Multiple Almost Periodic Solutions and Local Asymptotical Stability in a Harvesting System of Facultative Mutualism with Time Delays

Lijun Xu and Tianwei Zhang

Abstract—**For the last few years, by utilizing Mawhin's continuation theorem of coincidence degree theory, many scholars are concerning with the existence of multiple positive periodic solutions for the non-linear ecosystems with harvesting terms. In the real world, almost periodicity is usually more realistic and more general than periodicity, but there are scarcely any papers concerning on the issue of the multiplicity of positive almost periodic solutions and local asymptotical stability of non-linear ecosystems. By means of Mawhin's continuation theorem of coincidence degree theory and Lyapunov functional method, some sufficient conditions for the existence of at least four positive almost periodic solutions and local asymptotical stability to a delayed two-species system of facultative mutualism with harvesting terms are established.**

Index Terms—**Multiplicity, Almost periodicity, Coincidence degree, Facultative mutualism, Harvesting.**

I. INTRODUCTION

In theoretical population biology, mutualism has received very little attention compared to that given to predator-prey interactions or competition among species (see [1-6]). A facultative mutualist is one which benefits in some way from the association with another species but will survive in its absence, e.g., blue-green algae can grow and reproduce in the absence of zooplankton grazers, but growth and reproduction are enhanced by the presence of the zooplankton (see [1]). In [2], Liu et al. proposed a delayed two-species system modelling "facultative mutualism" as follows:

$$\begin{cases} y_1'(t) = y_1(t)\big[r_1(t) - a_1(t)y_1(t) \\ \qquad\qquad +b_1(t)y_1(t-\mu_1(t)) \\ \qquad\qquad +c_1(t)y_2(t-\nu_1(t))\big], \\ y_2'(t) = y_2(t)\big[r_2(t) - a_2(t)y_2(t) \\ \qquad\qquad +b_2(t)y_2(t-\mu_2(t)) \\ \qquad\qquad +c_2(t)y_1(t-\nu_2(t))\big], \end{cases} \tag{1}$$

where r_i, a_i, b_i, c_i, μ_i and ν_i are continuous periodic functions, $i = 1, 2$. It is obvious that the type of ecological interaction corresponding to system (1) is a two-species system of facultative mutualism, that is, each species can persist in the absence of the other, however, each species enhances the average growth rate of the other. The type of ecological interaction corresponding to system (1) is known as facultative mutualism. There exists an extensive literature concerned with the dynamics of mutualism [7-14] and other systems under the general formalism of competition and cooperation [15-17]). In [5], some sufficient conditions are derived for the existence and globally asymptotic stability of positive periodic solutions of system (1) by using Mawhin's continuous theorem of coincidence degree theory and constructing a suitable Lyapunov functional.

In the real world, there are all kinds of natural or artificially cultivable biological resources, e.g., ocean fish, grassland, forest and so on. For the purpose of long-term use of them, it must be reasonable development and scientific management. We should not only consider the current yield, but also consider the ecological balance and the long-term yield. In many earlier studies, it has been shown that the harvesting term has a strong impact on the development and management of biological resources. e.g., see [18-21]. So the study of the population dynamics with harvesting terms is becoming a very important subject in mathematical bio-economics [18-26]. In [22], Fang and Xiao investigated the existence of multiple periodic solutions of a delayed Lotka-Volterra competition systems with harvesting rate. In [23], Zhao et al. considered a kind of Lotka-Volterra network-like food-chain system and obtained the existence of multiple periodic solutions of the model. Compared with the harvesting predator-prey interactions or competition among species (see [18-26]), the study of the harvesting facultative mutualist is rare. The delayed two-species model of facultative mutualism with harvesting terms generally described as:

$$\begin{cases} y_1'(t) = y_1(t)\big[r_1(t) - a_1(t)y_1(t) \\ \qquad\qquad +b_1(t)y_1(t-\mu_1(t)) \\ \qquad\qquad +c_1(t)y_2(t-\nu_1(t))\big] - h_1(t), \\ y_2'(t) = y_2(t)\big[r_2(t) - a_2(t)y_2(t) \\ \qquad\qquad +b_2(t)y_2(t-\mu_2(t)) \\ \qquad\qquad +c_2(t)y_1(t-\nu_2(t))\big] - h_2(t), \end{cases} \tag{2}$$

where $h_i\,(i = 1, 2)$ denote harvesting rate.

For the last few years, by utilizing Mawhin's continuation theorem of coincidence degree theory, many scholars are concerning with the existence of multiple positive periodic solutions for some non-linear ecosystems with harvesting terms, see [22-26]. However, in real world phenomenon, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples, see Example 1) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity [27-36] is sometimes more realistic and more general than periodicity. Unlike the periodic oscillation, owing to the complexity of the almost

Manuscript received October 22, 2018; revised May 4, 2019.

Lijun Xu is with the School of Mathematics and Computer Science, Panzhihua University, Panzhihua, Sichuan 617000, China. (pzhxlj@126.com).

Tianwei Zhang is with the City College, Kunming University of Science and Technology, Kunming 650051, China. (zhang@kmust.edu.cn).

Correspondence author: Tianwei Zhang. (zhang@must.edu.cn).

periodic oscillation, it is hard to study the existence of positive almost periodic solutions of non-linear ecosystems by using Mawhin's continuation theorem [27-28]. Therefore, to the best of the authors' knowledge, so far, there are scarcely any papers concerning with the multiplicity of positive almost periodic solutions of system (2) by using Mawhin's continuation theorem. Stimulated by the above reason, the main purpose of this paper is to establish sufficient conditions for the existence of multiple positive almost periodic solutions to system (2) by applying Mawhin's continuation theorem of coincidence degree theory.

Example 1. Consider the following simple population model:

$$y'(t) = y(t)\left[|\sin\sqrt{2}t| - |\sin\sqrt{3}t|y(t)\right]. \qquad (3)$$

In system (3), $|\sin(\sqrt{2}t)|$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that Eq. (3) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of Eq. (3). Thus, it is significant to study the existence of positive almost periodic solutions of Eq. (3).

If $b_i \equiv 0$ and $\nu_i \equiv 0$, $i = 1, 2$, then system (2) reduces to the following form

$$\begin{cases} y_1'(t) = y_1(t)\big[r_1(t) - a_1(t)y_1(t) \\ \qquad\quad +c_1(t)y_2(t)\big] - h_1(t), \\ y_2'(t) = y_2(t)\big[r_2(t) - a_2(t)y_2(t) \\ \qquad\quad +c_2(t)y_1(t)\big] - h_2(t). \end{cases} \qquad (4)$$

By using the continuation theorem of coincidence degree, Hu and Zhang [3] established the existence of four positive periodic solutions for system (4).

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X}, \mathbb{Y})$ and $C^1(\mathbb{X}, \mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X}, \mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X}, \mathbb{X})$. Related to a continuous function f, we use the following notations:

$$f^l = \inf_{s\in\mathbb{R}} f(s), \quad f^u = \sup_{s\in\mathbb{R}} f(s),$$

$$|f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

Throughout this paper, we always make the following assumption for system (2):

(F_1) All the coefficients in system (2) are continuous nonnegative almost periodic functions.

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, we obtain sufficient condition for the existence of at least four positive almost periodic solutions to system (2) by way of Mawhin's continuous theorem. An example and numerical simulations are also given to illustrate our main result.

II. Preliminaries

Next, we present and prove several useful lemmas which will be used in later section.

Definition 1. ([27], [28]) $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t+\tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x, \epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x, \epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R}, \mathbb{R})$.

Lemma 1. *([27], [28]) If $x \in AP(\mathbb{R})$, then x is bounded and uniformly continuous on $\mathbb{R}$.*

Lemma 2. *([27], [28]) If $x \in AP(\mathbb{R})$, then $\int_0^t x(s)\,\mathrm{d}s \in AP(\mathbb{R})$ if and only if $\int_0^t x(s)\,\mathrm{d}s$ is bounded on $\mathbb{R}$.*

Lemma 3. *([27], [28])If $x \in C(\mathbb{R})$ is a strictly monotone function, then $x \bar{\in} AP(\mathbb{R})$.*

Lemma 4. *([27], [28]) Assume that $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ with $\dot{x} \in C(\mathbb{R})$, for $\forall\epsilon > 0$, we have the following conclusions:*

(1) *there is a point $\xi_\epsilon \in [0, +\infty)$ such that $x(\xi_\epsilon) \in [x^* - \epsilon, x^*]$ and $\dot{x}(\xi_\epsilon) = 0$;*
(2) *there is a point $\eta_\epsilon \in [0, +\infty)$ such that $x(\eta_\epsilon) \in [x_*, x_* + \epsilon]$ and $\dot{x}(\eta_\epsilon) = 0$.*

III. Almost periodic solutions

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [37].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 5. *([37]) Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;
(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;
(c) $\deg\{JQN, \Omega\cap\mathrm{Ker}L, 0\} \neq 0$, *where $J : \mathrm{Im}Q \to \mathrm{Ker}L$ is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

For $f \in AP(\mathbb{R})$, we denote by

$$\Lambda(f) = \left\{\varpi \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\}$$

the set of Fourier exponents of f.

Now we are in the position to present and prove our result on the existence of at least four positive almost periodic solutions for system (2).

Under the invariant transformation $(y_1, y_2)^T = (e^{x_1}, e^{x_2})^T$, system (2) reduces to

$$\begin{cases} x_1'(t) = r_1(t) - a_1(t)e^{x_1(t)} \\ \quad +b_1(t)e^{x_1(t-\mu_1(t))} \\ \quad +c_1(t)e^{x_2(t-\nu_1(t))} - \frac{h_1(t)}{e^{x_1(t)}} := F_1(t), \\ x_2'(t) = r_2(t) - a_2(t)e^{x_2(t)} \\ \quad +b_2(t)e^{x_2(t-\mu_2(t))} \\ \quad +c_2(t)e^{x_1(t-\nu_2(t))} - \frac{h_2(t)}{e^{x_2(t)}} := F_2(t). \end{cases} \tag{5}$$

Take $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \Big\{ z = (x_1, x_2)^T \in AP(\mathbb{R}, \mathbb{R}^2) : \forall \varpi \in \Lambda(x_1) \cup \Lambda(x_2), |\varpi| \geq \theta_0 \Big\},$$

$$\mathbb{V}_2 = \big\{ z = (x_1, x_2)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbb{R} \big\},$$

where θ_0 is a given positive constant. Define the norm

$$\|z\| = \max\left\{ \sup_{s\in\mathbb{R}} |x_1(s)|, \sup_{s\in\mathbb{R}} |x_2(s)| \right\}, \quad \forall z \in \mathbb{X} = \mathbb{Y}.$$

Then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|$.

Similar to the proof in papers [27], [28], we have the following lemmas.

Lemma 6. *Let $L : \mathbb{X} \to \mathbb{Y}$, $Lz = L(x_1, x_2)^T = (\dot{x}_1, \dot{x}_2)^T$, then L is a Fredholm mapping of index zero.*

Lemma 7. *Define $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ by*

$$Nz = N \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix},$$

$$Pz = P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m(x_1) \\ m(x_2) \end{bmatrix} = Qz, \quad \forall z \in \mathbb{X} = \mathbb{Y}.$$

Then N is L-compact on $\bar{\Omega}$(Ω is an open and bounded subset of $\mathbb{X}$).

Let

$$\rho_1 := \ln\left[\frac{r_1^u(a_2^l - b_2^u) + c_1^u r_2^u}{(a_1^l - b_1^u)(a_2^l - b_2^u) - c_1^u c_2^u}\right],$$

$$\rho_2 := \ln\left[\frac{r_2^u + c_2^u e^{\rho_1}}{a_2^l - b_2^u}\right],$$

$$\phi_1 := \frac{h_1^l}{r_1^u + b_1^u e^{\rho_1} + c_1^u e^{\rho_2}}, \quad \phi_2 := \frac{h_2^l}{r_2^u + b_2^u e^{\rho_2} + c_2^u e^{\rho_1}}.$$

Theorem 1. *Assume that (H_1) and the following condition hold:*

(H_2) $h_i^l > 0,\ i = 1, 2,$

(H_3) $a_1^l > b_1^u,$

(H_4) $(a_1^l - b_1^u)(a_2^l - b_2^u) > c_1^u c_2^u,$

(H_5) $r_1^l + b_1^l\phi_1 + c_1^l\phi_2 > 2\sqrt{a_1^u h_1^u},\ r_2^l + b_2^l\phi_2 + c_2^l\phi_1 > 2\sqrt{a_2^u h_2^u},$

then system (2) admits at least four positive almost periodic solutions.

Proof: It is easy to see that if system (5) has one almost periodic solution $(x_1, x_2)^T$, then $(y_1, y_2)^T = (e^{x_1}, e^{x_2})^T$ is a positive almost periodic solution of system (2). Therefore, to completes the proof it suffices to show that system (5) has one almost periodic solution.

In order to apply Lemma 5, we need to search for an appropriate open-bounded subset Ω.

Corresponding to the operator equation $Lw = \lambda w$, $\lambda \in (0, 1)$, we have

$$\begin{cases} x_1'(t) = \lambda\Big[r_1(t) - a_1(t)e^{x_1(t)} + b_1(t)e^{x_1(t-\mu_1(t))} \\ \quad +c_1(t)e^{x_2(t-\nu_1(t))} - \frac{h_1(t)}{e^{x_1(t)}}\Big], \\ x_2'(t) = \lambda\Big[r_2(t) - a_2(t)e^{x_2(t)} + b_2(t)e^{x_2(t-\mu_2(t))} \\ \quad +c_2(t)e^{x_1(t-\nu_2(t))} - \frac{h_2(t)}{e^{x_2(t)}}\Big]. \end{cases} \tag{6}$$

Suppose that $(x_1, x_2)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (6) for some $\lambda \in (0, 1)$, where $\mathrm{Dom}L = \{w = (x_1, x_2)^T \in \mathbb{X} : x_1, x_2 \in C^1(\mathbb{R}), x_1', x_2' \in C(\mathbb{R})\}$. By Lemma 4, for $\forall\epsilon \in (0, 1)$, there are two points $\xi_\epsilon^{(1)}, \xi_\epsilon^{(2)} \in [0, +\infty)$ such that

$$x_1'(\xi_\epsilon^{(1)}) = 0,\ x_1(\xi_\epsilon^{(1)}) \in [x_1^* - \epsilon, x_1^*];$$

$$x_2'(\xi_\epsilon^{(2)}) = 0,\ x_2(\xi_\epsilon^{(2)}) \in [x_2^* - \epsilon, x_2^*], \tag{7}$$

where $x_1^* = \sup_{s\in\mathbb{R}} x_1(s)$ and $x_2^* = \sup_{s\in\mathbb{R}} x_2(s)$.

Further, in view of (H_3)-(H_4), we may assume that the above ϵ is small enough so that

$$a_2^l > e^\epsilon b_2^u \quad \text{and} \quad (a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u) > e^{2\epsilon} c_1^u c_2^u.$$

From system (6), it follows from (7) that

$$\begin{aligned} & a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} + \frac{h_1(\xi_\epsilon^{(1)})}{e^{x_1(\xi_\epsilon^{(1)})}} \\ = {} & r_1(\xi_\epsilon^{(1)}) + b_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)} - \mu_1(\xi_\epsilon^{(1)}))} \\ & +c_1(\xi_\epsilon^{(1)})e^{x_2(\xi_\epsilon^{(1)} - \nu_1(\xi_\epsilon^{(1)}))}, \end{aligned} \tag{8}$$

$$\begin{aligned} & a_2(\xi_\epsilon^{(2)})e^{x_2(\xi_\epsilon^{(2)})} + \frac{h_2(\xi_\epsilon^{(2)})}{e^{x_2(\xi_\epsilon^{(2)})}} \\ = {} & r_2(\xi_\epsilon^{(2)}) + b_2(\xi_\epsilon^{(2)})e^{x_2(\xi_\epsilon^{(2)} - \mu_2(\xi_\epsilon^{(2)}))} \\ & +c_2(\xi_\epsilon^{(2)})e^{x_1(\xi_\epsilon^{(2)} - \nu_2(\xi_\epsilon^{(2)}))}. \end{aligned} \tag{9}$$

In view of (8), we have from (7) that

$$\begin{aligned} a_1^l e^{x_1^* - \epsilon} & \leq a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} \\ & \leq r_1(\xi_\epsilon^{(1)}) + b_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)} - \mu_1(\xi_\epsilon^{(1)}))} \\ & \quad +c_1(\xi_\epsilon^{(1)})e^{x_2(\xi_\epsilon^{(1)} - \nu_1(\xi_\epsilon^{(1)}))} \\ & \leq r_1^u + b_1^u e^{x_1^*} + c_1^u e^{x_2^*}. \end{aligned}$$

That is,

$$(a_1^l - e^\epsilon b_1^u)e^{x_1^*} \leq e^\epsilon r_1^u + e^\epsilon c_1^u e^{x_2^*}. \tag{10}$$

Similarly, we obtain from (9) that

$$(a_2^l - e^\epsilon b_2^u)e^{x_2^*} \leq e^\epsilon r_2^u + e^\epsilon c_2^u e^{x_1^*}. \tag{11}$$

Substituting (11) into (10) leads to

$$\begin{aligned} (a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u)e^{x_1^*} & \leq e^\epsilon r_1^u(a_2^l - e^\epsilon b_2^u) \\ & \quad +e^\epsilon c_1^u[e^\epsilon r_2^u + e^\epsilon c_2^u e^{x_1^*}], \end{aligned}$$

which implies that

$$\big[(a_1^l - e^\epsilon b_1^u)(a_2^l - e^\epsilon b_2^u) - e^{2\epsilon} c_1^u c_2^u\big]e^{x_1^*}$$

$$\leq e^{\epsilon}r_1^u(a_2^l - e^{\epsilon}b_2^u) + e^{2\epsilon}c_1^u r_2^u$$

is equivalent to

$$x_1^* \leq \ln\left[\frac{e^{\epsilon}r_1^u(a_2^l - e^{\epsilon}b_2^u) + e^{2\epsilon}c_1^u r_2^u}{(a_1^l - e^{\epsilon}b_1^u)(a_2^l - e^{\epsilon}b_2^u) - e^{2\epsilon}c_1^u c_2^u}\right].$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$x_1^* \leq \ln\left[\frac{r_1^u(a_2^l - b_2^u) + c_1^u r_2^u}{(a_1^l - b_1^u)(a_2^l - b_2^u) - c_1^u c_2^u}\right] := \rho_1. \tag{12}$$

Substituting (12) into (11) leads to

$$(a_2^l - e^{\epsilon}b_2^u)e^{x_2^*} \leq e^{\epsilon}r_2^u + e^{\epsilon}c_2^u e^{\rho_1}.$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$x_2^* \leq \ln\left[\frac{r_2^u + c_2^u e^{\rho_1}}{a_2^l - b_2^u}\right] := \rho_2. \tag{13}$$

Also, for $\forall \epsilon \in (0,1)$, there are two points $\eta_\epsilon^{(1)}, \eta_\epsilon^{(2)} \in [0,+\infty)$ such that

$$x_1'(\eta_\epsilon^{(1)}) = 0,\ x_1(\eta_\epsilon^{(1)}) \in [x_{1*}, x_{1*}+\epsilon];$$

$$x_2'(\eta_\epsilon^{(2)}) = 0,\ x_2(\eta_\epsilon^{(2)}) \in [x_{2*}, x_{2*}+\epsilon], \tag{14}$$

where $x_{1*} = \inf_{s\in\mathbb{R}} x_1(s)$ and $x_{2*} = \inf_{s\in\mathbb{R}} x_2(s)$.

From system (6), it follows from (14) that

$$\begin{aligned} &a_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)})} + \frac{h_1(\eta_\epsilon^{(1)})}{e^{x_1(\eta_\epsilon^{(1)})}} \\ &= r_1(\eta_\epsilon^{(1)}) + b_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)}-\mu_1(\eta_\epsilon^{(1)}))} \\ &\quad + c_1(\eta_\epsilon^{(1)})e^{x_2(\eta_\epsilon^{(1)}-\nu_1(\eta_\epsilon^{(1)}))}, \end{aligned} \tag{15}$$

$$\begin{aligned} &a_2(\eta_\epsilon^{(2)})e^{x_2(\eta_\epsilon^{(2)})} + \frac{h_2(\eta_\epsilon^{(2)})}{e^{x_2(\eta_\epsilon^{(2)})}} \\ &= r_2(\eta_\epsilon^{(2)}) + b_2(\eta_\epsilon^{(2)})e^{x_2(\eta_\epsilon^{(2)}-\mu_2(\eta_\epsilon^{(2)}))} \\ &\quad + c_2(\eta_\epsilon^{(2)})e^{x_1(\eta_\epsilon^{(2)}-\nu_2(\eta_\epsilon^{(2)}))}. \end{aligned} \tag{16}$$

In view of (15), we have from (12)-(14) that

$$\begin{aligned} \frac{h_1^l}{e^{x_{1*}+\epsilon}} &\leq \frac{h_1(\eta_\epsilon^{(1)})}{e^{x_1(\eta_\epsilon^{(1)})}} \\ &\leq r_1(\eta_\epsilon^{(1)}) + b_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)}-\mu_1(\eta_\epsilon^{(1)}))} \\ &\quad + c_1(\eta_\epsilon^{(1)})e^{x_2(\eta_\epsilon^{(1)}-\nu_1(\eta_\epsilon^{(1)}))} \\ &\leq r_1^u + b_1^u e^{\rho_1} + c_1^u e^{\rho_2}. \end{aligned}$$

That is,

$$e^{x_{1*}} \geq \frac{h_1^l}{e^{\epsilon}(r_1^u + b_1^u e^{\rho_1} + c_1^u e^{\rho_2})}$$

is equivalent to

$$x_{1*} \geq \ln\left[\frac{h_1^l}{e^{\epsilon}(r_1^u + b_1^u e^{\rho_1} + c_1^u e^{\rho_2})}\right].$$

Letting $\epsilon \to 0$ in the above inequality leads to

$$x_{1*} \geq \ln\left[\frac{h_1^l}{r_1^u + b_1^u e^{\rho_1} + c_1^u e^{\rho_2}}\right] := \rho_3. \tag{17}$$

By a parallel argument as that in (17), we can obtain that

$$x_{2*} \geq \ln\left[\frac{h_2^l}{r_2^u + b_2^u e^{\rho_2} + c_2^u e^{\rho_1}}\right] := \rho_4. \tag{18}$$

In view of (8) and (15), we have that

$$a_1(\xi_\epsilon^{(1)})e^{x_1(\xi_\epsilon^{(1)})} + \frac{h_1(\xi_\epsilon^{(1)})}{e^{x_1(\xi_\epsilon^{(1)})}} \geq r_1^l + b_1^l e^{\rho_3} + c_1^l e^{\rho_4},$$

$$a_1(\eta_\epsilon^{(1)})e^{x_1(\eta_\epsilon^{(1)})} + \frac{h_1(\eta_\epsilon^{(1)})}{e^{x_1(\eta_\epsilon^{(1)})}} \geq r_1^l + b_1^l e^{\rho_3} + c_1^l e^{\rho_4},$$

which yield that

$$a_1^u e^{2x_1(\xi_\epsilon^{(1)})} - (r_1^l + b_1^l e^{\rho_3} + c_1^l e^{\rho_4})e^{x_1(\xi_\epsilon^{(1)})} + h_1^u \geq 0,$$
$$a_1^u e^{2x_1(\eta_\epsilon^{(1)})} - (r_1^l + b_1^l e^{\rho_3} + c_1^l e^{\rho_4})e^{x_1(\eta_\epsilon^{(1)})} + h_1^u \geq 0,$$

which imply from (H_5) that

$$x_1(\xi_\epsilon^{(1)}) \geq \ln\kappa_+^1 \text{ or } x_1(\xi_\epsilon^{(1)}) \leq \ln\kappa_-^1,$$
$$x_1(\eta_\epsilon^{(1)}) \geq \ln\kappa_+^1 \text{ or } x_1(\eta_\epsilon^{(1)}) \leq \ln\kappa_-^1,$$

where

$$\kappa_\pm^1 := \frac{r_1^l + b_1^l\phi_1 + c_1^l\phi_2 \pm \sqrt{(r_1^l + b_1^l\phi_1 + c_1^l\phi_2)^2 - 4a_1^u h_1^u}}{2a_1^u}.$$

Letting $\epsilon \to 0$ in the above inequalities lead to

$$x_1^* \geq \ln\kappa_+^1 \quad \text{or} \quad x_1^* \leq \ln\kappa_-^1, \tag{19}$$

$$x_{1*} \geq \ln\kappa_+^1 \quad \text{or} \quad x_{1*} \leq \ln\kappa_-^1. \tag{20}$$

By similar arguments as that in (19)-(20), we obtain that

$$x_2^* \geq \ln\kappa_+^2 \quad \text{or} \quad x_2^* \leq \ln\kappa_-^2, \tag{21}$$

$$x_{2*} \geq \ln\kappa_+^2 \quad \text{or} \quad x_{2*} \leq \ln\kappa_-^2, \tag{22}$$

where

$$\kappa_\pm^2 := \frac{r_2^l + b_2^l\phi_2 + c_2^l\phi_1 \pm \sqrt{(r_2^l + b_2^l\phi_2 + c_2^l\phi_1)^2 - 4a_2^u h_2^u}}{2a_2^u}.$$

From (12)-(13) and (17)-(22), it follows that

$$\rho_3 \leq x_1(t) \leq \ln\kappa_-^1 \quad \text{or} \quad \ln\kappa_+^1 \leq x_1(t) \leq \rho_1, \tag{23}$$

$$\rho_4 \leq x_2(t) \leq \ln\kappa_-^2 \quad \text{or} \quad \ln\kappa_+^2 \leq x_2(t) \leq \rho_2. \tag{24}$$

Obviously, $\ln\kappa_\pm^1$, $\ln\kappa_\pm^2$, ρ_1, ρ_2, ρ_3 and ρ_4 are independent of λ. Let $\varepsilon_i = \frac{\ln\kappa_+^i - \ln\kappa_-^i}{4}$ $(i=1,2)$ and

$$\Omega_1 = \Big\{ w = (x_1,x_2)^T \in \mathbb{X} : 1 - \rho_3 < x_1(t) < \ln\kappa_-^1 + \varepsilon_1,$$
$$\rho_4 - 1 < x_2(t) < \ln\kappa_-^2 + \varepsilon_2, \forall t \in \mathbb{R} \Big\},$$

$$\Omega_2 = \Big\{ w = (x_1,x_2)^T \in \mathbb{X} : \ln\kappa_+^1 - \varepsilon_1 < x_1(t) < \rho_1 + 1,$$
$$\rho_4 - 1 < x_2(t) < \ln\kappa_-^2 + \varepsilon_2, \forall t \in \mathbb{R} \Big\},$$

$$\Omega_3 = \Big\{ w = (x_1, x_2)^T \in \mathbb{X} : 1 - \rho_3 < x_1(t) < \ln \kappa_-^1 + \varepsilon_1,$$
$$\ln \kappa_+^2 - \varepsilon_2 < x_2(t) < \rho_2 + 1, \forall t \in \mathbb{R} \Big\},$$
$$\Omega_4 = \Big\{ w = (x_1, x_2)^T \in \mathbb{X} : \ln \kappa_+^1 - \varepsilon_1 < x_1(t) < \rho_1 + 1,$$
$$\ln \kappa_+^2 - \varepsilon_2 < x_2(t) < \rho_2 + 1, \forall t \in \mathbb{R} \Big\}.$$

Then Ω_1, Ω_2, Ω_3 and Ω_4 are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, $i, j = 1, 2, 3, 4$. Therefore, Ω_1, Ω_2, Ω_3 and Ω_4 satisfy condition (a) of Mawhin's continuous theorem.

Now we show that condition (b) of Mawhin's continuous theorem holds, i.e., we prove that $QNw \neq 0$ for all $w = (x_1, x_2)^T \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap \mathbb{R}^2$, $i = 1, 2, 3, 4$. If it is not true, then there exists at least one constant vector $w_0 = (x_1^0, x_2^0)^T \in \partial\Omega_i \ (i = 1, 2, 3, 4)$ such that

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^{x_1^0} + \bar{b}_1 e^{x_1^0} + \bar{c}_1 e^{x_2^0} - \frac{\bar{h}_1}{e^{x_1^0}} = 0, \\ \bar{r}_2 - \bar{a}_2 e^{x_2^0} + \bar{b}_2 e^{x_2^0} + \bar{c}_2 e^{x_1^0} - \frac{\bar{h}_2}{e^{x_2^0}} = 0. \end{cases}$$

Similar to the arguments as that in (23)-(24), it follows that

$$\rho_3 \le x_1^0 \le \ln \kappa_-^1 \quad \text{or} \quad \ln \kappa_+^1 \le x_1^0 \le \rho_1,$$
$$\rho_4 \le x_2^0 \le \ln \kappa_-^2 \quad \text{or} \quad \ln \kappa_+^2 \le x_2^0 \le \rho_2.$$

Then $w_0 \in \Omega_1 \cap \mathbb{R}^2$ or $w_0 \in \Omega_2 \cap \mathbb{R}^2$ or $w_0 \in \Omega_3 \cap \mathbb{R}^2$ or $w_0 \in \Omega_4 \cap \mathbb{R}^2$. This contradicts the fact that $w_0 \in \partial\Omega_i \ (i = 1, 2, 3, 4)$. This proves that condition (b) of Mawhin's continuous theorem holds.

Finally, we will show that condition (c) of Mawhin's continuous theorem is satisfied. Let us consider the homotopy

$$H(\iota, w) = \iota QNw + (1 - \iota)\Phi w, \ \ (\iota, w) \in [0, 1] \times \mathbb{R}^2,$$

where

$$\Phi w = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \bar{r}_1 - \bar{a}_1 e^{x_1} + \bar{b}_1 e^{x_1} + \bar{c}_1 e^{\rho_2} - \frac{\bar{h}_1}{e^{x_1}} \\ \bar{r}_2 - \bar{a}_2 e^{x_2} + \bar{b}_2 e^{x_2} + \bar{c}_2 e^{\rho_1} - \frac{\bar{h}_2}{e^{x_2}} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, w) \neq 0$ on $\partial\Omega_i \cap \mathrm{Ker}L$, $\forall \iota \in [0, 1]$, $i = 1, 2, 3, 4$. Further, $\Phi w = 0$ has four distinct solutions:

$$(x_1^*, x_2^*)^T = (\ln x_-^1, \ln x_-^2)^T,$$
$$(y_1^*, y_2^*)^T = (\ln x_+^1, \ln x_+^2)^T,$$
$$(u_1^*, u_2^*)^T = (\ln x_+^1, \ln x_-^2)^T,$$
$$(v_1^*, v_2^*)^T = (\ln x_-^1, \ln x_+^2)^T,$$

where

$$x_\pm^1 = \frac{\bar{r}_1 + \bar{c}_1 e^{\rho_2} \pm \sqrt{(\bar{r}_1 + \bar{c}_1 e^{\rho_2})^2 - 4(\bar{a}_1 - \bar{b}_1)\bar{h}_1}}{2(\bar{a}_1 - \bar{b}_1)},$$
$$x_\pm^2 = \frac{\bar{r}_2 + \bar{c}_2 e^{\rho_1} \pm \sqrt{(\bar{r}_2 + \bar{c}_2 e^{\rho_1})^2 - 4(\bar{a}_2 - \bar{b}_2)\bar{h}_2}}{2(\bar{a}_2 - \bar{b}_2)}.$$

It is easy to verify that

$$\rho_3 < \ln x_-^1 < \ln \kappa_-^1 < \ln \kappa_+^1 < \ln x_+^1 < \rho_1,$$
$$\rho_4 < \ln x_-^2 < \ln \kappa_-^2 < \ln \kappa_+^2 < \ln x_+^2 < \rho_2.$$

Therefore

$$(x_1^*, x_2^*)^T \in \Omega_1, \quad (v_1^*, v_2^*)^T \in \Omega_2,$$
$$(u_1^*, u_2^*)^T \in \Omega_3, \quad (y_1^*, y_2^*)^T \in \Omega_4.$$

By the invariance property of homotopy, we have

$$\deg\big(JQN, \Omega_i \cap \mathrm{Ker}L, 0\big) = \deg\big(QN, \Omega_i \cap \mathrm{Ker}L, 0\big)$$
$$= \deg\big(\Phi, \Omega_i \cap \mathrm{Ker}L, 0\big) \neq 0, \quad i = 1, 2, 3, 4,$$

where $\deg(\cdot, \cdot, \cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 5 are satisfied. Therefore, system (5) has four almost periodic solutions, that is, system (2) has at least four positive almost periodic solutions. This completes the proof. ■

From the proof of Theorem 2, we have that

Corollary 1. *Assume that (H_1)-(H_5) hold. Suppose further that r_i, a_i, b_i, c_i, μ_i and ν_i of system (2) are continuous nonnegative periodic functions with periods α_i, β_i, γ_i, σ_i, ψ_i and ω_i, respectively, $i = 1, 2$, then system (2) admits at least four positive almost periodic solutions.*

Corollary 2. *Assume that (H_1)-(H_5) hold. Suppose further that r_i, a_i, b_i, c_i, μ_i and ν_i of system (2) are continuous nonnegative ω-periodic functions, $i = 1, 2$, then system (2) admits at least four positive ω-periodic solutions.*

Remark 1. In system (2), when $b_i \equiv 0$ and $\nu_i \equiv 0$, $i = 1, 2$, Hu and Zhang [3] obtained Corollary 2, but they couldn't obtain Corollary 1. Therefore, our result extends their work.

IV. Local asymptotical stability

In this section, we will construct some suitable Lyapunov functions to study the local asymptotical stability of system (2).

Theorem 2. *Assume that $\mu_i(t) \equiv 0$ and $\nu_i(t) \equiv 0$ are constant functions, $i = 1, 2$. Suppose further that*

(H_6) *there exist two positive constants y_1^* and y_2^* such that $\Theta = \min_{i=1,2}\{r_i^l - 2a_i^u y_i^* - 2b_i^u y_i^* - 2c_i^u y_i^*\} > 0$.*

Then system (2) is locally asymptotically stable.

Proof: Let $\mathbb{B} := \{y = (y_1, y_2)^T \in \mathbb{R}^2 : 0 \le y_i \le y_i^*, i = 1, 2\}$. Assume that $y(t) = (y_1(t), y_2(t))^T \in \mathbb{B}$ and $y^*(t) = (y_1^*(t), y_2^*(t))^T \in \mathbb{B}$ are any two solutions of system (2). In view of system (2), for $t \in \mathbb{R}^+$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} (y_1(t) - y_1^*(t))' &= r_1(t)\big[y_1(t) - y_1^*(t)\big] \\ &\quad - a_1(t)\big[y_1(t) + y_1^*(t)\big]\big[y_1(t) - y_1^*(t)\big] \\ &\quad - b_1(t)y_1(t)\big[y_1(t) - y_1^*(t)\big] \\ &\quad - b_1(t)y_1^*(t)\big[y_1(t) - y_1^*(t)\big] \\ &\quad + c_1(t)y_2^*(t)\big[y_1(t) - y_1^*(t)\big] \\ &\quad + c_1(t)y_1^*(t)\big[y_2(t) - y_2^*(t)\big] \\ &\ge r_1^l\big[y_1(t) - y_1^*(t)\big] \\ &\quad - a_1^u 2y_1^*\big|y_1(t) - y_1^*(t)\big| \\ &\quad - b_1^u y_1^*\big|y_1(t) - y_1^*(t)\big| \\ &\quad - b_1^u y_1^*\big|y_1(t) - y_1^*(t)\big| \end{aligned}$$

$$
\begin{aligned}
&-c_1^u(t)y_2^*\big|y_1(t)-y_1^*(t)\big|\\
&-c_1^u y_1^*\big|y_2(t)-y_2^*(t)\big|,
\end{aligned}
$$

similarly,

$$
\begin{aligned}
(y_2(t)-y_2^*(t))' \ \ge\ & r_2^l\big[y_2(t)-y_2^*(t)\big]\\
&-a_2^u 2y_2^*\big|y_2(t)-y_2^*(t)\big|\\
&-b_2^u y_2^*\big|y_2(t)-y_2^*(t)\big|\\
&-b_2^u y_2^*\big|y_2(t)-y_2^*(t)\big|\\
&-c_2^u(t)y_1^*\big|y_2(t)-y_2^*(t)\big|\\
&-c_2^u y_2^*\big|y_1(t)-y_1^*(t)\big|.
\end{aligned}
$$

Let

$$V(t)=\sum_{i=1}^{2}|y_i(t)-y_i^*(t)|.$$

Hence we can obtain from (H_6) that

$$
\begin{aligned}
&D^+V(t)\\
&\ge \sum_{i=1}^{2}(r_i^l-2a_i^u y_i^*-2b_i^u y_i^*-2c_i^u y_i^*)|y_i(t)-y_i^*(t)|\\
&=\Theta\sum_{i=1}^{2}|y_i(t)-y_i^*(t)|.
\end{aligned}
$$

Integrating the last inequality from T_0 to t leads to

$$V(T_0)+\Theta\sum_{i=1}^{2}\int_{T_0}^{t}|y_i(s)-y_i^*(s)|\,\Delta s\le V(t)<+\infty,$$

that is,

$$\sum_{i=1}^{2}\int_{T_0}^{+\infty}|y_i(s)-y_i^*(s)|\,\Delta s<+\infty,$$

which implies that

$$\sum_{i=1}^{2}\lim_{s\to+\infty}|y_i(s)-y_i^*(s)|=0.$$

Thus, system (2) is locally asymptotically stable. This completes the proof. ■

V. An example

Example 2. Consider the following almost periodic model of facultative mutualism with harvesting terms:

$$
\begin{cases}
y_1'(t) \ =\ y_1(t)\big[1-a_1(t)y_1(t)\\
\qquad\qquad +b_1(t)y_1(t-|\sin\sqrt{3}t|)\\
\qquad\qquad +y_2(t-2)\big]-0.05,\\
y_2'(t) \ =\ y_2(t)\big[1-a_2(t)y_2(t)\\
\qquad\qquad +b_2(t)y_2(t-2)\\
\qquad\qquad +y_1(t-\cos^2 t)\big]-0.069,
\end{cases}
\tag{25}
$$

where

$$\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1+0.5|\cos\sqrt{2}t|\\2.3\end{pmatrix},$$

$$\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}0.2\\0.2+0.1\sin^2(\sqrt{7}t)\end{pmatrix}.$$

Corresponding system (2), $r_1^l=r_1^u=r_2^l=r_2^u=1$, $c_1^l=c_1^u=c_2^l=c_2^u=1$, $a_1^l=1$, $a_1^u=1.5$, $a_2^l=a_2^u=2.3$, $b_1^l=b_1^u=0.2$, $b_2^l=0.2$, $b_2^u=0.3$, $h_1^l=h_1^u=0.05$, $h_2^l=h_2^u=0.069$. It is easy to see that $a_2^l>b_2^u$, which implies that (H_3) holds. Further, we have

$$(a_1^l-b_1^u)(a_2^l-b_2^u)=1.6>1=c_1^u c_2^u.$$

So (H_4) holds. By a easy calculation, we can obtain

$$\rho_1=\ln 5,\quad \rho_2=\ln 3,\quad \phi_1=\phi_2=0.01.$$

Then

$$
\begin{aligned}
r_1^l+b_1^l\phi_1+c_1^l\phi_2=1.012>2\sqrt{0.075}=2\sqrt{a_1^u h_1^u},\\
r_2^l+b_2^l\phi_2+c_2^l\phi_1=1.012>2\sqrt{0.1587}=2\sqrt{a_2^u h_2^u},
\end{aligned}
$$

which imply that (H_5) holds. Therefore, all the conditions in Theorem 2 are satisfied. By Theorem 2, system (25) admits at least four positive almost periodic solutions $(y_1^i(t),y_2^i(t))$, $i=1,2,3,4$ (see Figures 1-2).

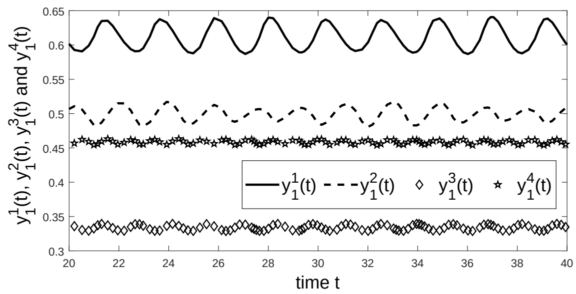

Fig. 1 Four positive almost periodic oscillations of state variable y_1 of system (25)

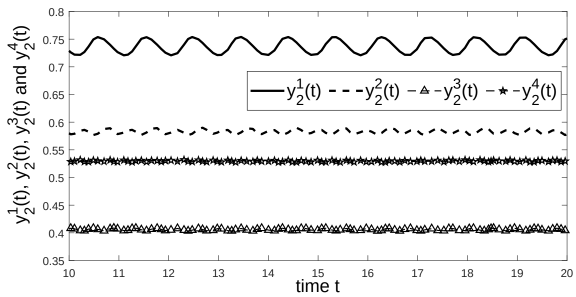

Fig. 2 Four positive almost periodic oscillations of state variable y_2 of system (25)

Remark 2. In system (25), corresponding to Corollary 1, $\alpha_i(i=1,2)$, β_2, γ_1, $\sigma_i(i=1,2)$, ψ_2 and ω_1 are arbitrary constants, $\beta_1=\frac{\sqrt{2}\pi}{2}$, $\gamma_2=\frac{\sqrt{7}\pi}{7}$, $\psi_1=\frac{\sqrt{3}\pi}{3}$ and $\omega_2=\pi$. To the best of our knowledge, through all coefficients of system (25) are periodic functions, it is impossible to sure the existence of positive periodic solutions of system (25) by today's literature. However, by Theorem 2 or Corollary 1, we obtain the existence of at least four positive almost periodic solutions of system (25).

VI. Conclusions

By using a fixed point theorem of coincidence degree theory, some criterions for the multiplicity of positive almost periodic solution to a kind of two-species harvesting model of facultative mutualism with both discrete and distributed delays are obtained. Theorem 1 gives the sufficient conditions for the multiplicity of positive almost periodic solution of system (2). The results obtained in this paper are new and different from the results in [5, 13]. Therefore, The method used in this paper provides a possible method to study the multiplicity of positive almost periodic solution of the models in biological populations.

Acknowledgements

This work are supported by Natural Science Foundation of Ningxia Province under Grant NZ15255.

References

[1] K.G. Porter, Enhancement of algal growth and productivity by grazing zooplankton, Science 192 (1976) 1332-1334.
[2] Z.J. Liu, J.H. Wu, R.h. Tan, Y.P. Chen, Modeling and analysis of a periodic delayed two-species model of facultative mutualism, Appl. Math. Comput. 217 (2010) 893-903.
[3] D.W. Hu, Z.Q. Zhang, Four positive periodic solutions for a Lotka-Volterra cooperative system with harvesting terms, Nonlinear Anal.: RWA 11 (2010) 1115-1121.
[4] L.L. Wang, "Dynamic Analysis on an Almost Periodic Predator-Prey Model with Impulses Effects," Engineering Letters, vol. 26, no.3, pp333-339, 2018.
[5] L.L. Wang, P.L. Xie, "Permanence and Extinction of Delayed Stage-Structured Predator-Prey System on Time Scales," Engineering Letters, vol. 25, no.2, pp147-151, 2017.
[6] D. Jana, S. Chakraborty, N. Bairagi, "Stability, nonlinear oscillations and bifurcation in a delay-induced predator-prey system with harvesting," Engineering Letters, vol. 20, no.3, pp238-246, 2012.
[7] B.S. Goh, Stability in models of mutualism, Am. Nat. 113 (1979) 261-275.
[8] C. Wolin, L. Lawlor, Models of facultative mutualism: density effect, Am. Nat. 124 (1984) 843-862.
[9] D.H. Boucher, The Biology of Mutualism, Croom Helm, London, 1985.
[10] X.Z. He, K. Gopalsamy, Persistence, attractivity and delay in facultative mutualism, J. Math. Anal. Appl. 215 (1997) 154-173.
[11] D. Mukherjee, Permanence and global attractivity for facultative mutualism system with delay, Math. Methods Appl. Sci. 26 (2003) 1-9.
[12] Z.J. Liu, L.S. Chen, Periodic solutions of a discrete time nonautonomous two-species mutualistic system with delays, Adv. Complex Syst. 9 (2006) 87-98.
[13] Z.J. Liu, R.H. Tan, Y.P. Chen, L.S. Chen, On the stable periodic solutions of a delayed two-species model of facultative mutualism, Appl. Math. Comput. 196 (2008) 105-117.
[14] F.D. Chen, Permanence for the discrete mutualism model with time delays, Math. Comput. Model. 47 (2008) 431-435.
[15] M.W. Hirsch, Systems of differential equations which are competitive or cooperative. I: Limit sets, SIAM J. Math. Anal. 13 (1982) 167-179.
[16] S.K. Kim, D.H. Kim, T.W. Yoon, Adaptive observer for estimating the parameters of an HIV model with mutants, International Journal of Control, Automation and Systems 13 (2015) 126-137.
[17] S.P. Lu, On the existence of positive periodic solutions of a Lotka Volterra cooperative population model with multiple delays, Nonlinear Anal.: TMA 68 (2008) 1746-1753.
[18] F. Brauer, A.C. Soudack, H.S. Jarosch, Stabilization and destabilization of predator-prey systems under harvesting and nutrient enrichment, International Journal of Control 23 (1976) 553-573.
[19] F. Brauer, A.C. Soudack, Response of predator-prey systems to nutrient enrichment and proportional harvesting, International Journal of Control 27 (1978) 65-86.
[20] T.K. Kar, U.K. Pahari, Non-selective harvesting in prey-predator models with delay, Commun. Nonlinear Sci. Numer. Simulat. 11 (2006) 499-509.
[21] T.K. Kar, A. Ghorai, Dynamic behaviour of a delayed predator-prey model with harvesting, Appl. Math. Comput. 217 (2011) 9085-9104.
[22] H. Fang, Y.F. Xiao, Existence of multiple periodic solutions for delay Lotka-Volterra competition patch systems with harvesting, Appl. Math. Model. 33 (2009) 1086-1096.
[23] L.G. Xu, Exponential p-stability of singularly perturbed impulsive stochastic delay differential systems, International Journal of Control, Automation and Systems 2011, 9:966.
[24] Y.M. Fu, A.G. Wu, G.R. Duan, Almost disturbance decoupling for a class of inherently nonlinear systems, International Journal of Control, Automation and Systems 7 (2009) 325-330.
[25] Y.H. Fan, L.L. Wang, Multiplicity of periodic solutions for a delayed ratio-dependent predator-prey model with monotonic functional response and harvesting terms, Appl. Math. Comput. 244 (2014) 878-894.
[26] F.Y. Wei, Existence of multiple positive periodic solutions to a periodic predator-prey system with harvesting terms and Holling III type functional response, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 2130-2138.
[27] T.W. Zhang, Multiplicity of positive almost periodic solutions in a delayed Hassell-Varley type predator-prey model with harvesting on prey, Math. Meth. Appl. Sci. 37 (2013) 686-697.
[28] T.W. Zhang, Almost periodic oscillations in a generalized Mackey-Glass model of respiratory dynamics with several delays, Int. J. Biomath. 7 (2014) 1-22.
[29] A.M. Fink, Almost Periodic Differential Equation, Spring-Verlag, Berlin, Heidleberg, New York, 1974.
[30] C.Y. He, Almost Periodic Differential Equations, Higher Education Publishing House, Beijing, 1992 (in Chinese).
[31] Y.Q. Li, T.W. Zhang, Existence and multiplicity of positive almost periodic solutions for a non-autonomous SIR epidemic model, Bull. Malays. Math. Sci. Soc. 39 (2016) 359-379.
[32] Y.Q. Li, T.W. Zhang, Almost periodic dynamics in a nonautonomous bi-directional consumer-resource system with time delays, Comp. Appl. Math. 35 (2016) 559-577.
[33] L.Y. Pang, T.W. Zhang, Permanence and almost periodic solution of a discrete impulsive Richards growth equation with variable delays and feedback control, Appl. Math. Comput. 248 (2014) 399-420.
[34] T.W. Zhang, Y.K. Li, Y. Ye, On the existence and stability of a unique almost periodic solution of Schoener's competition model with pure-delays and impulsive effects, Communications in Nonlinear Science and Numerical Simulation 17 (2012) 1408-1422.
[35] T. W. Zhang, X. R. Gan, Existence and permanence of almost periodic solutions for Leslie-Gower predator-prey model with variable delays, Elect. J. Differ. Equa. 2013 (2013) 1-21.
[36] T. W. Zhang, X. R. Gan, Almost periodic solutions for a discrete fishing model with feedback control and time delays, Commun. Nonlinear Sci. Numer. Simulat..19 (2014) 150-163.
[37] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer, Berlin, 1977.