Folded Hypercubes with Cycles Embedding in Hybrid Conditionally Faulty

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***Abstract*—A network is defined as g-conditionally faulty if there are g fault-free neighbors is found in every vertex at least, where $g \geq 2$. An folded hypercube FQ_n with n-dimension, a famous variation of an n-dimensional hypercube Q_n, can be established from Q_n through putting in an edge to every pair of vertices which has complementary addresses. Let FF_v represents the faulty vertex set and FF_e represents the faulty edge set in FQ_n, respectively, and let $F_{FQ_n}(e)$ represents the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|FF_v| + |FF_e| \leq 2n - 7$. We prove the properties of embedding fault-tolerant cycles in $FQ_n - FF_v - FF_e$ as follows:**

1) **For $n \geq 4$ and $|F_{FQ_n}(e)| \leq n - 2$, $FQ_n - FF_v - FF_e$ consists of the fault-free cycle for every even length from 4 to $2^n - 2|FF_v|$;**
2) **For $n = 4$ and $n \geq 8$ where n is even, and $|F_{FQ_n}(e)| \leq n - 3$, $FQ_n - FF_v - FF_e$ consists of the fault-free cycle for every odd length from $n + 1$ to $2^n - 2|FF_v| - 1$.**

This study has been submitted to HAL an open archive for the sustainability (https://hal.archives-ouvertes.fr/hal-01579266v2).

***Index Terms*—conditionally faulty, fault-free, folded hypercubes, hypercubes, interconnection networks.**

I. Introduction

To choose an appropriate *interconnection network* (referred to as *network*) is one of significant works for the design in parallel computing and distributed systems. At present, many network topologies are presented in the literature [1], [2], [3]. The *hypercube* proposed by Bhuyan and Agrawal [4] is a famous network model with several outstanding characteristics including regularity, symmetry, low degree, short mean internode distance, small diameter, smaller edge complexity, and recursive structure. These characteristics are highly useful for the development and design of large-scale parallel or distributed systems [5]. Therefore, many variants of hypercube are presented including El-Amawy and Latifi [6], Esfahanian et al. [7], Chen et al. [8], and Preparata and Vuillemin [9]. The *folded hypercube* is one of the variants that has become a focus of research. Folded hypercube can be established from a hypercube through putting in an edge to every pair of vertices which has the longest distance, i.e., a pair of vertices has complementary addresses. It has been proved helpful for improving the performance of the system on conventional hypercube in numerous measurements, for examples, connectivity, diameter, faulty diameter, and many more. (Please refer to El-Amawy and Latifi [6], and Wang [10])

The ability of efficiently simulate algorithms for the design of other architectures is a major characteristic of an interconnection network. We can formulate such simulation as *network embedding*. Let G represents *guest network* and H represents *host network*. To embed a G into a H is defined as a one-to-one mapping f from the vertex set G to the vertex set H. Under f, an edge in G is corresponded to a path in H [5]. According to the embedding strategy, we can simulate the influence for a guest network on a host network. Therefore, we can develop the algorithms for a guest network and applied them to the host network.

Cycles (rings) are considered as the most basic networks available for parallel and distributed computation. When we want to design simple algorithms with low communication costs, cycles are suitable one. There are many valid algorithms designed on cycles to solve all kinds of algebra and graph problems [5], [11], [12], [13]. In arbitrary networks, cycles are able to be employed for distributed computing in control/data flow structures. These usages encourage the embedding of cycles for networks.

Because the vertices and/or edges in the network may be occasionally broken, the network's fault tolerance must be considered. The literature has shown a lot of studies for the issue of fault-tolerant cycle embedding in an n-dimensional folded hypercube FQ_n in [3], [10], [14], [15], [16], [17], [18], [19], [20], [21], [22]. Let FF_v represents the faulty vertex set and FF_e represents the faulty edge set in FQ_n, respectively. In 2001, Wang proposed that $FQ_n - FF_e$ consists of a Hamiltonian cycle of length 2^n if $|FF_e| \leq n-1$ [10]. In 2006, Xu and Ma presented that every edge of FQ_n lies on the cycle for every even length from 4 to 2^n; if n is even; every edge of FQ_n also lies on the cycle for every odd length from $n+1$ to $2^n - 1$ [23]. In addition, Xu et al. in 2006 stretched his result as aforementioned to show that every fault-free edge of $FQ_n - FF_e$ lies on the cycle for every even length from 4 to 2^n; if n is even, every fault-free edge of $FQ_n - FF_e$ also lies on the cycle for every odd length from $n+1$ to $2^n - 1$, where $|FF_e| \leq n-1$ [22]. Let $f \in FF_v$ be any faulty vertex in FQ_n. Hsieh et al. in 2009 presented that $FQ_n - \{f\}$ consists of the fault-free cycle for every even length from 4 to $2^n - 2$ if $n \geq 3$, and if $n \geq 2$ is even, $FQ_n - \{f\}$ consists of the fault-free cycle of every odd length from $n+1$ to $2^n - 1$ [18]. Furthermore, Cheng et al. in 2013 presented that every fault-free edge of $FQ_n - \{f\}$ lies on the cycle for every odd length from $n+1$ to $2^n - 3$ for $n \geq 2$ where n is even [14]. Kuo in 2015 spread Cheng et al.s result [14] spread to get that every fault-free edge of

Manuscript received August 20, 2018; revised February 17, 2019; revised May 12, 2019. This work was supported in part by the Ministry of Science and Technology (MOST) in Taiwan under grant MOST107-2622-E-324-002-CC3, MOST107-2221-E-324-020, MOST107-2821-C-324-001-ES, MOST107-2218-E-005-023, and the Chaoyang University of Technology (CYUT) and Higher Education Sprout Project, Ministry of Education, Taiwan, under the project name: "The R&D and the cultivation of talent for Health-Enhancement Products."

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(Advance online publication: 12 August 2019)

$FQ_n - \{f\}$ lies on a cycle for every even length from 4 to $2^n - 2$ if $n \geq 3$, and if $n \geq 2$ is even, every fault-free edge of $FQ_n - \{f\}$ also lies on the cycle for every odd length from $n+1$ to $2^n - 1$ [3]. However, the independent reliability is owned by each component in a network. If a component of a network is independently broken, the probability is low for all breakdowns. Due to this reason, Harary in 1983 first presented the opinion of *conditional connectivity* [24]. Subsequently, Latifi et al. in 1994 determined the *conditional vertex-faults* which requires that each vertex of a network contains at least g fault-free neighbors, $g \geq 2$ [25]. For this thesis, we focus on $g = 4$ and define that a network is 4-conditionally faulty if its every vertex contains at least four fault-free neighbors. Let $F_{FQ_n}(e)$ represents the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|FF_v| + |FF_e| \leq 2n - 7$. We prove the properties of embedding fault-tolerant cycles in $FQ_n - FF_v - FF_e$, as follows:

1) For $n \geq 4$ and $|F_{FQ_n}(e)| \leq n-2$, $FQ_n - FF_v - FF_e$ consists of the fault-free cycle for every even length from 4 to $2^n - 2|FF_v|$;
2) For $n = 4$ and $n \geq 8$ is even, and $|F_{FQ_n}(e)| \leq n-3$, $FQ_n - FF_v - FF_e$ consists for a fault-free cycle of every odd length from $n+1$ to $2^n - 2|FF_v| - 1$.

Please note, the terms of network, node, and edge is interchangeable for graph, vertex, and link, respectively used throughout this paper. The following gives the organization of remainder for this paper. Some necessary definitions and notations are presented in Section II. The major result is shown in Section III. In the last, concluding remarks are concluded in Section IV.

II. Preliminaries

Let a graph is defined as $G = (V, E)$. $G = (V, E)$ is an ordered pair which V is the *vertex set* and is a finite set, and E is the *edge set* and is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. The *vertex set* and the *edge set* can be also represents $V(G)$ and $E(G)$, respectively. When $(u, v) \in E$, the vertices u and v are *adjacent*. For the edge $e = (u, v)$, u and v are called the *end-vertices* of e. We call u adjacent to v, and vice versa. A graph $G = (V_0 \cup V_1, E)$ is bipartite if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(x, y)|x \in V_0$ and $y \in V_1\}$. A path $P[v_0, v_k] = \langle v_0, v_1, \ldots, v_k \rangle$ is a sequence of different vertices with any two follow-up vertices are adjacent. v_0 and v_k are called as the *end-vertices* of the path. Furthermore, a *subpath* may be involved by a path, represented as $\langle v_0, v_1, \ldots, v_i, P[v_i, v_j], v_j, v_{j+1}, \ldots, v_k \rangle$, where $P[v_i, v_j] = \langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j \rangle$. The number of edges on the path represents the length of the path. When $v_0 = v_k$ and $v_0, v_1, \ldots, v_{k-1}$ are different, a path $\langle v_0, v_1, \ldots, v_k \rangle$ forms a *cycle*. A vertex is thought *fault-free* if it is not faulty. An edge is thought *fault-free* if the two end-vertices and their edge are not faulty. Vertex u is a *fault-free* neighbor of v if u and (u, v) are not faulty. A path (cycle) is *fault-free* if it has no faulty edges and faulty vertices. The faulty vertex and/or faulty edge set incident to the end-vertices of any edge $e \in E(G)$ can be denoted as $F_G(e)$. Other graph-theoretic terminologies and notations are not described here can refer to West et al. in 2001 [26].

An *n-dimensional hypercube* Q_n (n-cube for short) is denoted as an undirected graph. $V(Q_n)$ contains 2^n vertices labelled as binary strings of length n. Each edge $e = (u, v) \in E(Q_n)$ connects two vertices u and v if and only if u and v differ in exactly one bit of their labels, i.e., $u = b_n b_{n-1} \ldots b_k \ldots b_1$ and $v = b_n b_{n-1} \ldots \overline{b_k} \ldots b_1$, where $\overline{b_k}$ is the *one's complement* of b_k, i.e., $\overline{b_k} = 1 - i$ if and only if $b_k = i$ for $i \in \{0, 1\}$. e is called as an edge of *dimension* k. Obviously, each vertex connects to exactly n other vertices. Furthermore, it exists 2^{n-1} edges in each dimension and $|E(Q_n)| = n \cdot 2^{n-1}$.

Let $x = x_n x_{n-1} \ldots x_1$ and $y = y_n y_{n-1} \ldots y_1$ be two n-bit binary strings; and let $y = x^{(k)}$, where $1 \leq k \leq n$, if $y_k = 1 - x_k$ and $y_i = x_i$ for all $i \neq k$, $1 \leq i \leq n$. In addition, let $y = \bar{x}$ if $y_i = 1 - x_i$ for all $1 \leq i \leq n$. The *Hamming distance* $d_H(x, y)$ between vertex x and vertex y is the number of different bits in the corresponding strings of the vertices. The *Hamming weight* $hw(x)$ of x is the number of i's such that $x_i = 1$. Note that Q_n is a bipartite graph with two partite sets $\{x|\ hw(x)$ is odd$\}$ and $\{x|\ hw(x)$ is even$\}$. Let $d_{Q_n}(x, y)$ be the *distance* between vertex x and vertex y in Q_n. Clearly, $d_{Q_n}(x, y) = d_H(x, y)$.

An *n-dimensional folded hypercube* FQ_n can be established from an n-cube by putting in an *complementary edge* to every pair of vertices which has the longest distance, i.e., for a vertex whose address is $b = b_n b_{n-1} \ldots b_1$, it now has one more edge to vertex $\bar{b} = \bar{b}_n \bar{b}_{n-1} \ldots \bar{b}_1$, except its original n edges. Thus, FQ_n has 2^{n-1} more edges than Q_n. These augmented edges *skips* are represented as E_s. So the complete edge set of a folded hypercube $E(FQ_n)$ can be represented as $E(Q_n) \cup E_s$. Therefore, the edges of FQ_n can be formally defined as that $E(FQ_n) = E(Q_n) \cup E_s = \{e = (u, v)|d_H(u, v) = 1 \in E(Q_n)$ or $d_H(u, v) = n \in E_s\}$. It has been indicated that FQ_n is $(n+1)$-regular, $(n+1)$-connected, vertex-transitive, and edge-transitive in Xu et al. [22]. Furthermore, FQ_n has been indicated that for any odd $n \geq 3$ is bipartite in Lewinter and Widulski [27].

For convenience, FQ_n can be denoted as $\underbrace{**\ldots**}_{n} = *^n$, where $* \in \{0, 1\}$ means the "*don't care*" symbol. A regular hypercube Q_n can be partitioned into two subcubes Q_{n-1} along dimension i, where $1 \leq i \leq n$. The subcubes are defined as $Q^0_{n-1} = *^{n-i}0*^{i-1}$ and $Q^1_{n-1} = *^{n-i}1*^{i-1}$, in which the values of the ith bits of the vertices are 0 and 1, respectively. Formally, Q^0_{n-1}(respectively, Q^1_{n-1}) is a subgraph of FQ_n induced by $\{x_n \ldots x_i \ldots x_1 \in V(FQ_n)|\ x_i = 0\}$(respectively, $\{x_n \ldots x_i \ldots x_1 \in V(FQ_n)|\ x_i = 1\}$).

Definition 1: [28] An *i-partition on* $FQ_n = *^n$, where $1 \leq i \leq n$, partitions FQ_n along dimension i into two $(n-1)$-cubes $*^{n-i}0*^{i-1}$ (Q^0_{n-1}) and $*^{n-i}1*^{i-1}$ (Q^1_{n-1}). Furthermore, all edges in E_s are between Q^0_{n-1} and Q^1_{n-1}.

Let F_v(respectively, FF_v) and F_e(respectively, FF_e) represent the faulty vertex set and the faulty edge set in Q_n(respectively, FQ_n). By Definition 1, if we perform an i-partition on FQ_n to form two $(n-1)$-cubes Q^0_{n-1} and Q^1_{n-1}, we derived that $F^0_v = FF_v \cap V(Q^0_{n-1})$, $F^1_v = FF_v \cap V(Q^1_{n-1})$, $F^0_e = FF_e \cap E(Q^0_{n-1})$ and $F^1_e = FF_e \cap E(Q^1_{n-1})$. Finally, some previously results of path (cycle) embedding in hypercubes and folded hypercubes are considered in the remainder of this section. These results are beneficial for our method.

Lemma 1: Saad and Schultz in 1988 [29] Let u and v be any two vertices in Q_n and $d_{Q_n}(u,v) = d$. Then, there exist n internally disjoint paths joining u and v in Q_n, where d paths of them are of length d and lie in a d-dimensional subcube.

Lemma 2: Ma et al. in 2007 [30] Let u and v be any two fault-free vertices in Q_n. Then, $Q_n - F_v - F_e$ contains a fault-free path of every length l with $d_{Q_n}(u,v)+2 \le l \le 2^n - 2|F_v| - 1$ and $2|(l - d_{Q_n}(u,v))$, where $|F_v| + |F_e| \le n-2$ and $n \ge 3$.

Lemma 3: Xu and Ma in 2006 [23] For $n \ge 3$, every edge of FQ_n lies on a cycle of every even length from 4 to 2^n; and for $n \ge 2$ is even, every edge of FQ_n lies on a cycle of every odd length from $n+1$ to $2^n - 1$.

Lemma 4: Hsieh et al. in 2009 [18] For $n \ge 3$, $FQ_n - FF_v$ contains a fault-free cycle of every even length from 4 to $2^n - 2$; and for $n \ge 2$ is even, $FQ_n - FF_v$ contains a fault-free cycle of every odd length from $n+1$ to $2^n - 1$, where $|FF_v| = 1$.

Lemma 5: Xu et al. in 2006 [22] For $n \ge 3$, every edge of $FQ_n - FF_e$ lies on a fault-free cycle of every even length from 4 to 2^n; and for $n \ge 2$ is even, every edge of $FQ_n - FF_e$ lies on a fault-free cycle of every odd length from $n+1$ to $2^n - 1$, where $|FF_e| \le n-1$.

Lemma 6: Cheng and Guo in 2013 [31] Let $F_{Q_n}(e)$ denote the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(Q_n)$. Suppose that Q_n is 3-conditionally faulty and $|F_{Q_n}(e)| \le n-2$. Then, every fault-free edge of $Q_n - F_v - F_e$ lies on a cycle of every even length from 4 to $2^n - 2|F_v|$ if $|F_v| + |F_e| \le 2n-7$, where $n \ge 5$.

Lemma 7: Suppose that Q_n is 3-conditionally faulty and $|F_v| \le 2n-7$, where $n \ge 7$. Then, Q_n can be partitioned along some dimension $j \in \{1,2,\ldots,n\}$ to form two $(n-1)$-cubes Q^0_{n-1} and Q^1_{n-1} such that both Q^0_{n-1} and Q^1_{n-1} are 2-conditionally faulty with $|F^0_v| \le 2n-9$ and $|F^1_v| \le 2n-9$. The proof of Lemma 7 is given in the Appendix section.

Lemma 8: Suppose that FQ_n is 4-conditionally faulty and $|FF_v| \le 2n-7$, where $n \ge 8$. Then, FQ_n can be partitioned along some dimension $j \in \{1,2,\ldots,n\}$ to form two $(n-1)$-cubes Q^0_{n-1} and Q^1_{n-1} such that both Q^0_{n-1} and Q^1_{n-1} are 3-conditionally faulty, $|F^0_v| \le 2n-9$ and $|F^1_v| \le 2n-9$.

Proof: According to the definition of FQ_n, $E(FQ_n) = E(Q_n) \cup E_s$ and $V(FQ_n) = V(Q_n)$. If we eliminate all edges in E_s, then $FQ_n - E_s \cong Q_n$. Since FQ_n is 4-conditionally faulty, $FQ_n - E_s$ would be certainly 3-conditionally faulty. By Lemma 7, $FQ_n - E_s \cong Q_n$ can be partitioned along some dimension $j \in \{1,2,\ldots,n\}$ to form two $(n-1)$-cubes Q^0_{n-1} and Q^1_{n-1} such that both Q^0_{n-1} and Q^1_{n-1} are 3-conditionally faulty, $|F^0_v| \le 2n-9$ and $|F^1_v| \le 2n-9$. Then, the lemma holds. ■

III. Cycles Embedding in a Faulty Folded Hypercube

Let $F_{FQ_n}(e)$ represent the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty, we show that

1) For $n \ge 4$, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length l with $4 \le l \le 2^n - 2|FF_v|$, where $|FF_v| + |FF_e| \le 2n-7$ and $|F_{FQ_n}(e)| \le n-2$;
2) For $n = 4$ and $n \ge 8$ is even, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every odd length l with $n+1 \le l \le 2^n - 2|FF_v| - 1$, where $|FF_v| + |FF_e| \le 2n-7$ and $|F_{FQ_n}(e)| \le n-3$.

Lemma 9: Suppose that FQ_n is 4-conditionally faulty, $|FF_v| + |FF_e| \le 2n-7$ and $|F_{FQ_n}(e)| \le n-2$ for $n \ge 4$. Then, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length l with $4 \le l \le 2^n - 2|FF_v|$.

Proof: The cases for $n = 4$ and $n \ge 5$ are considered.

CASE 1.$n = 4$. In this case, $|FF_v| + |FF_e| \le 1$. If $|FF_v| = |FF_e| = 0$, by Lemma 3, FQ_4 contains a cycle of every even length l with $4 \le l \le 16$. If $|FF_v| = 1$ and $|FF_e| = 0$, by Lemma 4, $FQ_4 - FF_v$ contains a fault-free cycle of every even length l with $4 \le l \le 14$. If $|FF_v| = 0$ and $|FF_e| = 1$, by Lemma 5, $FQ_4 - FF_e$ contains a fault-free cycle of every even length l with $4 \le l \le 16$.

CASE 2. $n \ge 5$. According to the definition of FQ_n, $E(FQ_n) = E(Q_n) \cup E_s$ and $V(FQ_n) = V(Q_n)$. If we eliminate all edges in E_s, then $FQ_n - E_s \cong Q_n$. Note that FQ_n is 4-conditionally faulty, $FQ_n - E_s \cong Q_n$ would be certainly 3-conditionally faulty. Since $|F_v| + |F_e| \le |FF_v| + |FF_e| \le 2n-7$ and $|F_{Q_n}(e)| \le |F_{FQ_n}(e)| \le n-2$, by Lemma 6, $Q_n - F_v - F_e$ contains a fault-free cycle of every even length l with $4 \le l \le 2^n - 2|F_v|$, which implies that $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length l with $4 \le l \le 2^n - 2|FF_v|$.

By integrating the above two cases, the proof is completed. ■

Lemma 10: Suppose that FQ_n is 4-conditionally faulty, $|FF_v| + |FF_e| \le 2n-7$ and $|F_{FQ_n}(e)| \le n-3$ for $n = 4$ and $n \ge 8$ is even. Then, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every odd length l with $n+1 \le l \le 2^n - 2|FF_v| - 1$.

Proof: The cases for $n = 4$ and $n \ge 8$ is even are considered.

CASE 1. $n = 4$. In this case, $|FF_v| + |FF_e| \le 1$. If $|FF_v| = |FF_e| = 0$, by Lemma 3, FQ_4 contains a cycle of every odd length l with $5 \le l \le 15$. If $|FF_v| = 1$ and $|FF_e| = 0$, by Lemma 4, $FQ_4 - FF_v$ contains a fault-free cycle of every odd length l with $5 \le l \le 13$. If $|FF_v| = 0$ and $|FF_e| = 1$, by Lemma 5, $FQ_4 - FF_e$ contains a fault-free cycle of every odd length l with $5 \le l \le 15$.

CASE 2.$n \ge 8$ is even. If we assume that every faulty edge e in FF_e is regarded as one of the end-vertices of e is faulty, then $|FF_{v^+}| = |FF_v| + |FF_e| \le 2n-7$ in the worst case. Since $|FF_{v^+}| \le 2n-7$ and FQ_n is 4-conditionally faulty, by Lemma 8, FQ_n can be partitioned along some dimension $j \in \{1,2,\ldots,n\}$ to form two $(n-1)$-cubes Q^0_{n-1} and Q^1_{n-1} such that both Q^0_{n-1} and Q^1_{n-1} are 3-conditionally faulty, $|F^0_{v^+}| \le 2n-9$ and $|F^1_{v^+}| \le 2n-9$ which implies that $|F^0_v| + |F^0_e| \le 2n-9$ and $|F^1_v| + |F^1_e| \le 2n-9$, respectively. Without loss of generality, we may assume that $j = n$ and $|F^0_v| + |F^0_e| \ge |F^1_v| + |F^1_e|$. Therefore, $|F^0_v| + |F^0_e| \le 2n-9$ and $|F^1_v| + |F^1_e| \le n-4$. Since $|F_{Q_n}(e)| \le |F_{FQ_n}(e)| \le n-3$, we know

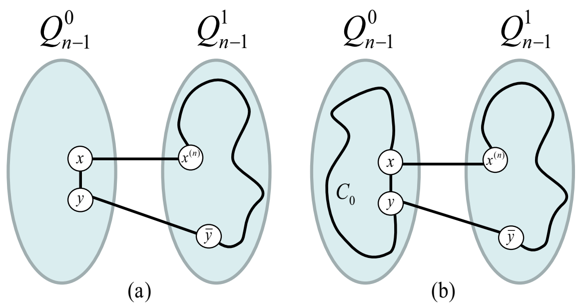

Fig. 1. An illustration of Case 2 in the proof of Lemma 10. (a) Case 2.2; (b) Case 2.3

that $|F_{Q^0_{n-1}}(e)| \leq n-3$ and $|F_{Q^1_{n-1}}(e)| \leq n-3$. Then, let (x, y) be any fault-free edge in Q^0_{n-1} such that either $\{x^{(n)}, \bar{y}\}$ or $\{\bar{x}, y^{(n)}\}$ is fault-free in Q^1_{n-1}. (If no such an edge exists, then $|FF_v| + |FF_e| \geq [2^{n-1}/2] = 2^{n-2} > 2n-7$ for $n \geq 8$ is even, which contradicts to the assumption that $|FF_v| + |FF_e| \leq 2n-7$.) Without loss of generality, we may assume that $\{x^{(n)}, \bar{y}\}$ in Q^1_{n-1} is fault-free. Then, we consider the cycle of every odd length l with $n+1 \leq l \leq 2^n - 2|FF_v| - 1$ in the following subcases.

Case 2.1. $l = n+1$. In Q^1_{n-1}, since $|F^1_v| + |F^1_e| \leq n-4$ and $d_H(x^{(n)}, \bar{y}) = n-2$, by Lemma 1, there exists a fault-free path $P[x^{(n)}, \bar{y}]$ of length $n-2$. Then, $\langle x, x^{(n)}, P[x^{(n)}, \bar{y}], \bar{y}, y, x\rangle$ forms a cycle of odd length $l = n+1$ in $FQ_n - FF_v - FF_e$.

Case 2.2. $l = n+3$. In Q^1_{n-1}, since $|F^1_v| + |F^1_e| \leq n-4$ and $d_H(x^{(n)}, \bar{y}) = n-2$, by Lemma 2, there exists a fault-free path $P[x^{(n)}, \bar{y}]$ of length $n-2+2 = n$. Then, $\langle x, x^{(n)}, P[x^{(n)}, \bar{y}], \bar{y}, y, x\rangle$ forms a cycle of odd length $l = n+3$ in $FQ_n - FF_v - FF_e$. (see Fig. 1(a))

Case 2.3. $n+5 \leq l \leq 2^n - 2|FF_v| - 1$. Since $|F^0_v| + |F^0_e| \leq 2n-9$, $|F_{Q^0_{n-1}}(e)| \leq n-3$, and Q^0_{n-1} is 3-conditionally faulty, by Lemma 6, (x, y) can lies on a fault-free cycle C_0 of every even length from 4 to $2^{n-1} - 2|F^0_v|$ in Q^0_{n-1}. Then, C_0 can be denoted as $\langle x, y, P[y, x], x\rangle$. Furthermore, since $|F^1_v| + |F^1_e| \leq n-4$ and $d_H(x^{(n)}, \bar{y}) = n-2$, by Lemma 2, there exists a fault-free path $P[x^{(n)}, \bar{y}]$ of every even length from $n-2+2 = n$ to $2^{n-1} - 2|F^1_v| - 2$ in Q^1_{n-1}. Therefore, $\langle x, x^{(n)}, P[x^{(n)}, \bar{y}], \bar{y}, y, P[y, x], x\rangle$ forms a cycle of every odd length l with $4-1+2+n \leq l \leq (2^{n-1} - 2|F^0_v|) - 1 + 2 + (2^{n-1} - 2|F^1_v| - 2)$ which implies $n+5 \leq l \leq 2^n - 2|FF_v| - 1$ in $FQ_n - FF_v - FF_e$. (see Fig. 1(b))

By integrating the above cases, the proof is completed. ■

By Lemmas 9 and 10, the following theorem is obtained.

Theorem 1: Let $F_{FQ_n}(e)$ denote the faulty vertex and/or faulty edge set which is incident to the end-vertices of any edge $e \in E(FQ_n)$. Suppose that FQ_n is 4-conditionally faulty and $|FF_v| + |FF_e| \leq 2n-7$. Then, for $n \geq 4$ and $|F_{FQ_n}(e)| \leq n-2$, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$; and furthermore, for $n = 4$ and $n \geq 8$ is even, and $|F_{FQ_n}(e)| \leq n-3$, $FQ_n - FF_v - FF_e$ also contains a fault-free cycle of every odd length from $n+1$ to $2^n - 2|FF_v| - 1$.

IV. Conclusion

Fault tolerance is one of the important research topics in the field of multi-process computer systems. Many studies focus on vertex-fault-tolerant or edge-fault-tolerant properties for certain specific networks. In this thesis, the 4-conditionally faulty folded hypercube with $|FF_v| + |FF_e| \leq 2n-7$ is considered. Then, $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every even length from 4 to $2^n - 2|FF_v|$ for $n \geq 4$ and $|F_{FQ_n}(e)| \leq n-2$; and $FQ_n - FF_v - FF_e$ contains a fault-free cycle of every odd length from $n+1$ to $2^n - 2|FF_v| - 1$ for $n = 4$ and $n \geq 8$ is even, and $|F_{FQ_n}(e)| \leq n-3$ are shown. Finally, this study has been submitted to HAL an open archive for the sustainability (https://hal.archives-ouvertes.fr/hal-01579266v2).

Appendix

Proof of Lemma 7

Proof: Since Q_n is 3-conditionally faulty, both Q^0_{n-1} and Q^1_{n-1} are 2-conditionally faulty. First, we consider the case that $|F_v| \leq 2n-8$. Let x and y be two faulty vertices, and let $j \in \{1, 2, \ldots, n\}$ such that $[x]_j \neq [y]_j$. Then we can partition Q_n along dimension j into two $(n-1)$-cubes $Q^{j:0}_n$ and $Q^{j:1}_n$ such that $|F^0_v| = |F_v(Q^{j:0}_n)| \leq 2n-9$ and $|F^1_v| = |F_v(Q^{j:1}_n)| \leq 2n-9$.

Next we consider the remaining case that $|F_v| = 2n-7$. For $n \geq 7$, we will show that we can partition Q_n along some dimension j into two $(n-1)$-cubes $Q^{j:0}_n$ and $Q^{j:1}_n$ such that $|F_v(Q^{j:0}_n)| \leq 2n-9$ and $|F_v(Q^{j:1}_n)| \leq 2n-9$. For $1 \leq k \leq n$, we define $q_k = 1$ if $[u]_k = [w]_k$ for any two faulty vertices $u, w \in F_v(Q_n)$, and $q_k = 0$ if otherwise. Let $q = \sum^n_{k=1} q_k$. Clearly, all faulty vertices are located in either $Q^{k:0}_n$ or $Q^{k:1}_n$ if $q_k = 1$. For convenience, let $\{1 \leq k \leq n | \; q_k = 0\} = \{i_1, \ldots, i_{n-q}\}$. Then both $|F_v(Q^{j:0}_n)| \geq 1$ and $|F_v(Q^{j:1}_n)| \geq 1$ for each $j \in \{i_1, \ldots, i_{n-q}\}$.

Suppose, by contradiction, that either $|F_v(Q^{j:0}_n)| = 1$ or $|F_v(Q^{j:1}_n)| = 1$ for every $j \in \{i_1, \ldots, i_{n-q}\}$. For $u \in F_v(Q_n)$, let $A(u) = \{1 \leq k \leq n | \; F_v(Q^{k:0}_n) = \{u\}$ or $F_v(Q^{k:1}_n) = \{u\}\}$. Since Q_n is vertex-transitive, we may assume that $\mathbf{e} = 0^n$ is a faulty vertex such that $|A(\mathbf{e})|$ achieves the maximum of the set $\{A(u) | \; u \in F_v(Q_n)\}$. For convenience, let $p = |A(\mathbf{e})|$. Obviously, we have $1 \leq p \leq n-q$. Moreover, let $A(\mathbf{e}) = \{i_1, \ldots, i_p\}$. For $u \in F_v(Q_n) - \{\mathbf{e}\}$, we see that $[u]_k = 1$ for each $k \in \{i_1, \ldots, i_p\}$. Further, let $B(k) = \{u \in F_v(Q_n) - \{\mathbf{e}\} | \; [u]_k \neq [\mathbf{e}]_k\}$ for $k \in \{i_{p+1}, \ldots, i_{n-q}\}$. Since we assumed, by contradiction, that either $|F_v(Q^{j:0}_n)| = 1$ or $|F_v(Q^{j:1}_n)| = 1$ for each $j \in \{i_1, \ldots, i_{n-q}\}$, we have $|B(j)| = 1$ for each $j \in \{i_{p+1}, \ldots, i_{n-q}\}$. Since Q_n is edge-transitive, without loss of generality we can assume that $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$ and $\{i_{p+1}, \ldots, i_{n-q}\} = \{p+1, \ldots, n-q\}$. Then we have $\left[(F_v(Q_n) - \{\mathbf{e}\}) - \bigcup_{k \in \{i_{p+1}, \ldots, i_{n-q}\}} B(k)\right] \subseteq$

$\{0^{n-p}1^p\}$. Accordingly, we obtain that $1 = |\{0^{n-p}1^p\}| \geq \left|(F_v(Q_n) - \{\mathbf{e}\}) - \bigcup_{k\in\{i_{p+1},\ldots,i_{n-q}\}} B(k)\right| \geq |F_v(Q_n)| - |\{\mathbf{e}\}| - \sum_{k\in\{i_{p+1},\ldots,i_{n-q}\}} |B(k)| = (2n-7)-1-(n-q-p)$ which implies $p+q \leq 9-n$. Recall that $n \geq 7$, $p \geq 1$ and $q \geq 0$. We thus have $n = 7$ or 8. Now we can identify all faulty vertices according to the values of p, q, and n.

Case 1: $(n,p,q) = (7,1,0)$.
We have $|F_v(Q_8)| = 2\cdot 7 - 7 = 7$, $[u]_1 = 1$ for each $u \in F_v(Q_7) - \{\mathbf{e}\}$ and $|B(j)| = 1$ for each $j \in \{2,3,4,5,6,7\}$. Thus we have $F(Q_7) = \{0000000, 0000011, 0000101, 0001001, 0010001, 0100001, 1000001\}$. Clearly, vertex 0000001 has seven faulty neighbors.

Case 2: $(n,p,q) = (7,1,1)$.
We have $[u]_1 = 1$ for each $u \in F_v(Q_7) - \{\mathbf{e}\}$ and $|B(j)| = 1$ for each $j \in \{2,3,4,5,6\}$. Thus we have $F(Q_7) = \{0000000, 0000011, 0000101, 0001001, 0010001, 0100001, 0000001\}$. Then vertex 0000001 has six faulty neighbors.

Case 3: $(n,p,q) = (7,2,0)$.
We have $[u]_1 = [u]_2 = 1$ for each $u \in F_v(Q_7) - \{\mathbf{e}\}$ and $|B(j)| = 1$ for each $j \in \{3,4,5,6,7\}$. Thus we have $F_v(Q_7) = \{0000000, 0000111, 0001011, 0010011, 0100011, 1000011, 0000011\}$. Then vertex 0000011 has five faulty neighbors.

Case 4: $(n,p,q) = (8,1,0)$.
We have $|F_v(Q_8)| = 2\cdot 8 - 7 = 9$, $[u]_1 = 1$ for each $u \in F_v(Q_8) - \{\mathbf{e}\}$ and $|B(j)| = 1$ for each $j \in \{2,3,4,5,6,7,8\}$. Thus we have $F_v(Q_8) = \{00000000, 00000011, 00000101, 00001001, 00010001, 00100001, 01000001, 10000001, 00000001\}$. Then vertex 00000001 has eight faulty neighbors.

In short, vertex $0^{n-p}1^p$ has at least $n-2$ faulty neighbors. This contradicts the requirement that every vertex has at least three faulty neighbors. Hence we can partition Q_n along some dimension j into two $(n-1)$-cubes $Q_n^{j:0}$ and $Q_n^{j:1}$ such that both $Q_n^{j:0}$ and $Q_n^{j:1}$ are conditional faulty, $|F(Q_n^{j:0})| \leq 2n-9$ and $|F(Q_n^{j:1})| \leq 2n-9$. ∎

ACKNOWLEDGMENT

The authors are grateful to the reviewers for their helpful comments and suggestions to improve the presentations and contributions of the paper. In addition, the author is also grateful to the editors and assistants for their hard work in dealing with this paper. Finally, this work was supported in part by the Ministry of Science and Technology (MOST) in Taiwan under grant MOST107-2622-E-324-002-CC3, MOST107-2221-E-324-020, MOST107-2821-C-324-001-ES, MOST107-2218-E-005-023, and the Chaoyang University of Technology (CYUT) and Higher Education Sprout Project, Ministry of Education, Taiwan, under the project name: The R&D and the cultivation of talent for Health-Enhancement Products.

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