On R-hued Coloring of Some Perfect and Circulant Graphs

Chenxu Yang, Xingchao Deng, Ruifang Shao

***Abstract*—A r-*hued* k-*coloring* of a graph G is a proper coloring with k colors such that for every vertex v with degree $d(v)$ in G, the neighbors of v must be colored by at least $min\{d(v), r\}$ different colors. The r-*hued chromatic number*, $\chi_r(G)$, of G is the minimum k for which G has a r-*hued* k-*coloring*. In this paper, we study the r-hued coloring of some perfect and circulant graphs.**

***Index Terms*—r-hued coloring, r-hued number, perfect graphs, tree, circulant graphs.**

I. INTRODUCTION

IN this paper, we consider graphs which are connected, finite, undirected and simple. A k-coloring of G is proper if no two distinct adjacent vertices have the same color. For any integers a and b with $a \leq b$, we use the notation $[a, b]$ for the set $\{a, a+1, \cdots, b\}$; and $[k]$ for $[1, k]$. Let $i \pmod k$ denote the remainder of i module k. The smallest integer k such that G has a proper k-coloring is known as the chromatic number of G, denoted by $\chi(G)$. For every $v \in V(G)$, $N_G(v)$ denotes the neighbor set of v in G and $N_G[v] = N_G(v) \cup \{v\}$. In [1], Lai et al. proposed r-hued Coloring of Graphs based on multi-agentsystems(MAS). An MAS can be modeled as a graph for which a typical vertex represents a situation in which the typical individual has a great variety in the type of relations. Thus, the overall interactions would not be so limited but more hued. This motivates the definition of the hued coloring. A (k, r)-coloring c of a graph G is a proper k-coloring of G such that for every $v \in V(G)$, we have $|c(N_G(v))| \geq \min\{d(v), r\}$. where a typical vertex is adjacent to more than one vertex with different colors. The r-*hued chromatic number*, $\chi_r(G)$, of G is the minimum k for which G has a r-*hued* k-*coloring*. By definition, $\chi_1(G) = \chi(G)$. The *2-hued chromatic number* of G is named the dynamic chromatic number, denoted by $\chi_2(G)$ or $\chi_d(G)$. It is easy to know that $\chi(G) \leq \chi_2(G)$.

Recently, the r-hued coloring of a graph G has been studied by many research groups, see [2], [4], [5], [6], [7], [8], [9], [10] and [11]. It is shown in [3] that for $n \geq 3$, if $3|n$, then $\chi_2(C_n) = 3$, if $n = 5$, then $\chi_2(C_n) = 5$, and $\chi_2(C_n) = 4$ otherwise. In [1], it is proved that for every graph G, if $\Delta(G) \leq 3$, $\chi_2(G) \leq 4$ unless $\chi_2(G) = 5$ for $G = C_5$; and if $\Delta(G) \geq 4$, then $\chi_2(G) \leq \Delta(G) + 1$. Moreover, Song et al. in [8] proposed the following conjecture.

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Chenxu Yang and Ruifang Shao are College of Mathematical Science, Tianjin Normal University.

Corresponding author: Xingchao Deng is with the College of Mathematical Science Tianjin Normal University, Tianjin, 300387, China. Email: dengyuqiu1980@126.com.

Conjecture 1.1[8] when G is a planar graph, then

$$\chi_r(G) \leq \begin{cases} r+3, & \text{if } 1 \leq r \leq 2, \\ r+5, & \text{if } 3 \leq r \leq 7, \\ \lfloor \frac{3r}{2} \rfloor + 1, & \text{if } r \geq 8. \end{cases} \quad (1)$$

Observation 1 $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$. Equality holds for trees.

Observation 2 If $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G)$.

Observation 3 For any graph G, $\chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_r(G) \leq \cdots \leq \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi(G^2)$.

The $l-th$ power of a graph G, denoted by $G^{(l)}$, is a graph with the same vertex set of G such that two vertices are adjacent if and only if their distance is at most l in G.

Conjecture 1.2[11] Let G be a planar graph of maximum degree Δ. The chromatic number of its square is

$$\chi(G^2) \leq \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta+5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases} \quad (2)$$

Recently, C. Thomassen [12] proved that the conjecture is correct for $\Delta = 3$.

This conjecture has also been generalized to the *list coloring*.

Conjecture 1.3[13] Let G be a planar graph with maximum degree Δ, then the list chromatic number of its square is

$$ch(G^2) \leq \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta+5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases} \quad (3)$$

Cranston and Kim [14] proved that the square of any connected graph (with $\Delta \leq 3$)(not necessarily planar) is 8-choosable, except for the Petersen graph. Havet et al. [4] proved the conjecture asymptotically:

Theorem 1.4[4] For sufficient large Δ, the square of every planar graph G has list chromatic number at most $(1 + o(1)) \frac{3}{2} \Delta$.

In [1], Lai et al. obtained a theorem analogous of Brooks Theorem for dynamic chromatic number. The above conjectures and some results in [16], [17], [18], [19], [20], [21], [22] make us consider the *r-hued coloring* of power graphs.

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II. R-HUED COLORING OF SOME PERFECT GRAPHS

Let $C_n(P_n)$ denote cycle(path) with n vertices, respectively. Since any three successive vertices induce a K_3 in P_n^2 and C_m^2, we have $\chi_2(C_m^2) = \chi(C_m^2)$ and $\chi_2(P_n^2) = \chi(P_n^2)$.

Theorem 2.1 For any integers $n \geq 3$ and $2 \leq l \leq n-1$, we have $\chi_2(P_n^l) = \chi(P_n^l) = l+1$.

Proof Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Clearly, $\chi_2(P_n^2) \geq 3$ since v_1, v_2, v_3 induce a subgraph K_3. It is obvious that the coloring $c : c(v_i) \equiv i \pmod 3$ is a 2-hued coloring, thus $\chi_2(P_n^2) \leq 3$. Therefore, we obtain that $\chi_2(P_n^2) = 3$.

Similarly, we have $\chi_2(P_n^l) = \chi(P_n^l) = l+1$ since any $l+1$ successive vertices induce a K_{l+1} and $c : c(v_i) \equiv i \pmod{l+1}$ is a 2-hued coloring of P_n^l. □

Lemma 2.2(*Bezout's Theorem*)[7] For any relatively prime positive integers a and b, then there are integers x and y such that $m = ax + by$.

Moreover, for large enough integer m, there are nonnegative integers x and y satisfy the above equation. This result can be proved as follows.

W.l.o.g., assume that $b > a$. Since there are integers x_0 and y_0 such that $ax_0 + by_0 = 1$ with $-a < y_0 < a$ and $-b < x_0 < b$ by relatively primeness of a and b, one of x_0 and y_0 is positive and another is negative. Suppose that x_0 is negative, then y_0 is positive, and there are nonnegative integers q and r with $0 \leq r < a$ such that $m = qa + r$ by division algorithm. So we have

$$m = qa + r = qa + rax_0 + rby_0 = (q + rx_0)a + (ry_0)b.$$

Let $x = q + rx_0$ and $y = ry_0$. Then $y \geq 0$, and $x \geq 0$ when $q \geq ab > rb \geq -rx_0$, so we have the desired result when $m \geq a^2b + a > a^2b + r$.

Theorem 2.3 Let $n \geq 3$ be an integer. Then

$$\chi_3(P_n^2) = \begin{cases} 3, & n = 3, \\ 4, & otherwise. \end{cases}$$

and

$$\chi_r(P_n^2) = \begin{cases} 3, & n = 3, \\ 4, & n = 4, \\ 5, & otherwise. \end{cases}$$

for $r \geq 4$.

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of P_n^2. For $n = 3$, $P_3^2 = K_3$, so $\chi_3(P_3^2) = \chi_r(P_3^2) = 3$. For $n = 4$, it is obvious that $\Delta(P_4^2) = 3$. By Observations 1-2, $\chi_3(P_4^2) \geq 4$, so $\chi_r(P_4^2) \geq 4$. But $\chi_3(P_4^2) = \chi_4(P_4^2) \leq 4$ because P_4^2 has four vertices, hence $\chi_3(P_4^2) = \chi_r(P_4^2) = 4$.

Assume that $n \geq 5$. By Observations 1-2, we have $\chi_3(P_n^2) \geq 4$ and $\chi_r(P_n^2) \geq 5$ for $r \geq 4$ since $\Delta(P_n^2) = 4$. Consider the folowing 4-coloring c of P_n^2 :

$$c : \{v_1, v_2, \cdots, v_n\} \to [4], c(v_i) \equiv i \pmod 4.$$

It is clear that $|c(N(v_i))| = 3$ and c is a 3-hued 4-coloring of P_n^2. Hence $\chi_3(P_n^2) = 4$. Similarly, the coloring c with $c(v_i) \equiv i \pmod 5$ is a r-hued 5-coloring of P_n^2, thus $\chi_r(P_n^2) = 5$ for $r \geq 4$. □

Let T be a tree with maximum degree $\triangle$ and

$$f(r) = \begin{cases} \Delta + 1, & \text{if } r \leq \Delta, \\ \Delta + 2, & \text{if } r = \Delta + 1, \\ min\{2\Delta + 1, r\}, & \text{if } r \geq \Delta + 2. \end{cases}$$

Theorem 2.4 If T is a tree with $|V(T)| \geq 5$, which is not a path, then $\chi_r(T^2) \leq f(r)$.

Proof We argue by induction on n= $|V(T)|$. For $n = 5$, let T be a tree with $\Delta(T) = 3$. W.l.o.g., we may assume that $d(v_2) = 3,\ d(v_5) = d(v_3) = d(v_1) = 1,\ d(v_4) = 2$. When $r \leq \Delta$, we let $c(v_5) = c(v_1) = 2,\ c(v_1) = 1, c(v_3) = 3,\ c(v_4) = 4$. If $r > \Delta$, we give the following coloring: $c(v_5) = c(v_1) = 2,\ c(v_1) = 1,\ c(v_3) = 3,\ c(v_4) = 4$. It is easy to see that c is a r-hued coloring of T^2 with 4 colors. Moreover, if T is a star with 5 vertices, then T^2 is a K_5 which needs 5 colors in any r-hued coloring. Now the coloring giving each vertex different colors.

Assume that $n \geq 5$ and the theorem holds for smaller values of n. Let T be a tree on n vertices and v_0 be a leaf adjacent to v_1 with minimized degree among all vertices which are adjacent to leaves in T.

By induction, $\chi_r((T - v_0)^2) \leq f(r)$. Let c be a r-hued coloring of $(T - v_0)^2$ with at most $f(r)$ colors. Since v_0 is adjacent to at most $d(v_1) \leq \Delta(T)$ vertices, we can choose $c^*(v_0) \in \{1, 2, 3, \ldots, f(r)\} \backslash \{c(N_T(v))\}$ and $c^*(v) = c(v)$, for $v \in V(T - v_0)$, then c^* is a r-hued coloring of T^2 with at most $f(r)$ colors. □

Theorem 2.5 If T is a tree with $|V(T)| \geq 3$, then T^2 is a perfect graph.

Proof We prove by induction on $n = |V(T)|$. For $n = 3$, it is easy to see that $\chi(H)$=$\omega(H)$ for any induced subgraph H of T^2. Thus the theorem is valid for $n = 3$.

Assume that $n \geq 4$ and the theorem holds for trees with at most $n-1$ vertices. Let T be a tree with n vertices and v_0 be a leaf of T with its neighbor degree minimized in T. Since $|V(T - v_0)| < n$, by induction, we have $\chi(H) = \omega(H)$ for any induced subgraph H of $(T - v_0)^2$.

By the definition of perfect graph, we will prove that $\chi(H) = \omega(H)$ for every induced subgraph H of T^2 by the following two cases.

Case 1. $v_0 \notin V(H)$. Then H is an induced subgraph of $(T - v_0)^2$. By induction we know that $\chi(H) = \omega(H)$.

Case 2. $v_0 \in V(H)$. The vertex v_0 is adjacent to v and has neighbors in $N_G(v)$ in T^2, where $N_G(v) = \{v_1, \ldots, v_s\}$, $s \leq \Delta$. W.o.l.g., v_i is adjacent to w_i and $N_G(w_i) = \{w_{i1},\ w_{i2},\ w_{i3},\ \ldots,\ w_{it}\}$, where $w_{i1} = v_i$ and $t \leq \Delta$, as Figure 1 showen.

Case 2.1 Since $v_0 \in V(H)$ and $\{N[v] \backslash v_0\} \bigcap V(H) = \varnothing$, we have that $V(H) \subset \{v_0 \cup V(T) \setminus N[v]\}$. In this case, v_0 is an isolated vertex of H, so $\chi(H) = \omega(H)$.

Case 2.2 When $v_0 \in V(H)$ and $V(H) \bigcap \{N[v] \backslash v_0\} = \{v, v_0, \ldots, v_{s1}\}$, $s_1 \leq s$. By induction, we know that

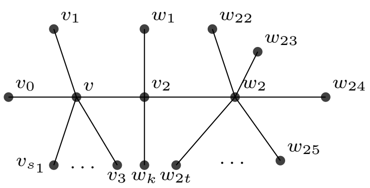

Figure 1: The graph in Case 2 of Theorem 2.5

$\chi(H - v_0) = \omega(H - v_0)$. If $\omega(H) = \omega(H - v_0)$, then we have $\chi(H) = \omega(H)$. Otherwise $\omega(H) = \omega(H - v_0) + 1$, then the maximal clique of H is K_{s1+1}. In this case, we also have $\chi(H) = \omega(H)$.

Case 2.3 $v_0 \in V(H)$ and $[N[v] \backslash v_0] \bigcap V(H) \neq \varnothing$, $V(H) = [\{N[v] \backslash v\} \bigcup \{v_0\} \bigcup_{i=1}^{t} N_T[N_T[v_i]]$, this case is obvious correct. Thus we proved that any induced subgraph H of T^2 satisfy $\chi(H) = \omega(H)$. □

By Theorems 2.4 and 2.5, one can easily obtain that $\chi(T^2) = \chi_1(T^2) = \chi_2(T^2) = \cdots = \chi_r(T^2) = \Delta(T) + 1$, if $r \leq \Delta(T)$.

Theorem 2.6 If T is a tree with $\Delta(T) \leq 3$, then T^2 is a planar graph.

Proof Since $\Delta(T) \leq 3$, we add a vertex to T which is adjacent to a 2-vertex, one by one. By the process, T has only 3-vertices and 1-vertices. Assume that T is such a tree. Let $d(z) = 3$, where z is adjacent to three vertices u, v, w, as shown in Figure 2. To prove the theorem, we only need to give a method to embed T^2 into plane.

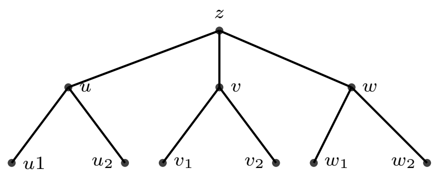

Figure 2: The first graph in Theorem 2.6

Now, we embed edges of T^2 in plane using the following method. Firstly, we consider the edge $zw \in T$. The plane are divided into two parts, the upper half plane and lower half plane. $zv,\ zu$ can be embed in the upper half plane according to clockwise around the root z. $ww_1,\ ww_2$ can be embed in the lower half plane according to the clockwise around the root w, in turn. The construction is illustrated in Figure 3. For any two edges $zx,\ zy \in E(T)$, then $xy \in E(T^2)$, so we obtain a new triangle uvw as shown in Figure 3. The triangles zvu and zuw are denoted by $zv-$face, $zu-$face respectively. Thus we complete the embedding of the children nodes of z.

We implement the process one by one for $v \in V(T)$. Note that $d(v) = 1$ or $d(v) = 3$. If $d(v) = 1$, we complete the embedding progress. When $d(v) = 3$, we continue the following process:

W.o.l.g., suppose $vv_1, vv_2 \in E(T)$, we can embed the edges incident with v into the $zv_1 -$ face and similarly for u, the construction is illustrated in Figure 4. When $zv, zu \in$

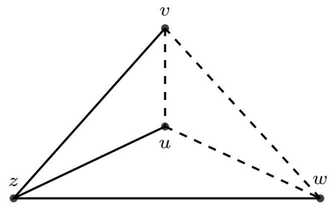

Figure 3: The second graph in Theorem 2.6

$E(T)$ are embedded, $vv_1, vv_2(uu_1, uu_2)$ can be embedded in $zv -$ face($zu -$ face) according to clockwise around the root $v(u)$, in turn. For any two edges $zx, zy \in E(T)$, $xy \in E(T^2)$, so we obtain two new triangles v_1v_2z and u_1u_2z. The construction is illustrated in Figure 4.

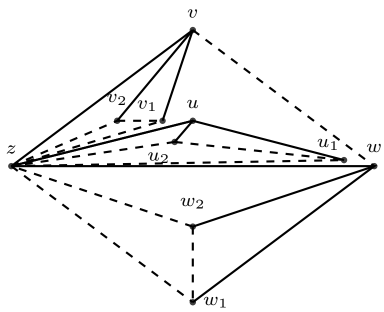

Figure 4: The third graph in Theorem 2.6

Consider the vertex v, the edges vv_1, v_1u_2, v_2v and vv_2, v_2z, zv produce two triangles which are denoted by $zv_1 -$ face and $zv_2 -$ face, respectively.

Suppose we have embedded the $(n-2)$-th vertex $v_{(n-2)}$. If every 3-vertex is embedded, then the process can be finished. If we have a vertex v_{n-2} with $d(v_{n-2}) = 3$, w.o.l.g., v_{n-2} is adjacent to x, v_{n-1}, v_n and x is its parent, the others are its children. So the edges $v_{n-2}v_{n-1}, v_{n-2}v_n, v_{n-1}v_n$ can be embedded in the $xv_{n-2} -$ face. We can get a new triangle $xx_{n-1}x_n$ which is denoted by $v_{(n-2)}$-face. We note that the edges are not crossing and the process can be finished with finite steps. For any $u, v \in V(T),\ d(u,v) = 2$, we add the edge uv into the T, so we get the graph T^2. Thus T^2 is a planar graph. □

By Theorems 2.4 and 2.6, we know that T^2 confirms Conjecture 1.1, when T is a tree with $\Delta(T) \leq 3$.

III. R-HUED COLORING OF SOME CIRCULANT GRAPHS

Firstly, we study the r-hued coloring of C_n^2.

Theorem 3.1 Let $m \geq 3$ be an integer, then

$$\chi(C_m^2) = \chi_2(C_m^2) = \begin{cases} 3, & \text{if } 3|\text{m}, \\ 5, & \text{if m} = 5, \\ 4, & \text{otherwise.} \end{cases}$$

Proof Let $\{v_1, v_2, \cdots, v_m\}$ be the vertex set of C_m^2. It is obvious that $\chi_2(C_m^2) \geq 3$ by Observation 1.

Case 1. $m = 3k$ for $k \geq 1$ or $m = 5$.
Clearly, $c : c(v_i) \equiv i \pmod 3$ is a 2-hued 3-coloring, so $\chi(C_m^2) = \chi_2(C_m^2) = 3$.
It is obvious that $C_5^2 = K_5$, so $\chi(C_5^2) = \chi_2(C_5^2) = 5$.

Case 2. $m = 3k + 1$ for $k \geq 1$.

It is obvious that the coloring $c : \{v_1, v_2, \cdots, v_{3k}\} \to [3]$ with $c(v_i) \equiv i \pmod 3$ and $c(v_{3k+1}) = 4$ is a hued coloring of C_m^2, so $\chi_2(C_m^2) \leq 4$. Since adjacent vertices must receive different colors, we have $\chi(C_m^2) \geq 4$. Hence $\chi(C_m^2) = \chi_2(C_m^2) = 4$.

Case 3. $m = 3k + 2$ for $k \geq 1$.

Suppose we color the graph by three colors, Since any consecutive three vertices must be colored by different colors, we obtain that $c(v_i) \equiv i(\pmod 3)$ for $i \leq 3k$ and $c(v_{3k+1}), c(v_{3k+2}) \notin [3]$, a contradicition. So we need at least four colors, i.e., $\chi_2(C_m^2) \geq 4$.

(i) Assume that $m \equiv 2 \pmod 4$, i.e., $m = 4t + 2$.

Since $m = 3k + 2$, we have that $m \geq 14$ in this case. Thus, we can define a 4-coloring c with $c(v_i) \equiv i \pmod 4$ for $i \leq 4t - 4$, and $c(v_{4t}) = c(v_{4t-3}) = 1$, $c(v_{4t+1}) = c(v_{4t-2}) = 2$ and $c(v_{4t+2}) = c(v_{4t-1}) = 3$. Clearly, this is a 2-hued coloring of C_m^2, so $\chi_2(C_m^2) \leq 4$, thus $\chi_2(C_m^2) = 4$.

(ii) Assume that $m \equiv 3$ or $4 \pmod 4$.

It is not difficult to verify that the coloring $c : \{v_1, v_2, \cdots, v_{3k+2}\} \to [4]$ with $c(v_i) \equiv i \pmod 4$ is a 2-hued coloring of C_m^2, so $\chi_2(C_m^2) \leq 4$. Hence $\chi_2(C_m^2) = 4$.

(iii) Assume that $m \equiv 1 \pmod 4$, *i.e.*, $m = 4t + 1$.

In this case $m \geq 17$, so we have $m = 3x + 4y$ for some positive integers x, y by Lemma 2.2. Define a 4-coloring c with $c(v_i) \equiv i \pmod 3$ for $i \leq 3x$ and $c(v_i) \equiv i \pmod 4$ for $3x+1 \leq i \leq m$. It is clear that c is a hued coloring of C_m^2, so $\chi_2(C_m^2) \leq 4$, hence $\chi_2(C_m^2) = 4$.□

Since any $l + 1$ successive vertices induce a K_{l+1} in C_m^l, we have $\chi_2(C_m^l) = \chi(C_m^l) \geq l + 1$. It is obvious that $\chi_2(C_m^l) = \chi(C_m^l) = m$ for $m \leq l+1$ since $C_m^l = K_m$.

Theorem 3.2 Let $m > l + 1 \geq 4$ be integers with $m = k(l + 1) + t$ and $0 \leq t \leq l$. Then

$$\chi(C_m^l) = \chi_2(C_m^l) = \begin{cases} l+1, & t = 0, \\ l+2, & k \geq t, \\ l+1+t, & k = 1 \text{ and } t \in [l], \\ l+2+q, & t > k \text{ and } t = kq + r. \end{cases}$$

Proof Let $\{v_1, v_2, \cdots, v_m\}$ be the vertex set of C_m^l. Clearly, $\chi_2(C_m^l) \geq l + 1$ since any adjacent vertices must receive different colors.

Case 1. $t = 0$, then $m = k(l + 1)$. The coloring c with $c(v_i) \equiv i \pmod{l + 1}$ is a 2-hued coloring, so $\chi_2(C_m^l) \leq l + 1$. Hence $\chi(C_m^l) = \chi_2(C_m^l) = l + 1$ in this case.

Case 2. $t \neq 0$ and $k \geq t$. If we use $l + 1$ colors to color C_m^l, then we can assume that the coloring c is that $c(v_i) \equiv i \pmod{l + 1}$ for $1 \leq i \leq k(l + 1)$. Then $c(v_{k(l+1)+1}) \notin [1, l + 1]$ since some adjacent vertex can receive the same color. Hence $\chi_2(C_m^l) \geq l + 2$.
Since $k \geq t$, $m = (k - t)(l + 1) + t(l + 2)$. We define a coloring $c : c(v_i) \equiv i(\pmod{l+1}$ for $1 \leq i \leq (k-t)(l+1)$, and $c(v_i) \equiv i \pmod{l + 2}$ for $(k - t)(l + 1) + 1 \leq i \leq m$. It is clear that c is a hued coloring, so $\chi_2(C_m^l) \leq l + 2$. Therefore $\chi(C_m^l) = \chi_2(C_m^l) = l + 2$ in this case.

Case 3. $t \in [l]$ and $k = 1$, then $m = l + 1 + t$. A 2-hued coloring c with $c(v_i) = i$ for $i \in [l + 1]$ would satisfy $c(v_{(l+1)+1}) = l + 1 + 1$, $c(v_{(l+1)+2}) = l + 1 + 2$, $\cdots$, $c(v_{(l+1)+t}) = l + 1 + t$ since any $l + 1$ successive vertices must receive different colors. Thus we obtain that $\chi(C_m^l) = \chi_2(C_m^l) = l + 1 + t$.

Case 4. $t \in [l]$ and $2 \leq k < t$, then $t = kq + r$ with $q \geq 1$, and $m = (k-r)(l+1+q)+r(l+2+q)$ by Lemma 2.2.

Subcase 4.1 $r = 0$, then $m = k(l + 1 + q)$. The coloring $c : c(v_i) \equiv i \pmod{l + 1 + q}$ for $1 \leq i \leq k(l + 1 + q)$ is an optimal 2-hued coloring, so $\chi_2(C_m^l) \leq l + 1 + q$. Let $m = p_1(l + 1) + p_2(l + 2) + \cdots + p_s(l + s)$, where $p_j \geq 0$ for $1 \leq j < s$ and $p_s > 0$, such that s is as small as possible. Then we can use $l + s$ colors to color C_m^l (it is enough to define $c(v_i) \equiv i \pmod{l + 1}$ for $i \in [p_1(l + 1)]$, $c(v_i) \equiv (i - p_1(l + 1)) \pmod{l + 2}$ for $i - p_1(l + 1) \in [p_2(l + 2)]$, $\cdots$, and $c(v_i) \equiv (i - \sum_{j=1}^{s-1} p_1(l + j)) \pmod{l + s})$ for $i - \sum_{j=1}^{s-1} p_1(l + j) \in [p_s(l + s)]$). It is clear that $s = q + 1$, hence $\chi(C_m^l) = \chi_2(C_m^l) = l + 1 + q$.

Subcase 4.2 Suppose that $0 < r < k$. We consider the coloring c with $c(v_i) \equiv i \pmod{l + 1 + q}$ for $1 \leq i \leq (k - r)(l + 1 + q)$ and $c(v_i) \equiv i \pmod{l + 2 + q})$ for $(k-r)(l+1+q)+1 \leq i \leq m$. Hence $\chi_2(C_m^l) \leq l+2+q$. We obtain that $\chi(C_m^l) = \chi_2(C_m^l) = l + 2 + q$ similarly. □

Theorem 3.3 Let $m \geq 3$ be an integer and $S = Z^+ \backslash \{3, 5, 6, 7, 11\}$. Then

$$\chi_3(C_m^2) = \begin{cases} 3, & m = 3, \\ 4, & m \equiv i \pmod 4, i \in [4] \text{ and m} \in \text{S} \\ 5, & m = 5, 6, 7, 11. \end{cases}$$

Proof Let $\{v_1, v_2, \cdots, v_m\}$ be a vertex set of C_m^2.

Case 1. $m \in [3, 5]$. Clearly, C_m^2 is one of K_3, K_4 and K_5, so $\chi_3(C_3^2) = 3, \chi_3(C_4^2) = 4$, and $\chi_3(C_5^2) = 5$.

Case 2. $m \equiv i \pmod 4$ for $i \in [4]$, $i.e.,\ m = 4k + i$. Since $m \geq 5,\ \Delta(C_m^2) = 4$. By Observation 1, $\chi_3(C_m^2) \geq 4$.

(1) $m \equiv 0 \pmod 4$, $i.\ e.,\ m = 4k + 4$ for some $k \geq 1$. Consider a 4-coloring of c of C_m^2,

$$c : \{v_1, v_2, \cdots, v_m\} \to [4], \text{ with } c(v_i) = i \pmod 4.$$

It is clear that $|c(N(v_i))| = 3$ and c is a 3-hued 4-coloring of C_m^2. Hence $\chi_3(C_m^2) = 4$ in this case.

(2) $m \equiv 1 \pmod 4$, $i.e.,\ m = 4k + 1$ for some $k \geq 2$.

Consider a 4-coloring c of C_m^2. $c(v_i) = i \pmod 4$ for $i \leq 4(k-1)$, moreover $c(v_{4k-3}) = 2,\ c(v_{4k-2}) = 1,\ c(v_{4k-1}) = 3,\ c(v_{4k}) = 2$ and $c(v_{4k+1}) = 4$. It is clear that $|c(N(v_i))| = 3$ and c is a 3-hued 4-coloring of C_m^2. Hence $\chi_3(C_m^2) = 4$ in this case.

(3) $m \equiv 2 \pmod 4$, $i.e.,\ m = 4k + 2$ for some $k \geq 2$.

Consider the following 4-coloring c of C_m^2. $c(v_i) = i \pmod 4$ for $i \leq 4(k-2)$, moreover $c(v_{4k-7}) = 1,\ c(v_{4k-6}) = 2,\ c(v_{4k-5}) = 3,\ c(v_{4k-4}) = 1,\ c(v_{4k-3}) = 4,\ c(v_{4k-2}) = 2,\ c(v_{4k-1}) = 1,\ c(v_{4k}) = 3,\ c(v_{4k+1}) = 2$, and $c(v_{4k+2}) = 4$. It is clear that $|c(N(v_i))| = 3$ and c is a 3-hued 4-coloring of C_m^2. Hence $\chi_3(C_m^2) = 4$ in this case.

(4) $m \equiv 3 \pmod 4$, $i.e.,\ m = 4k + 3$ for some $k \geq 3$.

Consider a 4-coloring of c of C_m^2. $c(v_i) = i \pmod 4$ for $i \leq 4(k-2)$, and $c(v_{4k-7}) = 1, c(v_{4k-6}) = 2, c(v_{4k-5}) = 3, c(v_{4k-4}) = 1,\ c(v_{4k-3}) = 4,\ c(v_{4k-2}) = 2,\ c(v_{4k-1}) = 1,\ c(v_{4k}) = 3, c(v_{4k+1}) = 2,\ c(v_{4k+2}) = 4$, and $c(v_{4k+3}) = 3$. It is clear that $|c(N(v_i))| = 3$ and c is a 3-hued 4-coloring of C_m^2. Hence $\chi_3(C_m^2) = 4$ in this case.

Case 3. $m = 6, 7,$ or 11. Since $m \geq 5,\ \Delta(C_m^2) = 4$. By Observation 1, $\chi_3(C_m^2) \geq 4$.

(1) When $m = 6$, since any adjacent vertices must be colored by different colors, we have the 3-hued 4-coloring c of C_6^2 as follows. $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = 1$, $c(v_5) = 4$, $c(v_6) = 3$ or $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = 4$, $c(v_5) = 2$, $c(v_6) = 3$.

For the former coloring, we have $|c(N(v_2))| = 2$ which contradicts the definition of 3-hued coloring, so we need at least five colors. For the later coloring, we have $|c(N(v_4))| = 2$ which contradicts the definition of 3-hued coloring, so we need at least five colors. If $c(v_6) = 5$, then the upper coloring is a 3-hued 5-coloring of C_6^2, hence $\chi_3(C_6^2) = 5$ in this case.

(2) When $m = 7$, since any adjacent vertices must be colored different colors, we have the 3-hued 4-coloring c of C_7^2 in the following.

(a) $c(v_1) = 1,\ c(v_2) = 2,\ c(v_3) = 3,\ c(v_4) = 1,\ c(v_5) = 4,\ c(v_6) = 2,\ c(v_7) = 3$.
(b) $c(v_1) = 1,\ c(v_2) = 2,\ c(v_3) = 3,\ c(v_4) = 4,\ c(v_5) = 1,\ c(v_6) = 2,\ c(v_7) = 3$.
(c) $c(v_1) = 1,\ c(v_2) = 2,\ c(v_3) = 3,\ c(v_4) = 4,\ c(v_5) = 1,\ c(v_6) = 3,\ c(v_7) = 4$.
(d) $c(v_1) = 1,\ c(v_2) = 2,\ c(v_3) = 3,\ c(v_4) = 4,\ c(v_5) = 2,\ c(v_6) = 3,\ c(v_7) = 4$.

For (a) and (b), we have $|c(N(v_1))| = 2$ which contradicts the definition of 3-hued coloring, so we need at least five colors. In (c) and (d), we have $|c(N(v_5))| = 2$ which contradicts the definition of 3-hued coloring, so we need at least five colors. If $c(v_6) = 5$, then the upper coloring is a 3-hued 5-coloring of C_7^2, hence $\chi_3(C_7^2) = 5$ in this case.

(3) When $m = 11$, we have $\chi_3(C_{11}^2) = 5$ similar argument as in (1), (2). ☐

Theorem 3.4 Let $m \geq 3$ and $r \geq 4$ be two integers and $N = \{11, 12, 16, 17, 18\}$. Then

$$\chi_r(C_m^2) = \begin{cases} m, & m \in [3, 9], \\ 5, & 5 \mid m, \\ 6, & 5 \nmid m, \text{and } m \geq 20 \text{ or } m \in N, \\ 7, & m \in \{13, 14, 19\}. \end{cases}$$

Proof Let $\{v_1, v_2, \cdots, v_m\}$ be the vertex set of C_m^2.

Claim 1. We have the following claim by lemma 2.2. For any integer $m \geq 20$, we have nonnegative integers p and q, such that $m = 5p + 6q$.

Suppose that $m = 5k + t$ with nonnegative integer $t \leq 4$, then $k \geq 4$, so $m = 5(k-t)+6t$, hence we have $m = 5p+6q$ with $p = k - t$ and $q = t$.

Claim 2. By the definition of r-hued coloring, we observe the following fact. Any five successive vertices must receive different colors.

We will prove the theorem by the following four cases.

Case 1. $m \in \{3, 4, 5\}$. The graph C_m^2 induces K_3, K_4, K_5, respectively. Thus $\chi_r(C_3^2) = 3, \chi_r(C_4^2) = 4$, and $\chi_r(C_5^2) = 5$.

Case 2. $m \in [6, 9]$, the coloring c of C_m^2 with $c(v_i) \equiv i \pmod m$ is a r-hued coloring, so $\chi_r(C_m^2) \leq m$. For any r-hued coloring c, w.o.l.g., assume that $c(v_i) = i$ for $i \in [5]$, then $c(v_6) \notin [5]$, let $c(v_6) = 6$, then $c(v_7) \notin [6]$, $c(v_7) = 7$, and $c(v_j) = j$ for $j \in [6, m]$ similarly, hence $\chi_r(C_m^2) = m$.

Case 3. $5|m$. It is easy to see that $\chi_r(C_m^2) \geq 5$ by Claim 2. The coloring c with $c(v_i) \equiv i \pmod 5$ is a r-hued coloring of C_m^2, so $\chi_r(C_m^2) \leq 5$. Hence $\chi_r(C_m^2) = 5$.

Case 4. $m \not\equiv 0 \pmod 5$ and $m = 5k + t$ with $t \in [4]$.

Clearly, $\chi_r(C_m^2) \geq 5$ by Claim 2. If we use five colors to color G, w.o.l.g., assume that $c(v_i) \equiv i \pmod 5$ for $i \in [5k]$, then $c(v_{5k+1}) \notin [5]$ by Claim 2. Thus $\chi_r(C_m^2) \geq 6$ in this case.

Subcase 4.1 $m \in \{11, 12, 16, 17, 18\}$ and $m \geq 20$. Assume that $m = 5k + t$ with $t \in [1, 4]$. By Claim 1, we have $m = 5p + 6q$ where p and q are nonnegative integers. We consider the coloring c of C_m^2 with $c(v_i) \equiv i(\bmod 5)$ for $1 \leq i \leq 5p$ and $c(v_i) \equiv i(\bmod 6)$ for $5p + 1 \leq i \leq m$, which is a hued coloring, so $\chi_r(C_m^2) \leq 6$. Hence $\chi_r(C_m^2) = 6$.

Subcase 4.2 $m \in \{13, 14, 19\}$.

(i) $m = 13$. If we use five colors to color C_m^2 with $c(v_i) \equiv i \pmod 5$ for $1 \leq i \leq 10$, then $c(v_{11}),\ c(v_{12})$ and $c(v_{13})$ are pairwise distinct and $\{c(v_{11}),\ c(v_{12})\ c(v_{13})\} \cap \notin [1, 5] = \phi$, since any adjacent vertices can not receive the same color. Thus there are at least eight colors in the coloring. If we use six colors to color C_m^2 with $c(v_i) \equiv i \pmod 6$ for $1 \leq i \leq 12$, then $c(v_{13}) \notin [6]$ by Claim 2, hence there are at least seven colors in this coloring. If we use seven colors to color C_m^2 with $c(v_i) \equiv i \pmod 7$ for $1 \leq i \leq 13$, then it is a r-hued coloring which is optimal, so $\chi_r(C_m^2) \leq 7$. Hence $\chi_r(C_m^2) = 7$.

(ii) $m \in \{14, 19\}$. We can obtain that $\chi_r(C_m^2) = 7$ similarly. □

IV. REMARKS

In this paper, we study the r-hued chromatic number of power of trees and cycles. By Theorems 2.3 and 2.5, we know that T^2 confirms the conjecture of Song et al. in [8], when T is a tree with $\Delta(T) \leq 3$. For the power of trees, we obtained that $\chi(T^2) = \chi_1(T^2) = \chi_2(T^2) = \cdots = \chi_r(T^2) = \Delta(T) + 1$, if $r \leq \Delta(T)$. We proved that T^2 is a perfect graph in Theorem 2.4. But we know that similar results do not hold for all perfect graphs. Thus the following question is interesting.

Question 4.1 Which perfect graphs satisfy $\chi(G) = \chi_1(G) = \chi_2(G) = \cdots = \chi_r(G) = \omega(G)$, when $r \leq \omega(G) - 1$.

Question 4.2 Characterize perfect graphs satisfying the condition of Question 4.1.

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