A New Approach to Rough Lattices and Rough Fuzzy Lattices Based on Fuzzy Ideals

Kuanyun Zhu, Jingru Wang* and Yongwei Yang

Abstract—**This paper aims to consider the relationships between rough sets, fuzzy sets and distributive lattices. In fact, we consider a distributive lattice as a universal set and we apply the notion of fuzzy ideals of a distributive lattice for definitions lower and upper approximations in a distributive lattice. A new congruence relation induced by a fuzzy ideal of a distributive lattice is introduced. Moreover, rough lattices (ideals, filters) and rough fuzzy lattices (ideals, filters) based on fuzzy ideals are investigated. In particular, we consider the relationships between lattices (ideals, filters) and rough lattice (ideals, filters), fuzzy lattices (ideals, filters) and rough fuzzy lattices (ideals, filters), respectively. Some characterizations are also discussed.**

Index Terms—**Fuzzy ideal; t-level set; Rough lattice (ideal, filter); Rough fuzzy lattice (ideal, filter)**

I. Introduction

It is well known that, in the real world, classical methods are not always successful in dealing with the problems in economy, engineering and social science, because of various types of uncertainties presented in these problems. As far as known, there are several theories to describe uncertainty, for example, fuzzy set theory [24], rough set theory [10] and other mathematical tools. Over these years, a lot of experts and scholars are looking for some different ways to solve the problem of uncertainty.

The Pawlak's rough sets which was introduced by Pawlak [10] in the early 1980s, is an extension of classical set theory and could be regarded as a mathematical tool for several assessment and decision problems of imprecision, vagueness and uncertainty data in information technology. The Pawlak's rough sets is built on the basis of a classification mechanism, it is classified by an equivalence relation in a specific universe and constitutes a partition of the universe. From the viewpoint of granular computing, an equivalence class can be viewed as a knowledge granule which be induced by an indiscernibility relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applications. Then some more general models have been proposed, such as [30], [31], [34]. Nowadays, rough sets has been applied to many areas, such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on, see [2], [7], [21]. In particular, many researchers applied this theory to algebraic structures, such as [4], [5], [20], [6], [1], [13]. In 2016, Wang and Zhan [17] investigated rough semigroups and rough fuzzy semigroups based on fuzzy ideals. In particular, Davvaz [5] constructed a t-level relation based on a fuzzy ideal and showed that $U(\mu, t)$ is a congruence relation on rings, and the author investigated roughness in rings based on fuzzy ideals. Zhan et al. [25] investigated roughness in n-ary semigroups based on fuzzy ideals. In 2017, Zhan et al. [26], [27] investigated roughness in hemirings based on strong h-ideals and roughness in non-associative po-semihypergroups based on pseudohyperorder relations, respectively. In addition, Zhan et al. [28] also studied rough soft n-ary semigroups based on a novel congruence relation and corresponding decision making. Wang et al. [18] studied soft rough semigroups and corresponding decision making applications. In 2018, Yang and Hu [22] discussed the communication between fuzzy information systems by using fuzzy covering mappings and fuzzy covering-based rough sets. Shao et al. [15] discussed the connections between two-universe rough sets and formal concepts. Wang and Zhan [19] investigated Z-soft rough fuzzy semigroups and its decision making. Rehman [14] studied generalized roughness in LA-semigroups. Shao et al. [16] investigated multi-granlation rough filters and rough fuzzy filters in pseudo-BCI algebras. Prasertpong and Siripitukdet [11] discussed rough sets induced by fuzzy relations approach in semigroups. Zhang and Zhan [32] combined rough sets with soft sets, they introduced the concept of rough soft BCK-algebras. In 2019, Hussain et al. [9] studied rough pythagorean fuzzy ideals in semigroups. Prasertpong and Siripitukdet [12] presented generalized rough sets in approximation spaces based on portions of successor classes induced by arbitrary binary relations between two universes. Yu et al. [23] investigated decision-theoretic rough set in lattice-valued decision information system. Zhan et al. [29] combined intuitionistic fuzzy sets with rough sets, they introduced intuitionistic fuzzy rough graphs. Zhang and Zhan [33] explored the relationships among several types of fuzzy soft β-covrings based fuzzy rough sets.

Based on [5], in this paper, we consider the relationships between rough sets, fuzzy sets and distributive lattices. A new congruence relation $U(\mu, t)$ induced by a fuzzy ideal μ of a distributive lattice L is introduced, and we give a definition of a t-level relation of a fuzzy ideal. Next, we present a definition for lower and upper approximations of a subset of a lattice with respect to a fuzzy ideal. Further, the properties of rough lattices and rough fuzzy lattices based on fuzzy ideals are investigated. We also characterize rough lattices (ideals, filters), fuzzy lattices (ideals, filters) and rough fuzzy lattices (ideals, filters) of lattices.

This paper is organized as follows. In Section II, we recall some concepts and results of lattices, fuzzy sets and rough

Manuscript received December 27, 2018; revised April 11, 2019, May 14, 2019 and June 11 2019. This work was supported in part by Higher Education Key Scientific Research Program Funded by Henan Province (No. 18A110008, 18A110010, 18A630001) and Research and Cultivation Fund Project of Anyang Normal University (No. AYNUKP-2018-B25, AYNUKP-2018-B26).

K. Zhu is with the School of Information and Mathematics, Yangtze University, Jingzhou, P.R. China e-mail: kyzhu@whu.edu.cn

J. Wang is with the School of Business Administration, Hunan University, Changsha, P.R. China e-mail: wangjingru@hnu.edu.cn

Y. Yang is with the School of Mathematics and Statistics, Anyang Normal University, Anyang, P.R. China e-mail: yangyw@aynu.edu.cn

sets. In Section III, we study some operations of lower and upper approximations of distributive lattices. In Section IV, we investigate rough lattices (ideals, filters) based on fuzzy ideals. In Section V, we discuss rough fuzzy lattices (ideals, filters) based on fuzzy ideals.

II. Preliminaries

In this section, we recall some basic notions and results of lattices, fuzzy set theory and rough set theory.

A lattice L is a poset in which any two elements have a unique supremum and an infimum, where $x \vee y$= sup$\{x, y\}$ and $x \wedge y$= inf $\{x, y\}$, $x, y \in L$.

L is said to be distributive if it satisfies the distributive law: $\forall x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Definition 2.1: [3] Let L be a lattice and $\emptyset \subsetneq X \subseteq L$. Then X is a sublattice of L if $x, y \in L$, $x \vee y \in X$ and $x \wedge y \in X$.

Definition 2.2: [3] Let L be a lattice and $\emptyset \subsetneq I \subseteq L$. Then I is called an ideal of L if

(1) $a, b \in I$ implies $x \vee y \in I$,

(2) $a \in L, b \in I$ and $a \leq b$ imply $a \in I$.

Definition 2.3: [3] Let L be a lattice and $\emptyset \subsetneq F \subseteq L$. Then F is called a filter of L if

(1) $a, b \in F$ implies $x \wedge y \in F$,

(2) $a \in L, b \in F$ and $a \geq b$ imply $a \in F$.

Definition 2.4: [8] Let L be a lattice. A relation R is called an equivalence relation if for all $a, b, c \in L$,

(1) Reflexive: $(a, a) \in R$,

(2) Symmetry: $(a, b) \in R$ implies $(b, a) \in R$,

(3) Transitivity: $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$.

An equivalence relation R is called a congruence relation if for all $a, b, c, d \in L$, $(a, b) \in R$ and $(c, d) \in R$ imply $(a \vee c, b \vee d) \in R$ and $(a \wedge c, b \wedge d) \in R$.

Definition 2.5: [8] Let L be a lattice and μ be a fuzzy set of L. Then μ is called a fuzzy sublattice of L if for all $x, y \in L$, $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$.

Let μ be a fuzzy sublattice of a lattice L. Then

(1) μ is a fuzzy ideal of L, if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

(2) μ is a fuzzy filter of L, if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.

Proposition 2.6: [8] Let L be a lattice and μ be a fuzzy sublattice of L. Then

(1) μ is a fuzzy ideal of L, if and only if $x \leq y$ implies that $\mu(x) \geq \mu(y)$ for all $x, y \in L$.

(2) μ is a fuzzy filter of L, if and only if $x \leq y$ implies that $\mu(x) \leq \mu(y)$ for all $x, y \in L$.

Let μ be a fuzzy subset of L and $t \in [0, 1]$. Then the set $\mu_t = \{x \in L | \mu(x) \geq t\}$ is called a t-level subset of μ.

Let μ be a fuzzy subset of L and $t \in [0, 1]$. Then the set $\mu_t = \{x \in L | \mu(x) \geq t\}$ is called a t-level subset of μ.

Definition 2.7: [10] Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $R_*X = R^*X$; in the opposite case, i.e., if $R_*X - R^*X \neq \emptyset$, X is said to be a rough set, where two operators are defined as:

$$R_*X = \{x \in U | [x]_R \subseteq X\}$$

and

$$R^*X = \{x \in U | [x]_R \cap X \neq \emptyset\}.$$

III. A new congruence relation induced by a fuzzy ideal

In this section, we introduce a new congruence relation $U(\mu, t)$ induced by a fuzzy ideal μ. Further, we investigate the operations of lower and upper approximations of lattices. Let A and B be subsets of L. We define the join and meet of two non-empty subsets in a lattice as follows: $A \vee B = \{a \vee b | a \in A, b \in B\}$, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$.

Definition 3.1: Let μ be a fuzzy ideal of a lattice L. For each $t \in [0, 1]$, the set $U(\mu, t) = \{(x, y) \in L \times L | \bigvee_{a \vee x = a \vee y} \mu(a) \geq t$ for some $a \in L\}$ is called a t-level relation of μ.

Example 3.2: Let $L = \{0, a, b, c, 1\}$ be a lattice defined in Fig.1. It is easy to see that L is a distributive lattice. Let $\mu = \frac{1}{0} + \frac{0.8}{a} + \frac{0.6}{b} + \frac{0.4}{c} + \frac{0}{1}$. Then it is clear that μ is a fuzzy ideal of L. Choose $t = 0.9$, then we have $U(\mu, 0.9) = \{(0, 0), (a, a), (b, b), (c, c), (1, 1)\}$. Thus $U(\mu, 0.9)$ is a 0.9-level relation of μ.

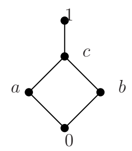

Fig. 1. Lattice L.

In the following, L is always a distributive lattice with the element 0. First, we proved $U(\mu, t)$ is a congruence relation on L.

Lemma 3.3: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $U(\mu, t)$ is a congruence relation on L.

Proof. It is easy to see that $\mu(0) = 1$. Now let $\forall x \in L$. Then $\bigvee_{a \vee x = a \vee x} \mu(a) = \bigvee \mu(a) \geq \mu(0) = 1 \geq t$. It follows from Definition 3.1 that $(x, x) \in U(\mu, t)$. Therefore, $U(\mu, t)$ is reflexive.

Obviously, $U(\mu, t)$ is symmetric.

Let $(x, y) \in U(\mu, t)$ and $(y, z) \in U(\mu, t)$. Then we have

$$\bigvee_{a \vee x = a \vee y} \mu(a) \geq t, \bigvee_{b \vee y = b \vee z} \mu(b) \geq t,$$

for $a, b \in L$. Thus,

$$(\bigvee_{a \vee x = a \vee y} \mu(a)) \wedge (\bigvee_{b \vee y = b \vee z} \mu(b)) \geq t.$$

Since μ is a fuzzy ideal of L, we have

$$\begin{aligned} & (\bigvee_{a \vee x = a \vee y} \mu(a)) \wedge (\bigvee_{b \vee y = b \vee z} \mu(b)) \\ = & \bigvee_{a \vee x = a \vee y, b \vee y = b \vee z} (\mu(a) \wedge \mu(b)) \\ = & \bigvee_{a \vee x = a \vee y, b \vee y = b \vee z} \mu(a \vee b). \end{aligned}$$

For $a \vee x = a \vee y, b \vee y = b \vee z$, we have

$$a \vee b \vee x = a \vee b \vee y,$$

$$a \vee b \vee y = a \vee b \vee z.$$

Thus,

$$a \vee b \vee x = a \vee b \vee z,$$

which implies $c \vee x = c \vee z$, where $c = a \vee b \in L$. Hence,

$$\bigvee_{a\vee x=a\vee y, b\vee y=b\vee z} \mu(a \vee b) \leq \bigvee_{c\vee x=c\vee z} \mu(c).$$

Thus $\bigvee_{c\vee x=c\vee z} \mu(c) \geq t$. Moreover, it follows from Definition 3.1 that $(x, z) \in U(\mu, t)$. Therefore, $U(\mu, t)$ is an equivalence relation on L.

Now we show that $U(\mu, t)$ is a congruence relation on L.

Let $(x, y) \in U(\mu, t)$ and $(u, v) \in U(\mu, t)$. Then for $a, b \in L$,

$$\bigvee_{a\vee x=a\vee y} \mu(a) \geq t, \bigvee_{b\vee u=b\vee v} \mu(b) \geq t.$$

Thus,

$$(\bigvee_{a\vee x=a\vee y} \mu(a)) \wedge (\bigvee_{b\vee y=b\vee z} \mu(b)) \geq t.$$

Further, we have

$$\begin{aligned} &\bigvee_{a\vee x=a\vee y} \mu(a) \wedge \bigvee_{b\vee u=b\vee v} \mu(b) \\ &= \bigvee_{a\vee x=a\vee y, b\vee u=b\vee v} (\mu(a) \wedge \mu(b)) \\ &= \bigvee_{a\vee x=a\vee y, u\vee y=b\vee v} \mu(a \vee b). \end{aligned}$$

For $a \vee x = a \vee y, b \vee u = b \vee v$, we have

$$a \vee b \vee (x \vee u) = a \vee b \vee (y \vee v),$$

that is

$$c \vee (x \vee u) = c \vee (y \vee v),$$

where $c = a \vee b \in L$. Hence,

$$\bigvee_{a\vee x=a\vee y, u\vee y=b\vee v} \mu(a \vee b) \leq \bigvee_{c\vee(x\vee u)=c\vee(y\vee v)} \mu(c).$$

Thus,

$$\bigvee_{c\vee(x\vee u)=c\vee(y\vee v)} \mu(c) \geq t.$$

This implies that $(x \vee u, y \vee v) \in U(\mu, t)$.
Let $(x_1, y_1) \in U(\mu, t)$ and $(x_2, y_2) \in U(\mu, t)$.
Then for $b, c \in L$,

$$\bigvee_{b\vee x_1=b\vee y_1} \mu(b) \geq t, \bigvee_{c\vee x_2=c\vee y_2} \mu(c) \geq t.$$

Thus,

$$(\bigvee_{b\vee x_1=b\vee y_1} \mu(b)) \wedge (\bigvee_{c\vee x_2=c\vee y_2} \mu(c)) \geq t.$$

For $b \vee x_1 = b \vee y_1$ and $c \vee x_2 = c \vee y_2$, we have $(b \vee x_1) \wedge (c \vee x_2) = (b \vee y_1) \wedge (c \vee y_2)$.

Since L is a distributive lattice, we have
$[(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)] \vee (x_1 \wedge x_2)$
$= [(b \wedge c) \vee (y_1 \wedge c) \vee (y_2 \wedge b)] \vee (y_1 \wedge y_2)$.
It follows from $(b \vee x_1) \wedge c = (c \vee y_1) \wedge c$
and $(c \vee x_2) \wedge b = (c \vee y_2) \wedge b$ that
$(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)$
$= (b \wedge c) \vee (y_1 \wedge c) \vee (y_2 \wedge b)$.
It follows from μ is a fuzzy ideal of L that
$\mu[(b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)]$
$= \mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b)$.
Since $b \wedge c \leq b$, $x_1 \wedge c \leq c$, $x_2 \wedge b \leq b$, we have

$$\mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b) \geq \mu(b) \wedge \mu(c).$$

Thus

$$\begin{aligned} &(\bigvee_{b\vee x_1=b\vee y_1} \mu(b)) \wedge (\bigvee_{c\vee x_2=c\vee y_2} \mu(c)) \\ &= \bigvee_{b\vee x_1=b\vee y_1, c\vee x_2=c\vee y_2} (\mu(b) \wedge \mu(c)) \\ &\leq \bigvee_{b\vee x_1=b\vee y_1, c\vee x_2=c\vee y_2} (\mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b)) \\ &\leq \bigvee_{t\vee(x_1\wedge x_2)=t\vee(y_1\wedge y_2)} (\mu(b \wedge c) \wedge \mu(x_1 \wedge c) \wedge \mu(x_2 \wedge b)) \\ &\leq \bigvee_{a\vee(x_1\wedge x_2)=a\vee(y_1\wedge y_2)} \mu(a), \end{aligned}$$

where $t = (b \wedge c) \vee (x_1 \wedge c) \vee (x_2 \wedge b)$
$= (b \wedge c) \vee (y_1 \wedge c) \vee (y_2 \wedge b)$.
This implies that

$$\bigvee_{a\vee(x_1\wedge x_2)=a\vee(y_1\wedge y_2)} \mu(a) \geq t.$$

It follows from Definition 3.1 that

$$(x_1 \wedge x_2, y_1 \wedge y_2) \in U(\mu, t).$$

Therefore, $U(\mu, t)$ is a congruence relation on L. □

Remark 3.4: In Lemma 3.3, we say x is congruent to y mod μ, written $x \equiv_t y$ (mod μ) if

$$\bigvee_{a\vee x=a\vee y} \mu(a) \geq t,$$

for $a \in L$.

It follows from Definition 3.1 and Lemma 3.3 that we can get many useful properties of these congruence relations.

We denote by $[x]_{(\mu,t)}$ the equivalence class of $U(\mu, t)$ containning x of L.

Lemma 3.5: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then

(1) $[x]_{(\mu,t)} \vee [y]_{(\mu,t)} \subseteq [x \vee y]_{(\mu,t)}$,
(2) $[x]_{(\mu,t)} \wedge [y]_{(\mu,t)} \subseteq [x \wedge y]_{(\mu,t)}$.

Proof. It is straightforward. □

Definition 3.6: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then

(1) $U(\mu, t)$ is called a $\vee$-complete congruence if $[x]_{(\mu,t)} \vee [y]_{(\mu,t)} = [x \vee y]_{(\mu,t)}$, for all $x, y \in L$.
(2) $U(\mu, t)$ is called a $\wedge$-complete congruence if $[x]_{(\mu,t)} \wedge [y]_{(\mu,t)} = [x \wedge y]_{(\mu,t)}$, for all $x, y \in L$.

Example 3.7: Consider the Example 3.2. It is easy to check that $U(\mu, t)$ are both $\vee$-complete congruence and $\wedge$-complete congruence.

Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $U(\mu, t)$ is a congruence relation. Thus, when $U = L$ and R is the above equivalence relation (congruence relation), then we use (L, μ, t) instead of approximation space (U, R).

Definition 3.8: Let μ be a fuzzy ideal of L, $t \in [0, 1]$, $U(\mu, t)$ be a t-level congruence relation of μ on L and $\emptyset \subsetneq A \subseteq L$. Then

$$\underline{U}(\mu, t, A) = \{x \in L | [x]_{(\mu,t)} \subseteq A\}$$

and

$$\overline{U}(\mu, t, A) = \{x \in L | [x]_{(\mu,t)} \cap A \neq \emptyset\}$$

are called the lower and upper approximations of the set A with respect to $U(\mu, t)$, respectively. It is easy to know that $\underline{U}(\mu, t, A) \subseteq A \subseteq \overline{U}(\mu, t, A)$.

Now, we investigate some operations of lower and upper approximations of the set A with respect to $U(\mu, t)$, respectively.

Proposition 3.9: Let A and B are nonempty subsets of L.

(1) If $U(\mu,t)$ is $\vee$-complete congruence, then $\overline{U}(\mu,t,A)\vee\overline{U}(\mu,t,B)=\overline{U}(\mu,t,A\vee B)$ and $\underline{U}(\mu,t,A)\vee\underline{U}(\mu,t,B)\subseteq\underline{U}(\mu,t,A\vee B)$.

(2) If $U(\mu,t)$ is $\wedge$-complete congruence, then $\overline{U}(\mu,t,A)\wedge\overline{U}(\mu,t,B)=\overline{U}(\mu,t,A\wedge B)$ and $\underline{U}(\mu,t,A)\wedge\underline{U}(\mu,t,B)\subseteq\underline{U}(\mu,t,A\wedge B)$

Proof. (1) It is obvious that $\overline{U}(\mu,t,A)\vee\overline{U}(\mu,t,B)\subseteq\overline{U}(\mu,t,A\vee B)$. Now we show that $\overline{U}(\mu,t,A\vee B)\subseteq\overline{U}(\mu,t,A)\vee\overline{U}(\mu,t,B)$. Let $x\in\overline{U}(\mu,t,A\vee B)$. Then $[x]_{(\mu,t)}\cap(A\vee B)\neq\emptyset$. Thus there exist $a\in A$ and $b\in B$ such that $(a\vee b,x)\in U(\mu,t)$. Since $U(\mu,t)$ is a $\vee$-complete congruence, we have $x\in[a\vee b]_{(\mu,t)}=[a]_{(\mu,t)}\vee[b]_{(\mu,t)}$. Thus there exist $c\in[a]_{(\mu,t)}$, $d\in[a]_{(\mu,t)}$ such that $x=c\vee d$. Since $a\in A$, $b\in B$, we have $[c]_{(\mu,t)}\cap A\neq\emptyset$, $[d]_{(\mu,t)}\cap B\neq\emptyset$. Thus, $c\in\overline{U}(\mu,t,A)$, $d\in\overline{U}(\mu,t,B)$. This means that $x=c\vee d\in\overline{U}(\mu,t,A)\vee\overline{U}(\mu,t,B)$. Thus $\overline{U}(\mu,t,A\vee B)\subseteq\overline{U}(\mu,t,A)\vee\overline{U}(\mu,t,B)$. Since $U(\mu,t)$ is $\vee$-complete congruence, it is clear that $\underline{U}(\mu,t,A)\vee\underline{U}(\mu,t,B)\subseteq\underline{U}(\mu,t,A\vee B)$.

(2) The proof is similar to that of (1). □

The following example shows that the containment in Proposition 3.9 is proper.

Example 3.10: Consider the lattice in Example 3.2. Let $\mu=\frac{1}{0}+\frac{0.6}{a}+\frac{0.8}{b}+\frac{0.4}{c}+\frac{0}{1}$. Then it is clear that μ is a fuzzy ideal of L. Choose $t=0.8$, then we have $U(\mu,0.8)=\{(0,0),(a,a),(b,b),(c,c),(1,1),(0,b),(a,c)\}$. If $A=\{a\}$ and $B=\{0,b,c\}$, then $\underline{U}(\mu,t,A)=\emptyset$, $\underline{U}(\mu,t,B)=\{0,b\}$. Thus we have $\underline{U}(\mu,t,A\vee B)=\{a,c\}$, $\underline{U}(\mu,t,A)\vee\underline{U}(\mu,t,B)=\emptyset$. Therefore $\underline{U}(\mu,t,A)\vee\underline{U}(\mu,t,B)\subsetneqq\underline{U}(\mu,t,A\vee B)$. Similarly, we have $\underline{U}(\mu,t,A)\wedge\underline{U}(\mu,t,B)\subsetneqq\underline{U}(\mu,t,A\wedge B)$.

Lemma 3.11: Let μ and ν be two fuzzy ideals of L, $\nu\subseteq\mu$ and $t\in[0,1]$. Then $[x]_{(\nu,t)}\subseteq[x]_{(\mu,t)}$.

Proof. Let $a\in[x]_{(\nu,t)}$. Then we have $(a,x)\in U(\nu,t)$. That is $\bigvee\limits_{b\vee a=b\vee x}\nu(b)\geq t, b\in L$. Since $\nu\subseteq\mu$, $\nu(b)\leq\mu(b)$. Thus we have $\bigvee\limits_{b\vee a=b\vee x}\mu(b)\geq\bigvee\limits_{b\vee a=b\vee x}\nu(b)\geq t$. This implies that $(a,x)\in U(\mu,t)$, that is $a\in[x]_{(\mu,t)}$. Therefore, $[x]_{(\nu,t)}\subseteq[x]_{(\mu,t)}$. □

Proposition 3.12: Let μ and ν be two fuzzy ideals of L and $t\in[0,1]$. If X is a non-empty subsets of L, then

(1) $\overline{U}(\mu\cap\nu,t,X)\subseteq\overline{U}(\mu,t,X)\cap\overline{U}(\nu,t,X)$.

(2) $\underline{U}(\mu,t,X)\cup\underline{U}(\nu,t,X)\subseteq\underline{U}(\mu\cap\nu,t,X)$.

Proof. It follows from Lemma 3.11 that we get the conclusions easily. □

The following example shows that the containment in Proposition 3.12 is proper.

Example 3.13: Consider the Example 3.10. Let $\nu=\frac{1}{0}+\frac{0.8}{a}+\frac{0.5}{b}+\frac{0.3}{c}+\frac{0}{1}$. *Then it is clear that* ν *is a fuzzy ideal of* L. *Choose* $t=0.8$, *then we have* $U(\nu,0.8)=\{(0,0),(a,a),(b,b),(c,c),(1,1),(0,a),(b,c)\}$. *Thus* $U(\mu\cap\nu,0.8)=\{(0,0),(a,a),(b,b),(c,c),(1,1)\}$. *If* $X=\{0,c\}$, *then* $\overline{U}(\mu\cap\nu,t,X)=\{0,c\}$, $\overline{U}(\mu,t,X)\cap\overline{U}(\nu,t,X)=\{0,a,b,c\}$. *Thus* $\overline{U}(\mu\cap\nu,t,X)\subsetneqq\overline{U}(\mu,t,X)\cap\overline{U}(\nu,t,X)$. *Further, If* $X=\{c,1\}$, *then* $\underline{U}(\mu,t,X)\cup\underline{U}(\nu,t,X)=\{1\}$, $\underline{U}(\mu\cap\nu,t,X)=\{1,c\}$. *Thus* $\underline{U}(\mu,t,X)\cup\underline{U}(\nu,t,X)\subsetneqq\underline{U}(\mu\cap\nu,t,X)$.

Let μ and ν be two fuzzy ideals of L and $t\in[0,1]$. The composition of $U(\mu,t)$ and $U(\nu,t)$ is defined as follows:

$U(\mu,t)*U(\nu,t)=\{(x,y)\in L\times L|\exists z\in L$ such that $(x,z)\in U(\mu,t)$ and $(z,y)\in U(\nu,t)\}$.

It is no difficult to see that $U(\mu,t)*U(\nu,t)$ is a congruence relation on L if and only if $U(\mu,t)*U(\nu,t)=U(\nu,t)*U(\mu,t)$. We denote this congruence relation by $U(\mu*\nu,t)$.

Proposition 3.14: Let μ and ν be two fuzzy ideals of L, $t\in[0,1]$ and $U(\mu*\nu,t)=U(\nu*\mu,t)$. If X is a sublattice of L, then

(1) $\overline{U}(\mu,t,X)\vee\overline{U}(\nu,t,X)\subseteq\overline{U}(\mu*\nu,t,X)$.

(2) $\overline{U}(\mu,t,X)\wedge\overline{U}(\nu,t,X)\subseteq\overline{U}(\mu*\nu,t,X)$.

Proof. (1) Let $m\in\overline{U}(\mu,t,X)\vee\overline{U}(\nu,t,X)$. Then $m=x\vee y$, where $x\in\overline{U}(\mu,t,X)$ and $y\in\overline{U}(\nu,t,X)$. Hence, there exist $a,b\in L$ such that $a\in[x]_{(\mu,t)}\cap X$ and $b\in[y]_{(\nu,t)}\cap X$, that is $a\in[x]_{(\mu,t)},a\in X$ and $b\in[y]_{(\nu,t)},b\in X$. Since X is a sublattice of L, we have $a\vee b\in X$. Since $U(\mu,t)$ and $U(\nu,t)$ are congruence relations on L, we have $(a\vee y,x\vee y)\in U(\mu,t)$ and $(a\vee b,a\vee y)\in U(\nu,t)$. Hence, $(x\vee y,a\vee b)\in U(\mu*\nu,t)$. Thus $a\vee b\in[x\vee y]_{(\mu*\nu,t)}$. Therefore, $m=x\vee y\in\overline{U}(\mu*\nu,t,X)$. That is $\overline{U}(\mu,t,X)\vee\overline{U}(\nu,t,X)\subseteq\overline{U}(\mu*\nu,t,X)$.

(2) The proof is similar to that of (1). □

The following example shows that the containment in Proposition 3.14 is proper.

Example 3.15: Consider the Example 3.13. If $X=\{c\}$, then X is a sublattice of L. Thus we have $\overline{U}(\mu,t,X)\vee\overline{U}(\nu,t,X)=\{c\}$ and $\overline{U}(\mu*\nu,t,X)=\{0,a,b,c\}$. Therefore $\overline{U}(\mu,t,X)\vee\overline{U}(\nu,t,X)\subsetneqq\overline{U}(\mu*\nu,t,X)$. Similarly, we have $\overline{U}(\mu,t,X)\wedge\overline{U}(\nu,t,X)\subsetneqq\overline{U}(\mu*\nu,t,X)$.

Theorem 3.16: Let μ and ν be two fuzzy ideals of L, $t\in[0,1]$ and $U(\mu*\nu,t)=U(\nu*\mu,t)$. If X is a non-empty subset of L, then

$$\underline{U}(\mu,t,\underline{U}(\nu,t,X))=\underline{U}(\nu,t,\underline{U}(\mu,t,X)).$$

Proof. Let $a\in\underline{U}(\mu,t,\underline{U}(\nu,t,X))$. Assume that $a\notin\underline{U}(\nu,t,\underline{U}(\mu,t,X))$, then there exists $b\in[a]_{(\nu,t)}$ but $b\notin\overline{U}(\mu,t,X)$. This implies that there exists $c\in[b]_{(\nu,t)}$ but $c\notin X$. So $(a,c)\in U(\nu*\mu,t)=U(\mu*\nu,t)$. Thus there exists $d\in L$ such that $(a,d)\in U(\mu,t)$ and $(d,c)\in U(\nu,t)$. Since $a\in\underline{U}(\mu,t,\underline{U}(\nu,t,X))$, we have $b\in\underline{U}(\mu,t,X)$. Thus we have $c\in[d]_{(\nu,t)}\subseteq X$. That is $c\in X$, which contradicts with $c\notin X$. From the above, we have $\underline{U}(\mu,t,\underline{U}(\nu,t,X))\subseteq\underline{U}(\nu,t,\underline{U}(\mu,t,X))$. Similarly, we have $\underline{U}(\mu,t,\underline{U}(\nu,t,X))\supseteq\underline{U}(\nu,t,\underline{U}(\mu,t,X))$. Therefore, $\underline{U}(\mu,t,\underline{U}(\nu,t,X))=\underline{U}(\nu,t,\underline{U}(\mu,t,X))$. □

Theorem 3.17: Let μ *and* ν *be two fuzzy ideals of* L *and* $t\in[0,1]$. *If* X *is a non-empty subset of* L, *then the following are equivalent:*

(1) $U(\mu*\nu,t)=U(\nu*\mu,t)$.

(2) $\overline{U}(\mu,t,\overline{U}(\nu,t,X))=\overline{U}(\nu,t,\overline{U}(\mu,t,X))$.

Proof. (1) $\Rightarrow$ (2) Let $a\in\overline{U}(\mu,t,\overline{U}(\nu,t,X))$. Then there exists $b\in\overline{U}(\nu,t,X)$ such that $(a,b)\in U(\mu,t)$. This means that there exists $c\in X$ such that $(b,c)\in U(\nu,t)$. Thus $(a,c)\in U(\mu*\nu,t)$. Since $U(\mu*\nu,t)=U(\nu*\mu,t)$, there exists $d\in L$ such that $(a,d)\in U(\nu,t)$ and $(d,c)\in U(\mu,t)$. Thus $c\in[d]_{(\nu,t)}\cap X$. This means that $d\in\overline{U}(\mu,t,X)$. Further, $d\in[a]_{(\nu,t)}\cap\overline{U}(\mu,t,X)$. So $a\in\overline{U}(\nu,t,\overline{U}(\mu,t,X))$. Therefore $\overline{U}(\mu,t,\overline{U}(\nu,t,X))\subseteq\overline{U}(\nu,t,\overline{U}(\mu,t,X))$. In a similar way, we have $\overline{U}(\mu,t,\overline{U}(\nu,t,X))\supseteq\overline{U}(\nu,t,\overline{U}(\mu,t,X))$. So $\overline{U}(\mu,t,\overline{U}(\nu,t,X))=\overline{U}(\nu,t,\overline{U}(\mu,t,X))$.

(2) $\Rightarrow$ (1) Let $(x,y)\in U(\mu*\nu,t)$. Then we have $x\in\overline{U}(\mu,t,\overline{U})(\nu,t,\{y\})=\overline{U}(\nu,t,\overline{U})(\mu,t,\{y\})$. This means that there exists $z\in\overline{U}(\mu,t,\{y\})$ such that $(x,z)\in U(\nu,t)$. Thus $(z,y)\in U(\mu,t)$. So $(x,y)\in U(\nu*\mu,t)$. Therefore,

$U(\mu * \nu, t) \subseteq U(\nu * \mu, t)$. In a similar way, we have $U(\mu * \nu, t) \supseteq U(\nu * \mu, t)$. Thus $U(\mu * \nu, t) = U(\nu * \mu, t)$. □

IV. Rough lattices (ideals, filters) of lattices based on fuzzy ideals

In this section, we investigate rough lattices (ideals, filters) of lattices based on fuzzy ideals.

Definition 4.1: In Definition 3.8, if $\underline{U}(\mu, t, A) \neq \overline{U}(\mu, t, A)$, then
(i) A is called a lower (upper) rough lattice (ideal, filter) w.r.t. $U(\mu, t)$ of L, if $\underline{U}(\mu, t, A)$ ($\overline{U}(\mu, t, A)$) is a sublattice (ideal, filter) of L;
(ii) A is called a rough lattice (ideal, filter) w.r.t. $U(\mu, t)$ of L, if $\underline{U}(\mu, t, A)$ and $\overline{U}(\mu, t, A)$ are sublattices (ideals, filters) of L.

Lemma 4.2: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. If A, B are nonempty subsets of L and $A \subseteq B$, then $\underline{U}(\mu, t, A) \subseteq \underline{U}(\mu, t, B)$ and $\overline{U}(\mu, t, A) \subseteq \overline{U}(\mu, t, B)$.
Proof. It is straightforward. □

Theorem 4.3: Let μ be a fuzzy ideal of L and $t \in [0, 1]$.
(1) If A is a sublattice of L, then A is an upper rough sublattice of L.
(2) If A is an ideal of L, then A is an upper rough ideal of L.
(3) If A is a filter of L, then A is an upper rough filter of L.

Proof. (1) Let A be a sublattice of L. Then $A \vee A \subseteq A$ and $A \wedge A \subseteq A$. It follows from Proposition 3.9 and Lemma 4.2 that $\overline{U}(\mu, t, A) \vee \overline{U}(\mu, t, A) \subseteq \overline{U}(\mu, t, A \vee A) \subseteq \overline{U}(\mu, t, A)$ and $\overline{U}(\mu, t, A) \wedge \overline{U}(\mu, t, A) \subseteq \overline{U}(\mu, t, A \wedge B) \subseteq \overline{U}(\mu, t, A)$. Thus, $\overline{U}(\mu, t, A)$ is a sublattice of L. It follows from Definition 4.1 that A is an upper rough sublattice of L.

(2) Let A be an ideal of L. Then A is a sublattice of L. It follows from (1) that $a \vee b \in \overline{U}(\mu, t, A)$ for all $a, b \in \overline{U}(\mu, t, A)$. Let $c \in L, d \in \overline{U}(\mu, t, A)$ and $c \leq d$. Then there exists $e \in [d]_{(\mu,t)} \cap A$. Now Let $f \in [c]_{(\mu,t)}$. Then $e \wedge f \in [d]_{(\mu,t)} \wedge [c]_{(\mu,t)} \subseteq [c \wedge d]_{(\mu,t)} = [c]_{(\mu,t)}$. Since A is an ideal of L, $e \wedge f \leq e$, we have $e \wedge f \in A$. Thus $[c]_{(\mu,t)} \cap A \neq \emptyset$. This means that $c \in \overline{U}(\mu, t, A)$. Thus, $\overline{U}(\mu, t, A)$ is an ideal of L. It follows from Definition 4.1 that A is an upper rough ideal of L.

(3) Let A be a filter of L. Then it follows from (1) that $a \wedge b \in \overline{U}(\mu, t, A)$ for all $a, b \in \overline{U}(\mu, t, A)$. Let $c \in L, d \in \overline{U}(\mu, t, A)$ and $c \geq d$. Then there exists $e \in [d]_{(\mu,t)} \cap A$. Now Let $f \in [c]_{(\mu,t)}$. Then $e \vee f \in [d]_{(\mu,t)} \vee [c]_{(\mu,t)} \subseteq [c \vee d]_{(\mu,t)} = [c]_{(\mu,t)}$. Since A is a filter of L, $e \vee f \geq e$, we have $e \vee f \in A$. Thus $[c]_{(\mu,t)} \cap A \neq \emptyset$. This means that $c \in \overline{U}(\mu, t, A)$. Thus, $\overline{U}(\mu, t, A)$ is a filter of L. It follows from Definition 4.1 that A is an upper rough filter of L. □

Theorem 4.4: Let $U(\mu, t)$ be a $\vee$-complete congruence relation on L.
(1) If A is a sublattice of L and $U(\mu, t)$ is a $\wedge$-complete congruence relation on L, then A is a lower rough sublattice of L.
(2) If A is an ideal of L, then A is a lower rough ideal of L.
(3) If A is a filter of L, then A is a lower rough filter of L.

Proof. (1) Let A be a sublattice of L. Then $A \vee A \subseteq A$ and $A \wedge A \subseteq A$. It follows from Proposition 3.9 and Lemma 4.2 that $\underline{U}(\mu, t, A) \vee \underline{U}(\mu, t, A) \subseteq \underline{U}(\mu, t, A \vee A) \subseteq \underline{U}(\mu, t, A)$ and $\underline{U}(\mu, t, A) \wedge \underline{U}(\mu, t, A) \subseteq \underline{U}(\mu, t, A \wedge A) \subseteq \underline{U}(\mu, t, A)$. Thus, $\underline{U}(\mu, t, A)$ is a sublattice of L. It follows from Definition 4.1 that A is a lower rough sublattice of L.

(2) Let A be an ideal of L. Then A is a sublattice of L. It follows from (1) that $a \vee b \in \underline{U}(\mu, t, A)$ for all $a, b \in \underline{U}(\mu, t, A)$. Let $a \in L, b \in \underline{U}(\mu, t, A)$ and $a \leq b$. Then $\{x \vee y | x \in [a]_{(\mu,t)}, y \in [b]_{(\mu,t)}\} = [a]_{(\mu,t)} \vee [b]_{(\mu,t)} = [a \vee b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq A$. Now let $x \in [a]_{(\mu,t)}$ and $y \in [b]_{(\mu,t)}$. Then $x \vee y \in [a]_{(\mu,t)} \vee [b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq A$. That is $x \vee y \in A$. Since A is an ideal of L and $x \leq x \vee y$, we have $x \in A$. This means that $[a]_{(\mu,t)} \in A$. Thus, $a \in \underline{U}(\mu, t, A)$. Hence, $\underline{U}(\mu, t, A)$ is an ideal of L. It follows from Definition 4.1 that A is a lower rough ideal of L.

(3) Let A be a filter of L. It follows from (1) that $a \wedge b \in \underline{U}(\mu, t, A)$ for all $a, b \in \underline{U}(\mu, t, A)$. Let $a \in L, b \in \underline{U}(\mu, t, A)$ and $a \geq b$. Then $\{x \wedge y | x \in [a]_{(\mu,t)}, y \in [b]_{(\mu,t)}\} = [a]_{(\mu,t)} \wedge [b]_{(\mu,t)} = [a \wedge b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq A$. Now let $x \in [a]_{(\mu,t)}$ and $y \in [b]_{(\mu,t)}$. Then $x \wedge y \in [a]_{(\mu,t)} \wedge [b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq A$. That is $x \wedge y \in A$. Since A is a filter of L and $x \geq x \wedge y$, we have $x \in A$. This means that $[a]_{(\mu,t)} \in A$. Thus, $a \in \underline{U}(\mu, t, A)$. Hence, $\underline{U}(\mu, t, A)$ is a filter of L. It follows from Definition 4.1 that A is a lower rough filter of L. □

Lemma 4.5: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then
(1) $[0]_{(\mu,t)}$ is an ideal of L.
(2) $[0]_{(\mu,t)}$ is a filter of L.

Proof. (1) Let $x, y \in [0]_{(\mu,t)}$. Then $x \vee y \in [0]_{(\mu,t)} \vee [0]_{(\mu,t)} = [0 \vee 0]_{(\mu,t)} = [0]_{(\mu,t)}$. Thus, $x \vee y \in [0]_{(\mu,t)}$. Now let $a \in L, b \in [0]_{(\mu,t)}$ and $a \leq b$. Then $(0, b) \in U(\mu, t)$. Since $U(\mu, t)$ is a congruence relation, we have $(0 \wedge a, b \wedge a) \in U(\mu, t)$, that is $(0, a) \in U(\mu, t)$. Thus, $a \in [0]_{(\mu,t)}$ Therefore $[0]_{(\mu,t)}$ is an ideal of L.

(2) The proof is similar to that of (1). □

Proposition 4.6: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $\underline{U}(\mu, t, [0]_{(\mu,t)}) = [0]_{(\mu,t)}$.

Proof. (1) It is easy to know that $\underline{U}(\mu, t, [0]_{(\mu,t)}) \subseteq [0]_{(\mu,t)}$. Now we show that $[0]_{(\mu,t)} \subseteq \underline{U}(\mu, t, [0]_{(\mu,t)})$. Let $x \in [0]_{(\mu,t)}$. Then $(0, x) \in U(\mu, t)$. Let $y \in [0]_{(\mu,t)}$. Then $(x, y) \in U(\mu, t)$. Since $U(\mu, t)$ is a congruence relation on L, we have $(0, y) \in U(\mu, t)$. This means that $y \in [0]_{(\mu,t)}$. Thus $[x]_{(\mu,t)} \subseteq [0]_{(\mu,t)}$, i.e. $x \in \underline{U}(\mu, t, [0]_{(\mu,t)})$, i.e. $x \in \underline{U}(\mu, t, [0]_{(\mu,t)})$. Therefore, $\underline{U}(\mu, t, [0]_{(\mu,t)}) = [0]_{(\mu,t)}$. □

Corollary 4.7: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $[0]_{(\mu,t)}$ is a lower rough ideal of L.

Proposition 4.8: Let μ be a fuzzy ideal of L and $t \in [0, 1]$. Then $[0]_{(\mu,t)} = \mu_t$.

Proof. We first show that $\mu_t \subseteq [0]_{(\mu,t)}$. Let $x \in \mu_t$. Then $\mu(x) \geq t$. Thus $\bigvee_{a \vee x = a \vee 0} \mu(a) \geq \mu(x) \geq t$. It follows from Definition 3.1 that $(0, x) \in U(\mu, t)$. That is $x \in [0]_{(\mu,t)}$. Therefore, $[0]_{(\mu,t)} \subseteq \mu_t$. Now we prove that $[0]_{(\mu,t)} \subseteq \mu_t$. Let $y \in [0]_{(\mu,t)}$. Then $(y, 0) \in U(\mu, t)$. That is $\bigvee_{a \vee y = a \vee 0} \mu(a) \geq t$, $a \in L$. For $a \vee y = a \vee 0$, we know that $y \leq a$. Since μ is a fuzzy ideal of L, we have $\mu(y) \geq \mu(a)$. Thus $\mu(y) \geq \bigvee_{a \vee y = a \vee 0} \mu(a) \geq t$, that is $y \in \mu_t$. Therefore, $[0]_{(\mu,t)} \subseteq \mu_t$. From the above, we obtain that $[0]_{(\mu,t)} = \mu_t$. □

V. Rough fuzzy lattices (ideals, filters) of lattices based on fuzzy ideals

In this section, we investigate rough fuzzy lattices (ideals, filters) of lattices based on fuzzy ideals.

Definition 5.1: Let μ be a fuzzy ideal of L, $t \in [0,1]$ and α be a fuzzy subset of L. Then the lower and upper approximation of α are defined as

$$\underline{U}(\mu,t,\alpha)(x) = \bigwedge\{\alpha(y)|y \in [x]_{(\mu,t)}\},$$

and

$$\overline{U}(\mu,t,\alpha)(x) = \bigvee\{\alpha(y)|y \in [x]_{(\mu,t)}\},$$

respectively.

(i) α is called a lower (upper) rough fuzzy lattice (ideal, filter) of L if $\underline{U}(\mu,t,\alpha)$ ($\overline{U}(\mu,t,\alpha)$) is a fuzzy lattice (ideal, filter) of L.

(ii) α is called a rough fuzzy lattice (ideal, filter) of L if $\underline{U}(\mu,t,\alpha)$ and $\overline{U}(\mu,t,\alpha)$ are fuzzy lattices (ideals, filters) of L .

Lemma 5.2: Let μ be a fuzzy ideal of L, $t \in [0,1]$ and α be a fuzzy subset of L. Then

(1) $\underline{U}(\mu,t,\alpha_t) = \underline{U}(\mu,t,\alpha)_t$.

(2) $\overline{U}(\mu,t,\alpha_t) = \overline{U}(\mu,t,\alpha)_t$.

Proof. It is straightforward. □

Theorem 5.3: Let μ be a fuzzy ideal of L and $t \in [0,1]$.

(1) If α is a fuzzy sublattice of L, then α is an upper rough fuzzy sublattice of L.

(2) If α is a fuzzy ideal of L, then α is an upper rough fuzzy ideal of L.

(3) If α is a fuzzy filter of L, then α is an upper fuzzy rough filter of L.

Proof. (1) Let α be a fuzzy sublattice of L. Then α_t is a sublattice of L. That is $\alpha_t \vee \alpha_t \subseteq \alpha_t$ and $\alpha_t \wedge \alpha_t \subseteq \alpha_t$. It follows from Proposition 3.9 and Lemmas 4.2 and 5.2 that $\overline{U}(\mu,t,\alpha)_t \vee \overline{U}(\mu,t,\alpha)_t = \overline{U}(\mu,t,\alpha_t) \vee \overline{U}(\mu,t,\alpha_t) \subseteq \overline{U}(\mu,t,\alpha_t \vee \alpha_t) \subseteq \overline{U}(\mu,t,\alpha_t) = \overline{U}(\mu,t,\alpha)_t$ and $\overline{U}(\mu,t,\alpha)_t \wedge \overline{U}(\mu,t,\alpha)_t = \overline{U}(\mu,t,\alpha_t) \wedge \overline{U}(\mu,t,\alpha_t) \subseteq \overline{U}(\mu,t,\alpha_t \wedge \alpha_t) \subseteq \overline{U}(\mu,t,\alpha_t) = \overline{U}(\mu,t,\alpha)_t$. Thus, $\overline{U}(\mu,t,\alpha)_t$ is a sublattice of L. This means that $\overline{U}(\mu,t,\alpha)$ is a fuzzy sublattice of L. It follows from Definition 5.1 that α is an upper rough fuzzy sublattice of L.

(2) Let α be a fuzzy ideal of L. Then α is a fuzzy sublattice of L. It follows from (1) that $a \vee b \in \overline{U}(\mu,t,\alpha_t)$ for all $a,b \in \overline{U}(\mu,t,\alpha_t)$. Let $c \in L, d \in \overline{U}(\mu,t,\alpha_t)$ and $c \leq d$. Then there exists $e \in [d]_{(\mu,t)} \cap \alpha_t$. Now Let $f \in [c]_{(\mu,t)}$. Then $e \wedge f \in [d]_{(\mu,t)} \wedge [c]_{(\mu,t)} \subseteq [c \wedge d]_{(\mu,t)} = [c]_{(\mu,t)}$. Since α_t is an ideal of L, $e \wedge f \leq e$, we have $e \wedge f \in \alpha_t$. Thus $[c]_{(\mu,t)} \cap \alpha_t \neq \emptyset$. This means that $c \in \overline{U}(\mu,t,\alpha_t)$. Thus, $\overline{U}(\mu,t,\alpha_t)$ is an ideal of L. It follows from Lemma 5.2 that $\overline{U}(\mu,t,\alpha_t) = \overline{U}(\mu,t,\alpha)_t$. Hence, $\overline{U}(\mu,t,\alpha)_t$ is an ideal of L, which implies $\overline{U}(\mu,t,\alpha)$ is an fuzzy ideal of L. It follows from Definition 5.1 that α is an upper rough fuzzy ideal of L.

(3) Let α be a filter of L. Then it follows from (1) that $a \wedge b \in \overline{U}(\mu,t,\alpha_t)$ for all $a,b \in \overline{U}(\mu,t,\alpha_t)$. Let $c \in L, d \in \overline{U}(\mu,t,\alpha_t)$ and $c \geq d$. Then there exists $e \in [d]_{(\mu,t)} \cap \alpha_t$. Now Let $f \in [c]_{(\mu,t)}$. Then $e \vee f \in [d]_{(\mu,t)} \vee [c]_{(\mu,t)} \subseteq [c \vee d]_{(\mu,t)} = [c]_{(\mu,t)}$. Since α_t is a filter of L, $e \vee f \geq e$, we have $e \vee f \in \alpha_t$. Thus $[c]_{(\mu,t)} \cap \alpha_t \neq \emptyset$. This means that $c \in \overline{U}(\mu,t,\alpha_t)$. Thus, $\overline{U}(\mu,t,\alpha_t)$ is a filter of L. It follows from Lemma 5.2 that $\underline{U}(\mu,t,\alpha_t) = \underline{U}(\mu,t,\alpha)_t$. Hence, $\underline{U}(\mu,t,\alpha)_t$ is a filter of L, which implies $\underline{U}(\mu,t,\alpha)$ is a fuzzy filter of L. It follows from Definition 5.1 that α is an upper rough fuzzy filter of L. □

Theorem 5.4: Let $U(\mu,t)$ be a $\vee$-complete congruence relation on L.

(1) If α is a fuzzy sublattice of L and $U(\mu,t)$ is a $\wedge$-complete congruence relation on L, then A is a lower rough fuzzy sublattice of L.

(2) If α is a fuzzy ideal of L, then α is a lower rough fuzzy ideal of L.

(3) If α is a filter of L, then α is a lower rough fuzzy filter of L.

Proof. (1) Let α be a fuzzy sublattice of L. Then α_t is a sublattice of L. That is $\alpha_t \vee \alpha_t \subseteq \alpha_t$ and $\alpha_t \wedge \alpha_t \subseteq \alpha_t$. It follows from Proposition 3.9, Lemmas 4.2 and 5.2 that $\underline{U}(\mu,t,\alpha)_t \vee \underline{U}(\mu,t,\alpha)_t = \underline{U}(\mu,t,\alpha_t) \vee \underline{U}(\mu,t,\alpha_t) \subseteq \underline{U}(\mu,t,\alpha_t \vee \alpha_t) \subseteq \underline{U}(\mu,t,\alpha_t) = \underline{U}(\mu,t,\alpha)_t$ and $\underline{U}(\mu,t,\alpha)_t \wedge \underline{U}(\mu,t,\alpha)_t = \underline{U}(\mu,t,\alpha_t) \wedge \underline{U}(\mu,t,\alpha_t) \subseteq \underline{U}(\mu,t,\alpha_t \wedge \alpha_t) \subseteq \underline{U}(\mu,t,\alpha_t) = \underline{U}(\mu,t,\alpha)_t$. Thus, $\underline{U}(\mu,t,\alpha)_t$ is a sublattice of L. This means that $\underline{U}(\mu,t,\alpha)$ is a fuzzy sublattice of L. It follows from Definition 5.1 that α is a lower rough fuzzy sublattice of L.

(2) Let α be a fuzzy ideal of L. Then α is a fuzzy sublattice of L. It follows from (1) that $a \vee b \in \underline{U}(\mu,t,\alpha_t)$ for all $a,b \in \underline{U}(\mu,t,\alpha_t)$. Let $a \in L, b \in \underline{U}(\mu,t,\alpha_t)$ and $a \leq b$. Then $\{x \vee y|x \in [a]_{(\mu,t)}, y \in [b]_{(\mu,t)}\} = [a]_{(\mu,t)} \vee [b]_{(\mu,t)} = [a \vee b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq \alpha_t$. Now let $x \in [a]_{(\mu,t)}$ and $y \in [b]_{(\mu,t)}$. Then $x \vee y \in [a]_{(\mu,t)} \vee [b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq \alpha_t$. That is $x \vee y \in \alpha_t$. Since α_t is an ideal of L and $x \leq x \vee y$, we have $x \in \alpha_t$. This means that $[a]_{(\mu,t)} \in \alpha_t$. Thus, $a \in \underline{U}(\mu,t,\alpha_t)$. Hence, $\underline{U}(\mu,t,\alpha_t)$ is an ideal of L. It follows from Lemma 5.2 that $\underline{U}(\mu,t,\alpha_t) = \underline{U}(\mu,t,\alpha)_t$, which implies $\underline{U}(\mu,t,\alpha)_t$ is a ideal of L. It follows from Definition 5.1 that α is a lower rough fuzzy ideal of L.

(3) Let α be a filter of L. It follows from (1) that $a \wedge b \in \underline{U}(\mu,t,\alpha_t)$ for all $a,b \in \underline{U}(\mu,t,\alpha_t)$. Let $a \in L, b \in \underline{U}(\mu,t,\alpha_t)$ and $a \geq b$. Then $\{x \wedge y|x \in [a]_{(\mu,t)}, y \in [b]_{(\mu,t)}\} = [a]_{(\mu,t)} \wedge [b]_{(\mu,t)} = [a \wedge b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq A$. Now let $x \in [a]_{(\mu,t)}$ and $y \in [b]_{(\mu,t)}$. Then $x \wedge y \in [a]_{(\mu,t)} \wedge [b]_{(\mu,t)} = [b]_{(\mu,t)} \subseteq \alpha_t$. That is $x \wedge y \in \alpha_t$. Since α is a fuzzy filter of L, α_t is a filter of L. Since $x \geq x \wedge y$, we have $x \in \alpha_t$. This means that $[a]_{(\mu,t)} \in \alpha_t$. Thus, $a \in \underline{U}(\mu,t,\alpha_t)$. Hence, $\underline{U}(\mu,t,\alpha_t)$ is a filter of L. It follows from Lemma 5.2 that $\underline{U}(\mu,t,\alpha_t) = \underline{U}(\mu,t,\alpha)_t$. Hence, $\underline{U}(\mu,t,\alpha)_t$ is a filter of L, which implies $\underline{U}(\mu,t,\alpha)$ is a fuzzy filter of L. It follows from Definition 5.1 that α is a lower rough fuzzy filter of L. □

VI. Conclusions

In this paper, we built up a connection between rough sets, fuzzy sets and lattices. Firstly, we introduced a new congruence relation induced by a fuzzy ideal of a distributive lattice, and then we presented a definition of lower and upper approximations of a subset of a distributive lattice with respect to a fuzzy ideal. Some properties of rough subsets in distributive lattices are investigated. Finally, we obtained that the notions of rough sublattices (ideals, filters), rough fuzzy sublattices (ideals, filters) are the extensions of sublattices (ideals, filters) and fuzzy sublattices (ideals, filters), respectively.

As an extension of this work, the following problems maybe considered:

(1) Roughness of distributive lattices based on fuzzy ideals.

(2) Rough prime ideals and rough fuzzy prime ideals in distributive lattices.

(3) Rough sets induced by fuzzy ideals in distributive lattices.

VII. Acknowledgements

The authors would like to express their sincere thanks to the Editors and anonymous reviewers for their most valuable comments and suggestions in improving this paper greatly.

VIII. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M.I. Ali, M. Shabir, S. Tanveer, "Roughness in hemirings", *Neural Comput. Appl.*, vol. 21, pp. 171-180, 2012.
[2] J.H. Dai, "Rough set approach to incomplete numerical data", *Inform. Sci.*, vol. 241, pp. 43-57, 2013.
[3] B.A. Davey, H.A. Priestley, "Introduction to lattices and order", *Cambridge University Press*, 2002.
[4] B. Davvaz, "Roughness in rings", *Inform. Sci.*, vol. 164, no. 1, pp. 147-163, 2004.
[5] B. Davvaz, "Roughness based on fuzzy ideals", *Inform. Sci.*, vol. 176, pp. 2417-2437, 2006.
[6] B. Davvaz, M. Mahdavipour, "Roughness in modules", *Inform. Sci.*, vol. 176, pp. 3658-3674, 2006.
[7] B. Davvaz, M. Mahdavipour, "Rough approximations in a general approximation space and their fundamental properties", *Int. J. Gen. Syst.*, vol. 37, pp. 373-386, 2008.
[8] A.A. Estaji, S. Khodaii, S. Bahrami, "On rough set and fuzzy sublattice", *Inform. Sci.*, vol. 181, pp. 3981-3994, 2011.
[9] A. Hussain, T. Mahmood, M.I. Ali, "Rough pythagorean fuzzy ideals in semigroups", *Comput. Appl. Math.*, vol. 38, no. 2, UNSP 67 , 2019.
[10] Z. Pawlak, "Rough sets", *Int. J. Inf. Comp. Sci.*, vol. 11, no. 5, pp. 341-356, 1982.
[11] R. Prasertpong, M. Siripitukdet, "On rough sets induced by fuzzy relations approach in semigroups", *Open Math.,* vol. 16, pp. 1634-1650, 2018.
[12] R. Prasertpong, M. Siripitukdet, "Applying Generalized Rough Set Concepts to Approximation Spaces of Semigroups", *IAENG Int. J. Appl. Math.,* vol. 49, no. 1, pp. 51-60, 2019.
[13] S. Rasouli, B. Davvaz, "Roughness in MV-algebras", *Inform. Sci.,* vol. 180, pp. 737-747, 2010.
[14] N. Rehman, C. Park, SIA. Shah, A. Ali, "On generalized roughness in LA-semigroups", *Math.,* vol. 6, no. 7, pp. 1-8, 2018.
[15] M. Shao, L. Guo, C. Wang, "Connections between two-universe rough sets and formal concepts", *J. Mach. Learn. Cyb.*, vol. 9, no. 11, pp. 1869-1877, 2018.
[16] S. Shao, X. Zhang, C. Bo, C. Park, "Multi-granlation rough filters and rough fuzzy filters in pseudo-BCI algebras", *J. Intell. Fuzzy Syst.*, vol. 34, no. 6, pp. 4377-4386, 2018.
[17] Q. Wang, J. Zhan, "Rough semigroups and rough fuzzy semigroups based on fuzzy ideals", *Open Math.,* vol. 14, pp. 1114-1121, 2016
[18] Q. Wang, J. Zhan, R.A. Borzooei, "A study on soft rough semigroups and corresponding decision making applications", *Open Math.,* vol. 15, pp. 1400-1413, 2017.
[19] Q. Wang, J. Zhan, "A study on Z-soft rough fuzzy semigroups and its decision making", *Int. J. Uncertain. Quan.,* vol. 8, no. 1, pp. 1-22, 2018.
[20] Q. Xiao, Z. Zhang, "Rough prime ideasl and rough fuzzy ideals in semigroups", *Inform. Sci.*, vol. 176, pp. 725-733, 2006.
[21] Y.Y. Yao, "Three-way decisions with probabilistis rough sets", *Inform. Sci.*, vol. 180, pp. 341-353, 2010.
[22] Y. Yang, B.Q. Hu, "Communication between fuzzy information systems using fuzzy covering-based rough sets", *Int. J. Approx. Reason.*, vol. 103, pp. 414-436, 2018.
[23] J. Yu, Y. Li, M. Chen, B. Zhang, W. Xu, "Decision-theoretic rough set in lattice-valued decision information system". *J. Intell. Fuzzy Syst.*, vol. 36, no. 4, pp. 3289-3301, 2019.
[24] L.A. Zadeh, "Fuzzy sets", *Inform. Control*, vol. 8, pp. 338-353, 1965.
[25] J. Zhan, X. Zhou, D. Xiang, "Roughness in n-ary semigroup based on fuzzy ideals", *J. Intell. Fuzzy Syst.*, vol. 30, pp. 2833-2841, 2016.
[26] J. Zhan, K. Zhu, L-F. Violeta, "Roughness in hemirings based on strong h-ideals". *J. Mult.-Valued Logic Soft Comput.,* vol. 29, pp. 43-66, 2017.
[27] J. Zhan, Y. Naveed, K. Madad, "Roughness in non-associative po-semihypergroups based on pseudohyperorder relations". *J. Mult.-Valued Logic Soft Comput.,* vol. 28, no. 2-3, pp. 153-177, 2017.
[28] J. Zhan, X. Zhou, D. Xiang, "Rough soft n-ary semigroups based on a novel congruence relation and corresponding decision making", *J. Intell. Fuzzy Syst.*, vol. 33, no. 2, pp. 693-703, 2017.
[29] J. Zhan, H.M. Malik, M. Akram, "Novel decision-making algorithms based on intuitionistic fuzzy rough environment", *J. Mach. Learn. Cyb.*, vol. 10, no. 6, pp. 1459-1485.
[30] X. Zhang, J. Dai, Y. Yu, "On the union and intersection operations of rough sets based on various approximation spaces", *Inform. Sci.*, vol. 292, pp. 214-229, 2015.
[31] X. Zhang, D. Maio, C. Liu, M. Le, "Constructive methods of rough approximation operators and multigranuation rough sets", *Knowl.-Based Syst.*, vol. 91, pp. 114-125, 2016.
[32] L. Zhang, X. Ma, J. Zhan, "Rough soft BCK-algebras and their decision making", *Ital. J. Pure Appl. Math.*, vol. 39, pp. 649-650, 2017.
[33] L. Zhang, J. Zhan, "Fuzzy soft β-covering based fuzzy rough sets and corresponding decision-making applications", *J. Mach. Learn. Cyb.*, vol. 10, no. 6, pp. 1487-1502, 2019.
[34] W. Zhu, "Generalized rough sets based on relations", *Inform. Sci.*, vol. 177, no. 22, pp. 4997-5011, 2008.