using the formula

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, \dots$$

Substituting equation (10) and equation (11) into equation (9) gives

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x, t) = F(x, t) \\ - E^{-1} \left[ v^w \left\{ E \left[ R \sum_{n=0}^{\infty} u_n(x, t) \right] + E \left[ \sum_{n=0}^{\infty} A_n \right] \right\} \right]. \end{aligned} \quad (12)$$

Then from equation (12), we have

$$u_0(x, t) = F(x, t). \quad (13)$$

And the recursive relation is given as:

$$u_{n+1} = -E^{-1} [v^w \{E[Ru_n(x, t)] + E[A_n]\}].$$

Here  $w = 1, 2, 3$  and  $n \geq 0$ .

The analytical solution  $u(x, t)$  can be approximated by a truncated series

$$u(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x, t).$$

### IV. APPLICATIONS

The effectiveness of the Elzaki transform and the Adomian polynomial are demonstrated by solving the following Sine Gordon Equations.

**Example 4.1:** Consider the homogeneous Sine-Gordon Equation [6]

$$u_{tt} - u_{xx} + \sin u = 0, \quad (14)$$

with initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = 4 \operatorname{sech} x.$$

To solve this problem, we used Taylor's series expansion of  $\sin u$ , that is,

$$\sin u \approx u - \frac{u^3}{3!} + \frac{u^5}{5!} = u - \frac{u^3}{6} + \frac{u^5}{120}.$$

Then equation (14) becomes:

$$u_{tt} - u_{xx} = - \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right]. \quad (15)$$

Applying Elzaki transform to both sides of equation (15)

$$E[u_{tt}] - E[u_{xx}] = -E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right]. \quad (16)$$

where

$$E[u_{tt}] = \frac{U(x, v)}{v^2} - u(x, 0) - v u_t(x, 0), \text{ and}$$

$$E[u_{xx}] = \frac{d^2}{dx^2} [U(x, v)] = \frac{d^2}{dx^2} E[u].$$

Applying these and the given initial conditions to equation (16) and simplifying, we obtain:

$$U(x, t) = 4v^3 \operatorname{sech} x + v^2 \frac{d^2}{dx^2} E[u] - v^2 E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right]. \quad (17)$$

Applying the inverse Elzaki transform to equation (17) and simplifying, give

$$u = 4t \operatorname{sech} x + E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u] - v^2 E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right] \right\}. \quad (18)$$

From equation (18), let

$$u_0 = 4t \operatorname{sech} x.$$

The recursive relation is given as:

$$u_{n+1} = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u_n] - v^2 E[A_n] \right\}, \quad (19)$$

where  $A_n$  is the Adomian polynomial to decompose the nonlinear terms by using the relation:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f \left[ \sum_{i=0}^{\infty} \lambda^i u_i \right]_{\lambda=0}. \quad (20)$$

Let the nonlinear term be represented by

$$f(u) = u - \frac{u^3}{6} + \frac{u^5}{120}. \quad (21)$$

Substituting equation (21) into equation (20) gives:

$$\begin{aligned} A_0 &= u_0 - \frac{u_0^3}{6} + \frac{u_0^5}{120}, \quad A_1 = u_1 - \frac{1}{2} u_1 u_0^2 + \frac{1}{24} u_1 u_0^4, \\ A_2 &= u_2 - \frac{1}{2} u_0 u_1^2 - \frac{1}{2} u_2 u_0^2 + \frac{1}{12} u_0^3 u_1^2 + \frac{1}{24} u_2 u_0^4, \quad \dots \end{aligned}$$

From equation (19), when  $n=0$ , we have

$$u_1 = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u_0] - v^2 E[A_0] \right\}. \quad (22)$$

Since  $u_0 = 4t \operatorname{sech} x$

$$\begin{aligned} u_1 &= E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[4t \operatorname{sech} x] - v^2 E \left[ 4t \operatorname{sech} x - \frac{64}{6} t^3 \operatorname{sech}^3 x \right] \right\} \\ &\quad - E^{-1} \left\{ v^2 E \left[ \frac{1024}{120} t^5 \operatorname{sech}^5 x \right] \right\}. \end{aligned}$$

By simplifying, we obtain

$$u_1 = \frac{4}{315} \operatorname{sech}^5 x \left[ -105 t^3 \cosh^2 x + 42 t^5 \cosh^2 x - 16 t^7 \right]. \quad (23)$$

When  $n = 1$ ,

$$u_2 = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u_1] - v^2 E[A_1] \right\}. \quad (24)$$

$A_1$  is computed as;

$$\begin{aligned} A_1 &= -\frac{4}{3} t^3 \operatorname{sech}^3 x + \frac{32}{3} t^5 \operatorname{sech}^5 x - \frac{128}{9} t^7 \operatorname{sech}^7 x + \frac{8}{15} t^5 \operatorname{sech}^3 x \\ &\quad - \frac{1408}{315} t^7 \operatorname{sech}^5 x + \frac{2304}{315} t^9 \operatorname{sech}^7 x - \frac{2048}{945} t^{11} \operatorname{sech}^9 x. \end{aligned} \quad (25)$$

Therefore,  $u_2$  is computed as:

$$\begin{aligned} u_2 &= \frac{4}{2027025} t^5 \operatorname{sech}^9 x \left[ -270270 \cosh^6 x + 405405 \cosh^4 x \right] \\ &\quad + \frac{4}{2027025} t^5 \operatorname{sech}^9 x \left[ 30030 t^4 \cosh^4 x - 128700 t^2 \cosh^4 x \right] \\ &\quad + \frac{4}{2027025} t^5 \operatorname{sech}^9 x \left[ 100100 t^4 \cosh^2 x - 33696 t^6 \cosh^2 x + 7040 t^8 \right]. \end{aligned} \quad (26)$$

The approximate series solution is

$$u(x, t) = u_0 + u_1 + u_2 + u_3 + u_4 + \dots$$

Substituting  $u_0, u_1$  and  $u_2$  computed above, we get

$$\begin{aligned} u(x, t) &= 4t \operatorname{sech} x - \frac{4}{3} t^3 \operatorname{sech}^3 x + \frac{8}{15} t^5 \operatorname{sech}^3 x - \frac{64}{315} t^7 \operatorname{sech}^5 x \\ &\quad - \frac{8}{15} t^5 \operatorname{sech}^3 x + \frac{4}{5} t^5 \operatorname{sech}^5 x + \frac{168}{2835} t^9 \operatorname{sech}^5 x - \frac{16}{63} t^7 \operatorname{sech}^5 x \\ &\quad + \frac{16}{81} t^9 \operatorname{sech}^7 x - \frac{2304}{34650} t^{11} \operatorname{sech}^7 x + \frac{2048}{147420} t^{13} \operatorname{sech}^9 x. \end{aligned}$$

To obtain the closed form solution from above, let us only consider

$$\begin{aligned} u(x, t) &= 4t \operatorname{sech} x - \frac{4}{3} t^3 \operatorname{sech}^3 x + \frac{4}{5} t^5 \operatorname{sech}^5 x + \dots, \\ u(x, t) &= 4 \left[ t \operatorname{sech} x - \frac{[t \operatorname{sech} x]^3}{3} + \frac{[t \operatorname{sech} x]^5}{5} \dots \right], \end{aligned} \quad (27)$$

where

$$\arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} \dots$$

Then equation (27) becomes:

$$u(x, t) = 4 \arctan(t \operatorname{sech} x). \quad (28)$$

This closed form solution for equation (14) agree with the one obtained by reduced differential transform method [6].

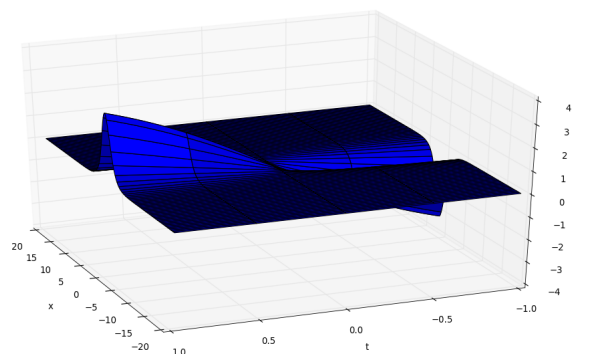


Fig. 1. The solution of the first sine-Gordon equation by ETM in Equation (14)

Figure 1 shows the shape of the solution to Example 4.1. The graph agrees with that obtained in [6] where reduced differential transform method is used.

**Example 4.2:** Consider the homogeneous Sine-Gordon Equation [6]

$$u_{tt} - u_{xx} + \sin u = 0, \quad (29)$$

with initial conditions

$$u(x, 0) = \pi + \alpha \cos(\beta x), \quad u_t(x, 0) = 0,$$

where  $\beta = \frac{\sqrt{2}}{2}$  and  $\alpha$  are constants.

To solve this problem, Taylor's series expansion of  $\sin u$  is used, that is,

$$\sin u \approx u - \frac{u^3}{3!} + \frac{u^5}{5!} = u - \frac{u^3}{6} + \frac{u^5}{120}.$$

Then equation (29) becomes:

$$u_{tt} - u_{xx} = - \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right]. \quad (30)$$

Applying the Elzaki transform to both sides of equation (30) gives:

$$E[u_{tt}] - E[u_{xx}] = -E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right], \quad (31)$$

where

$$E[u_{tt}] = \frac{U(x, v)}{v^2} - u(x, 0) - v u_t(x, 0), \text{ and}$$

$$E[u_{xx}] = \frac{d^2}{dx^2} [U(x, v)] = \frac{d^2}{dx^2} E[u].$$

So equation (31) becomes

$$\begin{aligned} \frac{U(x, v)}{v^2} - u(x, 0) - v u_t(x, 0) - \frac{d^2}{dx^2} E[u] \\ = -E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right]. \end{aligned} \quad (32)$$

Applying the given initial conditions to equation (32) and simplifying, gives

$$U(x, v) = v^2 [\pi + \alpha \cos(\beta x)] + v^2 \frac{d^2}{dx^2} E[u] - v^2 E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right]. \quad (33)$$

Applying the inverse Elzaki transform to equation (33) and simplifying, we obtain;

$$u(x, t) = [\pi + \alpha \cos(\beta x)] + E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u] - v^2 E \left[ u - \frac{u^3}{6} + \frac{u^5}{120} \right] \right\}. \quad (34)$$

From equation (34), let

$$u_0 = \pi + \alpha \cos(\beta x).$$

Then the recursive relation is given as:

$$u_{n+1} = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u_n] - v^2 E[A_n] \right\}, \quad (35)$$

where  $A_n$  is the Adomian polynomial to decompose the nonlinear terms by using the relation:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f \left[ \sum_{i=0}^{\infty} \lambda^i u_i \right]_{\lambda=0}. \quad (36)$$

Let the nonlinear term be represented by

$$f(u) = u - \frac{u^3}{6} + \frac{u^5}{120}. \quad (37)$$

By substituting equation (37) into equation (36), gives:

$$A_0 = u_0 - \frac{u_0^3}{6} + \frac{u_0^5}{120}, \quad A_1 = u_1 - \frac{1}{2} u_1 u_0^2 + \frac{1}{24} u_1 u_0^4, \\ A_2 = u_2 - \frac{1}{2} u_0 u_1^2 - \frac{1}{2} u_2 u_0^2 + \frac{1}{12} u_0^3 u_1^2 + \frac{1}{24} u_2 u_0^4, \quad \dots$$

From equation (35), when  $n=0$ , we get

$$u_1 = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u_0] - v^2 E[A_0] \right\}. \quad (38)$$

Since  $u_0 = \pi + \alpha \cos(\beta x)$ , we have:

$$u_1 = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[\pi + \alpha \cos(\beta x)] \right\} - E^{-1} \left\{ v^2 E \left[ [\pi + \alpha \cos(\beta x)] - \frac{1}{6} [\pi + \alpha \cos(\beta x)]^3 \right] \right\} - E^{-1} \left\{ v^2 E \left[ \frac{1}{120} [\pi + \alpha \cos(\beta x)]^5 \right] \right\}.$$

By simplifying, we obtain

$$u_1 = t^2 \left[ -\frac{1}{2} \alpha \beta^2 \cos(\beta x) - \frac{1}{2} \sin[\pi + \alpha \cos(\beta x)] \right].$$

From equation (35), when  $n=1$ , we get

$$u_2 = E^{-1} \left\{ v^2 \frac{d^2}{dx^2} E[u_1] - v^2 E[A_1] \right\}, \quad (39)$$

where

$$A_1 = u_1 \left[ 1 - \frac{1}{2} u_0^2 + \frac{1}{24} u_0^4 \right].$$

Since  $u_0 = \pi + \alpha \cos(\beta x)$ , we have

$$A_1 = \frac{1}{2} t^2 [-\alpha \beta^2 \cos(\beta x) \cos[\pi + \alpha \cos(\beta x)]] - \frac{1}{2} t^2 [\sin[\pi + \alpha \cos(\beta x)] \cos[\pi + \alpha \cos(\beta x)]]. \quad (40)$$

Then  $u_2$  is computed as:

$$u_2 = t^4 \left[ \frac{1}{24} \alpha \beta^4 \cos(\beta x) + \frac{1}{24} \alpha^2 \beta^2 \sin(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ -\frac{1}{24} \alpha^2 \beta^2 \cos^2(\beta x) \sin(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ \frac{1}{12} \alpha \beta^2 \cos(\beta x) \cos(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ \frac{1}{24} \sin(\pi + \alpha \cos(\beta x)) \cos(\pi + \alpha \cos(\beta x)) \right]. \quad (41)$$

The approximate analytical solution is given as:

$$u(x, t) = u_0 + u_1 + u_2 + \dots$$

Substituting  $u_0, u_1$  and  $u_2$  computed above, therefore:

$$u(x, t) = [\pi + \alpha \cos(\beta x)] + t^2 \left[ -\frac{1}{2} \alpha \beta^2 \cos(\beta x) - \frac{1}{2} \sin(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ \frac{1}{24} \alpha \beta^4 \cos(\beta x) + \frac{1}{24} \alpha^2 \beta^2 \sin(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ -\frac{1}{24} \alpha^2 \beta^2 \cos^2(\beta x) \sin(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ \frac{1}{12} \alpha \beta^2 \cos(\beta x) \cos(\pi + \alpha \cos(\beta x)) \right] + t^4 \left[ \frac{1}{24} \sin(\pi + \alpha \cos(\beta x)) \cos(\pi + \alpha \cos(\beta x)) \right]. \quad (42)$$

This series solution for equation (29) agree with the one obtained by reduced differential transform method [6].

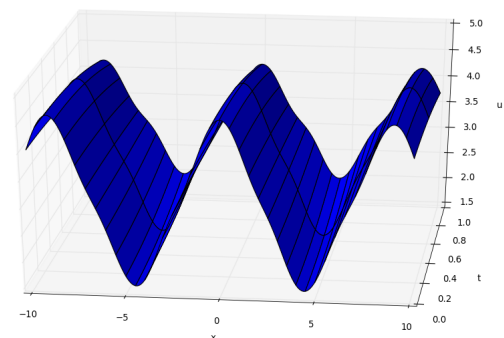


Fig. 2. The solution of the second sine-Gordon equation by ETM in Equation (29)

Figure 2 shows the shape of the solution to Example 4.2. The graph agrees with that obtained in [6] where reduced differential transform method is used.

**Example 4.3:** Consider a system of coupled Sine-Gordon Equations[6]

$$u_{tt} - u_{xx} = -\alpha^2 \sin[u(x, t) - v(x, t)], \\ v_{tt} - c^2 v_{xx} = \sin[u(x, t) - v(x, t)], \quad (43)$$

with initial conditions

$$u(x, 0) = A \cos(kx), \quad u_t(x, 0) = 0, \\ v(x, 0) = 0, \quad v_t(x, 0) = 0.$$

Applying Elzaki transform to both sides of equation (43) gives:

$$E[u_{tt}] - E[u_{xx}] = -\alpha^2 E[\sin[u(x, t) - v(x, t)]], \\ E[v_{tt}] - c^2 E[v_{xx}] = E[\sin[u(x, t) - v(x, t)]], \quad (44)$$

where

$$\begin{aligned} E[u_{tt}] &= \frac{U(x, w)}{w^2} - u(x, 0) - wu_t(x, 0), \\ E[u_{xx}] &= \frac{d^2}{dx^2}[U(x, w)] = \frac{d^2}{dx^2}E[u], \\ E[v_{tt}] &= \frac{V(x, w)}{w^2} - v(x, 0) - wv_t(x, 0), \\ E[v_{xx}] &= \frac{d^2}{dx^2}[V(x, w)] = \frac{d^2}{dx^2}E[v]. \end{aligned}$$

So equation (44) becomes,

$$\begin{aligned} \frac{U(x, w)}{w^2} - u(x, 0) - wu_t(x, 0) - \frac{d^2}{dx^2}E[u] &= -\alpha^2 E[\sin(u - v)], \\ \frac{V(x, w)}{w^2} - v(x, 0) - wv_t(x, 0) - c^2 \frac{d^2}{dx^2}E[v] &= E[\sin(u - v)]. \end{aligned} \quad (45)$$

Note that:

$$\sin[u - v] = [u - v] - \frac{[u - v]^3}{6} + \frac{[u - v]^5}{120}. \quad (46)$$

Applying equation (46) and the given initial conditions to equation (45) and simplifying gives

$$\begin{aligned} U(x, w) &= w^2[A \cos(kx)] + w^2 \frac{d^2}{dx^2}E[u] \\ &\quad - \alpha^2 w^2 E\left[[u - v] - \frac{[u - v]^3}{6} + \frac{[u - v]^5}{120}\right], \\ V(x, w) &= c^2 w^2 \frac{d^2}{dx^2}E[v] \\ &\quad + w^2 E\left[[u - v] - \frac{[u - v]^3}{6} + \frac{[u - v]^5}{120}\right]. \end{aligned} \quad (47)$$

Applying the inverse Elzaki transform to equation (47) gives:

$$\begin{aligned} u(x, t) &= A \cos(kx) + E^{-1} \left\{ w^2 \frac{d^2}{dx^2}E[u] \right\} \\ &\quad + E^{-1} \left\{ -\alpha^2 w^2 E\left[[u - v] - \frac{[u - v]^3}{6} + \frac{[u - v]^5}{120}\right] \right\}, \\ v(x, t) &= E^{-1} \left\{ c^2 w^2 \frac{d^2}{dx^2}E[v] \right\} \\ &\quad + E^{-1} \left\{ w^2 E\left[[u - v] - \frac{[u - v]^3}{6} + \frac{[u - v]^5}{120}\right] \right\}. \end{aligned} \quad (48)$$

In equation (48), let

$$\begin{aligned} u_0 &= A \cos(kx), \\ v_0 &= 0. \end{aligned}$$

Then the recursive relation is given as:

$$\begin{aligned} u_{n+1} &= E^{-1} \left\{ w^2 \frac{d^2}{dx^2}E[u_n] - \alpha^2 w^2 E[A_n] \right\}, \\ v_{n+1} &= E^{-1} \left\{ c^2 w^2 \frac{d^2}{dx^2}E[v_n] + w^2 E[A_n] \right\}, \end{aligned} \quad (49)$$

where  $A_n$  is the Adomian polynomial to decompose the nonlinear terms by using the relation:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f \left[ \sum_{i=0}^{\infty} \lambda^i (u_i, v_i) \right]_{\lambda=0}. \quad (50)$$

Let the nonlinear term be represented by

$$f(u, v) = [u - v] - \frac{[u - v]^3}{6} + \frac{[u - v]^5}{120}. \quad (51)$$

By substituting equation (51) in equation (50) gives

$$\begin{aligned} A_0 &= [u_0 - v_0] - \frac{[u_0 - v_0]^3}{6} + \frac{[u_0 - v_0]^5}{120}, \\ A_1 &= [u_1 - v_1] \left[ 1 - \frac{1}{2}[u_0 - v_0]^2 + \frac{1}{24}[u_0 - v_0]^4 \right], \quad \dots \end{aligned}$$

From equation (49), when  $n=0$ , we get:

$$\begin{aligned} u_1 &= E^{-1} \left\{ w^2 \frac{d^2}{dx^2}E[u_0] - \alpha^2 w^2 E[A_0] \right\}, \\ v_1 &= E^{-1} \left\{ c^2 w^2 \frac{d^2}{dx^2}E[v_0] + w^2 E[A_0] \right\}. \end{aligned} \quad (52)$$

Since  $u_0 = A \cos(kx)$  and  $v_0 = 0$ , we have:

$$\begin{aligned} u_1 &= E^{-1} \left\{ w^2 \frac{d^2}{dx^2}E[A \cos(kx)] \right\} \\ &\quad - \alpha^2 E^{-1} \left\{ w^2 E \left[ A \cos(kx) - \frac{A^3 \cos^3(kx)}{6} + \frac{A^5 \cos^5(kx)}{120} \right] \right\}, \end{aligned} \quad (53)$$

$$v_1 = E^{-1} \left\{ w^2 E \left[ A \cos(kx) - \frac{A^3 \cos^3(kx)}{6} + \frac{A^5 \cos^5(kx)}{120} \right] \right\}. \quad (54)$$

By simplifying equation (53) and equation (54), we obtain,

$$\begin{aligned} u_1 &= t^2 \left[ -\frac{1}{2} A k^2 \cos(kx) - \frac{1}{2} \alpha^2 \sin[A \cos(kx)] \right], \\ v_1 &= t^2 \left[ \frac{1}{2} \sin[A \cos(kx)] \right]. \end{aligned}$$

From equation (49), when  $n=1$ , we find that:

$$u_2 = E^{-1} \left\{ w^2 \frac{d^2}{dx^2}E[u_1] - \alpha^2 w^2 E[A_1] \right\}, \quad (55)$$

$$v_2 = E^{-1} \left\{ c^2 w^2 \frac{d^2}{dx^2}E[v_1] + w^2 E[A_1] \right\}, \quad (56)$$

where

$$A_1 = [u_1 - v_1] \left[ 1 - \frac{1}{2}[u_0 - v_0]^2 + \frac{1}{24}[u_0 - v_0]^4 \right].$$

Therefore:

$$\begin{aligned} A_1 &= t^2 \left[ -\frac{1}{2} A k^2 \cos(kx) \cos[A \cos(kx)] \right] \\ &\quad + t^2 \left[ -\frac{1}{2} \alpha^2 \sin[A \cos(kx)] \cos[A \cos(kx)] \right] \\ &\quad + t^2 \left[ -\frac{1}{2} \sin[A \cos(kx)] \cos[A \cos(kx)] \right]. \end{aligned} \quad (57)$$

Using equation (57) in equation (55) and in equation (56),  $u_2$  and  $v_2$  are computed as:

$$\begin{aligned} u_2 &= t^4 \left[ \frac{A k^4 \cos(kx)}{24} + \frac{\alpha^2 A^2 k^2 \sin[A \cos(kx)]}{24} \right] + \\ &\quad t^4 \left[ -\frac{\alpha^2 A^2 k^2 \cos^2(kx) \sin[A \cos(kx)]}{24} \right] + \\ &\quad t^4 \left[ \frac{\alpha^2 A k^2 \cos(kx) \cos[A \cos(kx)]}{12} \right] + \\ &\quad t^4 \left[ \frac{\alpha^4 \sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right] + \\ &\quad t^4 \left[ \frac{\alpha^2 \sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right]. \end{aligned} \quad (58)$$

$$\begin{aligned}
 v_2 = & t^4 \left[ -\frac{c^2 A^2 k^2 \sin[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ \frac{c^2 A^2 k^2 \cos^2(kx) \sin[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{c^2 A k^2 \cos(kx) \cos[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{A k^2 \cos(kx) \cos[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{\alpha^2 \sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{\sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right]. \quad (59)
 \end{aligned}$$

The approximate analytical solution is then given as:

$$\begin{aligned}
 u(x, t) &= u_0 + u_1 + u_2 + \dots \\
 v(x, t) &= v_0 + v_1 + v_2 + \dots
 \end{aligned}$$

Therefore, substituting  $u_0, v_0, u_1, v_1, u_2$  and  $v_2$  computed above, we have:

$$\begin{aligned}
 u(x, t) = & A \cos(kx) + t^2 \left[ -\frac{1}{2} A k^2 \cos(kx) - \frac{1}{2} \alpha^2 \sin[A \cos(kx)] \right] \\
 & + t^4 \left[ \frac{A k^4 \cos(kx)}{24} + \frac{\alpha^2 A^2 k^2 \sin[A \cos(kx)]}{24} \right] \\
 & + t^4 \left[ -\frac{\alpha^2 A^2 k^2 \cos^2(kx) \sin[A \cos(kx)]}{24} \right] \\
 & + t^4 \left[ \frac{\alpha^2 A k^2 \cos(kx) \cos[A \cos(kx)]}{12} \right] \\
 & + t^4 \left[ \frac{\alpha^4 \sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right] \\
 & + t^4 \left[ \frac{\alpha^2 \sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right]. \quad (60)
 \end{aligned}$$

This series solution  $u(x, t)$  in equation (43) agree with the one obtained by reduced differential transform method [6].

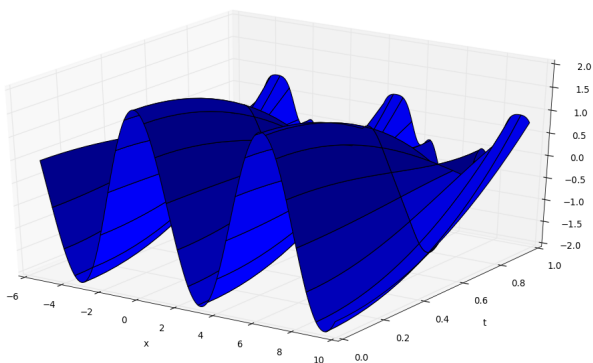


Fig. 3. The first solution of the coupled sine-Gordon equation by ETM in Equation (43)

Figure 3 shows the shape of the solution  $u(x, t)$  to Example 4.3. The graph agrees with that obtained in [6] where reduced

differential transform method is used.

$$\begin{aligned}
 v(x, t) = & t^2 \left[ \frac{1}{2} \sin[A \cos(kx)] \right] + \\
 & t^4 \left[ -\frac{c^2 A^2 k^2 \sin[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ \frac{c^2 A^2 k^2 \cos^2(kx) \sin[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{c^2 A k^2 \cos(kx) \cos[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{A k^2 \cos(kx) \cos[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{\alpha^2 \sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right] + \\
 & t^4 \left[ -\frac{\sin[A \cos(kx)] \cos[A \cos(kx)]}{24} \right]. \quad (61)
 \end{aligned}$$

The series solution  $v(x, t)$  in equation (43) agree with the one obtained by reduced differential transform method [6].

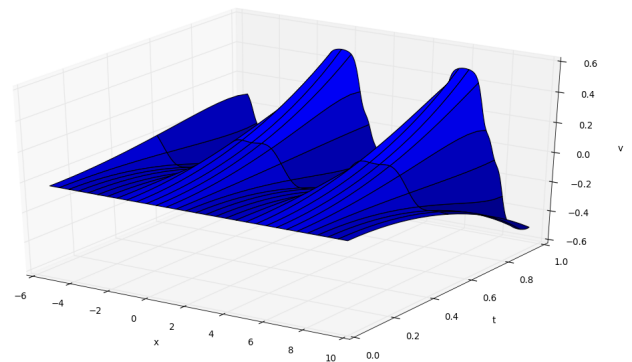


Fig. 4. The second solution of the coupled sine-Gordon equation by ETM in Equation (43)

Figure 4 shows the shape of the solution  $v(x, t)$  to Example 4.3. The graph agrees with that obtained in [6] where reduced differential transform method is used.

## V. DISCUSSION OF THE RESULTS

The scheme of the Elzaki transform has been effectively combined with Adomian polynomial to handle nonlinear partial differential equations. Example 4.1 is a single nonlinear Sine Gordon equation which was solved with the said method. The series solution obtained converges to the exact solution and this shows the power of the proposed method. Furthermore, example 4.2 is analogous to example 4.1 but it deals with a more complex initial condition where the solution obtained in this case when compared with that in [6] is found to be in agreement. Example 4.3 reveals that the method is also effective in solving coupled nonlinear partial differential equations as the result obtained in this example also agrees with that in the said reference.

Moreover, figures 1, 2, 3 and 4 show the graph of each of the equations considered so as to understand the behaviour/shape of each equation/system at any particular time and this could be interesting to the engineers in case of control analysis. Also, the solutions obtained may be significant for the explanation of some practical physical problems. The method has small computational size and is not affected by discretisation error as the solutions are presented in series form. Hence, the method of Elzaki transform combined with Adomian polynomial could be applied to solve nonlinear travelling wave equations.

## VI. CONCLUSION

We have analysed the approximate analytical solutions of different kinds of Sine Gordon equations using the combination of the

Elzaki transform method and the Adomian polynomial. The essence of obtaining the analytical solutions is to enable researchers to know the influence of each parameter on the equations under study. In conclusion, all the problems considered showed that the Elzaki transform method and the Adomian polynomial are very powerful integral transform methods in solving Sine Gordon equations. The solutions presented also agree with the solutions obtained when reduced differential transform method is used as provided in the reference. A three dimensional graph of all the problems considered were also plotted to give the shape of the solutions to Sine Gordon equations. A comparison with those given in the reference were made and they were found to agree. Solving nonlinear differential equations (whether partial or ordinary differential equations) is very easy by using this method.

## REFERENCES

- [1] M. Ablowitz, B. Herbst, and C. Schober, "On the numerical solution of the sine-gordon equation: I. integrable discretizations and homoclinic manifolds," *Journal of Computational Physics*, vol. 126, no. 2, pp. 299–314, 1996.
- [2] —, "On the numerical solution of the sine-gordon equation," *Journal of Computational Physics*, vol. 131, no. 2, pp. 354–367, 1997.
- [3] A. D. Polyanin and V. F. Zaitsev, *Handbook of nonlinear partial differential equations*. Chapman and Hall/CRC, 2016.
- [4] A.-M. Wazwaz, "A variable separated ode method for solving the triple sine-gordon and the triple sinh-gordon equations," *Chaos, Solitons & Fractals*, vol. 33, no. 2, pp. 703–710, 2007.
- [5] A. D. Polyanin and V. F. Zaitsev, *Handbook of nonlinear partial differential equations*. CRC press, 2004.
- [6] Y. Keskin, İ. Çağlar, and A. B. Koç, "Numerical solution of sine-gordon equation by reduced differential transform method," in *Proceedings of the World Congress on Engineering*, vol. 1, 2011, pp. 6–8.
- [7] A. Bekir and A. Boz, "Exact solutions for a class of nonlinear partial differential equations using exp-function method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 4, pp. 505–512, 2007.
- [8] Q. Wang, "An application of the modified adomian decomposition method for (n+ 1)-dimensional sine-gordon field," *Applied mathematics and computation*, vol. 181, no. 1, pp. 147–152, 2006.
- [9] S. S. Ray, "A numerical solution of the coupled sine-gordon equation using the modified decomposition method," *Applied mathematics and computation*, vol. 175, no. 2, pp. 1046–1054, 2006.
- [10] B. Batiha, M. S. M. Noorani, and I. Hashim, "Numerical solution of sine-gordon equation by variational iteration method," *Physics Letters A*, vol. 370, no. 5-6, pp. 437–440, 2007.
- [11] —, "Approximate analytical solution of the coupled sine-gordon equation using the variational iteration method," *Physica Scripta*, vol. 76, no. 5, p. 445, 2007.
- [12] A. Aslanov, "The homotopy-perturbation method for solving klein-gordon-type equations with unbounded right-hand side," *Zeitschrift für Naturforschung A*, vol. 64, no. 1-2, pp. 149–152, 2009.
- [13] A. Sadighi, D. Ganji, and B. Ganjavi, "Traveling wave solutions of the sine-gordon and the coupled sine-gordon equations using the homotopy-perturbation method," 2009.
- [14] M. Chowdhury and I. Hashim, "Application of homotopy-perturbation method to klein-gordon and sine-gordon equations," *Chaos, Solitons & Fractals*, vol. 39, no. 4, pp. 1928–1935, 2009.
- [15] N.-H. Kuo and C. Hu, "A study of the solutions of the combined sine-cosine-gordon equation," *Applied Mathematics and Computation*, vol. 215, no. 3, pp. 1015–1019, 2009.
- [16] T. M. Elzaki and S. A. Alkhateeb, "Modification of sumudu transform" elzaki transform" and adomian decomposition method," *Applied Mathematical Sciences*, vol. 9, no. 13, pp. 603–611, 2015.
- [17] T. M. Elzaki, "The new integral transform elzaki transform," *Global Journal of Pure and Applied Mathematics*, vol. 7, no. 1, pp. 57–64, 2011.
- [18] —, "Application of new transform elzaki transform to partial differential equations," *Global Journal of Pure and Applied Mathematics*, vol. 7, no. 1, pp. 65–70, 2011.
- [19] —, "On the connections between laplace and elzaki transforms," *Advances in Theoretical and Applied Mathematics*, vol. 6, no. 1, pp. 1–11, 2011.
- [20] —, "On the elzaki transform and ordinary differential equation with variable coefficients," *Advances in Theoretical and Applied Mathematics*, vol. 6, no. 1, pp. 13–18, 2011.
- [21] —, "A solution for nonlinear systems of differential equations using a mixture of elzaki transform and differential transform method," in *International Mathematical Forum*, vol. 7, no. 13, 2012, pp. 625–630.
- [22] S. M. El-Sayed, "The decomposition method for studying the klein-gordon equation," *Chaos, Solitons & Fractals*, vol. 18, no. 5, pp. 1025–1030, 2003.
- [23] T. M. Elzaki, E. M. Hilal, J.-S. Arabia, and J.-S. Arabia, "Homotopy perturbation and elzaki transform for solving nonlinear partial differential equations," *Mathematical Theory and Modeling*, vol. 2, no. 3, pp. 33–42, 2012.
- [24] D. Ziane and M. H. Cherif, "Resolution of nonlinear partial differential equations by elzaki transform decomposition method," *J. Appro. Theo. Appl. Math.*, vol. 5, pp. 17–30, 2015.
- [25] O. E. Ige, M. Heilio, R. A. Oderinu, and T. M. Elzaki, "Adomian polynomial and elzaki transform method of solving third order korteweg-de vries equations," *Global Journal of Pure and Applied Mathematics*, vol. 15, no. 3, pp. 261–277, 2019.