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New Dual Orlicz Mixed Volume

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Abstract—**In this paper, we propose new dual Orlicz mixed volume and compute the corresponding variational formula. Further, we also establish dual Orlicz mixed Minkowski inequality, dual Orlicz mixed Brunn-Minkowski inequality and demonstrate the equivalence between them.**

Index Terms—**star body, dual Orlicz mixed volume, dual Orlicz mixed Minkowski inequality, dual Orlicz mixed Brunn-Minkowski inequality.**

I. Introduction

THE classical Brunn-Minkowski theory is the product of combining the Minkowski linear combination with volume, its essences are mixed volume, mixed area measure and basic Brunn-Minkowski inequalities. Its dual counterpart called dual Brunn-Minkowski theory can be developed from a few basic concepts: star body, radial function and dual mixed volume. We use $\mathcal{S}_o^n$ to denote the set of star bodies with respect to the origin in Euclidean space $\mathbf{R}^n$ and S^{n-1} the unit sphere. For $K_1, K_2, \cdots, K_n \in \mathcal{S}_o^n$, the dual mixed volume $\widetilde{V}(K_1, K_2, \cdots, K_n)$ is defined by (see [20])

$$\widetilde{V}(K_1, K_2, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho(K_1,u)\rho(K_2,u)\cdots\rho(K_n,u)dS(u), \quad (1.1)$$

where ρ denotes the radial function and $S(u)$ the Lebesgue measure on S^{n-1}. Its special case, dual mixed volume $\widetilde{V}_i(K,L)$ is given by

$$\widetilde{V}_i(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}\rho(L,u)^i dS(u),$$

where i is allowed to be any real number (see [20]). Specifically, $\widetilde{V}_0(K,L)$ just is volume $V(K)$ of the body K. As a key ingredient of dual Brunn-Minkowski theory, dual mixed volume has been widely studied (e.g., [21], [22], [38]).

Associated with the L_p harmonic radial combination, Lutwak [24] proposed L_p dual mixed volume: For $K, L \in \mathcal{S}_o^n$, $\varepsilon > 0$ and real $p \geq 1$, the L_p dual mixed volume $\widetilde{V}_{-p}(K,L)$ is defined by

$$-\frac{n}{p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon\to 0^+} \frac{V(K\tilde{+}_{-p}\varepsilon\circ L) - V(K)}{\varepsilon},$$

which actually can be written as (see [24])

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}dS(u).$$

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These together with succedent researches, including fundamental Minkowski and Brunn-Minkowski inequalities, generalized the dual Brunn-Minkowski theory to L_p case and established L_p dual Brunn-Minkowski theory. In fact, the more general combinations for real $p \neq 0$ (called L_p radial combination) were proposed by Gardner [6] and Grinberg *et al* [11]. For the (dual) Brunn-Minkowski theory and their L_p versions, we refer the reader to [7], [23], [24], [29].

In recent years, investigations toward to Orlicz case, called Orlicz Brunn-Minkowski theory. This groundbreaking work was launched by Lutwak, Yang, and Zhang in 2010 ([25], [26]). Gardner, Hug and Weil [8] provided a general framework for such theory, their work shows the relation to Orlicz spaces and norms. Ye [35] developed the basic setting for the dual Orlicz Brunn-Minkowski theory and gave a formula for the Orlicz L_ϕ dual mixed volume based on linear Orlicz φ-radial addition. Whereafter, Gardner, Hug, Weil and Ye [10] extended Ye's results from star bodies to star sets. The more developments of the Orlicz Brunn-Minkowski theory also see [1], [3], [4], [5], [9], [12], [13], [14], [15], [16], [17], [18], [19], [27], [28], [30], [32], [33], [34], [36], [37], [39], [41].

In 2014, Zhu *et al* [40] presented the following Orlicz radial combination: For $a, b \geq 0$ (not both zero) and $\phi \in \Phi$, the Orlicz radial combination $a\cdot K\tilde{+}_\phi b\cdot L$ of $K, L \in \mathcal{S}_o^n$ is defined by

$$\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u) = \sup\left\{t>0 : a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) \leq \phi(1)\right\}. \quad (1.2)$$

Here Φ denotes a set of convex functions $\phi : (0,\infty) \to (0,\infty)$ that are strictly decreasing and satisfy $\lim_{t\to\infty}\phi(t) = 0$ and $\lim_{t\to 0}\phi(t) = \infty$. It is easy to conclude from [29] that ϕ is continuous and both left derivative ϕ'_l and right derivative ϕ'_r are existent.

From this, they [40] also gave the dual Orlicz mixed volume $\widetilde{V}_\phi(K,L)$ by

$$\widetilde{V}_\phi(K,L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^n dS(u), \quad (1.3)$$

and proved two fundamental inequalities respectively.

Theorem 1.A *If $\phi \in \Phi$ and $K, L \in \mathcal{S}_o^n$, then*

$$\widetilde{V}_\phi(K,L) \geq V(K)\phi\left(\left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}\right), \quad (1.4)$$

equality holds if and only if K and L are dilates.

Theorem 1.B *If $\phi \in \Phi$, $a, b > 0$ and $K, L \in \mathcal{S}_o^n$, then*

$$\phi(1) \geq a\phi\left(\left(\frac{V(K)}{V(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{n}}\right) + b\phi\left(\left(\frac{V(L)}{V(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{n}}\right), \quad (1.5)$$

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equality holds if and only if K and L are dilates.

Quite recently, Chen and Guo [2] introduced the Orlicz mixed volume based on the mixed volume functional and Orlicz combination. In this paper, by applying the function $\widetilde{V}:(\mathcal{S}_o^n)^n\to\mathbf{R}$ to Orlicz radial combination, we consider the dual case. For $1\le m\le n-1$, we let $\mathbf{C}=(K_{m+1},\cdots,K_n)$ and write

$$\begin{aligned}\widetilde{V}_{(0)}(K)&=\widetilde{V}(\underbrace{K,\cdots,K}_{m},K_{m+1},\cdots,K_n)\\&=\widetilde{V}(K[m],\mathbf{C})\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^m c(u)dS(u),\end{aligned}\quad(1.6)$$

where $c(u)=\prod_{i=m+1}^n\rho(K_i,u)$, $u\in S^{n-1}$.

$$\begin{aligned}&\widetilde{V}_{(1)}(K,L)\\&=\widetilde{V}(\underbrace{K,\cdots,K}_{m-1},L,K_{m+1},\cdots,K_n)\\&=\widetilde{V}(K[m-1],L,\mathbf{C})\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{m-1}\rho(L,u)c(u)dS(u).\end{aligned}\quad(1.7)$$

We now define dual Orlicz mixed volume $\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})$ and derive its integral representation as follows.

Definition 1.1 *For $1\le m\le n-1$, $\phi\in\Phi$ and $K_{m+1},\cdots,K_n,K,L\in\mathcal{S}_o^n$, let $\mathbf{C}=(K_{m+1},\cdots,K_n)$ and define dual Orlicz mixed volume $\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})$ by*

$$\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})=\frac{\phi_r'(1)}{m}\lim_{\varepsilon\to0^+}\frac{\widetilde{V}_{(0)}(K_\varepsilon)-\widetilde{V}_{(0)}(K)}{\varepsilon},\quad(1.8)$$

where $K_\varepsilon=K\tilde{+}_\phi\varepsilon\cdot L$.

Theorem 1.1 *For $1\le m\le n-1$ and $K_{m+1},\cdots,K_n,K,L\in\mathcal{S}_o^n$, let $\mathbf{C}=(K_{m+1},\cdots,K_n)$ and $\phi\in\Phi$, then*

$$\begin{aligned}&\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})\\&=\frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right)\rho(K,u)^m c(u)dS(u).\end{aligned}\quad(1.9)$$

Furthermore, we respectively give significant dual Orlicz mixed Minkowski inequality, dual Orlicz mixed Brunn-Minkowski inequality and state the equivalence between them.

Theorem 1.2 *For $1\le m\le n-1$ and $K_{m+1},\cdots,K_n,K,L\in\mathcal{S}_o^n$, let $\mathbf{C}=(K_{m+1},\cdots,K_n)$. If $\phi\in\Phi$, then*

$$\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})\ge\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\right)^{\frac{1}{m}}\right)\widetilde{V}_{(0)}(K),\quad(1.10)$$

equality holds if and only if K and L are dilates.

Theorem 1.3 *For $a,b\ge0$ (not both zero), $1\le m\le n-1$, $K,L\in\mathcal{S}_o^n$. If $\phi\in\Phi$, then*

$$\begin{aligned}\phi(1)\ge{}&a\phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{m}}\right)\\&+b\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{m}}\right),\end{aligned}\quad(1.11)$$

equality holds if and only if K and L are dilates.

Theorem 1.4 *Let m and $\mathbf{C}$ be as above. If $a,b\ge0$ (not both zero), $K,L\in\mathcal{S}_o^n$ and $\phi\in\Phi$, then (1.10) and (1.11) are equivalent.*

Our work further enriches and develops the dual Orlicz Brunn-Minkowski theory, please see the next section for interrelated backgrounds.

II. Preliminaries

A. Radial Function and Star Bodies

Let $K\subset\mathbf{R}^n$ be a star-shaped with respect to the origin, its radial function ρ_K for $x\in\mathbf{R}^n\setminus\{o\}$ is defined by (see [7], [29])

$$\rho_K(x)=\rho(K,x)=\max\{c\ge0:cx\in K\},\ x\in\mathbf{R}^n\setminus\{o\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates of each other, if $\rho(K,u)=c\rho(L,u)$ for all $u\in S^{n-1}$ $(c>0)$. For general linear transformation $\Lambda\in GL(n)$, $\rho_{\Lambda K}(x)=\rho_K(\Lambda^{-1}x)$, $x\in\mathbf{R}^n\setminus\{o\}$, and for the standard unit ball B, we have $\rho(B,u)=1$, $u\in S^{n-1}$.

B. L_p Radial Combination

For $K,L\in\mathcal{S}_o^n$, real $p\ne0$ and $\lambda,\mu\ge0$ (not both zero), define L_p radial combination $\lambda\circ K\tilde{+}_p\mu\circ L$ by (see [6])

$$\rho(\lambda\circ K\tilde{+}_p\mu\circ L,\cdot)^p=\lambda\rho(K,\cdot)^p+\mu\rho(L,\cdot)^p.\quad(2.1)$$

For $p=1$, this is the classical case. If $p>0$, then (2.1) is Grinberg and Zhang's definition (see [11]), and $p\le-1$ the L_p harmonic radial combination (see [24]).

III. Results and Proofs

The proof of Theorem 1.1 requires the following two lemmas (see [40]).

Lemma 3.1 *Let $\phi\in\Phi$. If $a_i,b_i\ge0$ and $a_i\to a$, $b_i\to b$, as $i\to\infty$, then*

$$a_i\cdot K\tilde{+}_\phi b_i\cdot L\to a\cdot K\tilde{+}_\phi b\cdot L,\quad as\ i\to\infty,\quad(3.1)$$

for all $K,L\in\mathcal{S}_o^n$.

Lemma 3.2 *Let $\phi\in\Phi$ and $K,L\in\mathcal{S}_o^n$, then*

$$\lim_{\varepsilon\to0^+}\frac{\rho_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}=\frac{\rho_K(u)}{\phi_r'(1)}\phi\left(\frac{\rho(L,u)}{\rho(K,u)}\right),\quad(3.2)$$

uniformly for $u\in S^{n-1}$.

Proof of Theorem 1.1. According to (1.1) and (1.6), we know

$$\begin{aligned}&\frac{\widetilde{V}_{(0)}(K_\varepsilon)-\widetilde{V}_{(0)}(K)}{\varepsilon}\\&=\frac{\widetilde{V}(K_\varepsilon[m],\mathbf{C})-\widetilde{V}(K[m],\mathbf{C})}{\varepsilon}\\&=\sum_{i=1}^m\frac{1}{\varepsilon}\\&\quad\times\left(\widetilde{V}(K_\varepsilon[i],K[m-i],\mathbf{C})-\widetilde{V}(K_\varepsilon[i-1],K[m-i+1],\mathbf{C})\right)\end{aligned}$$

$$=\sum_{i=1}^{m}\frac{1}{n}\int_{S^{n-1}}\frac{1}{\varepsilon}c(u)$$
$$\times\Big(\rho_{K_\varepsilon}(u)^i\rho_K(u)^{m-i}-\rho_{K_\varepsilon}(u)^{i-1}\rho_K(u)^{m-i+1}\Big)dS(u)$$
$$=\sum_{i=1}^{m}\frac{1}{n}\int_{S^{n-1}}\rho_{K_\varepsilon}(u)^{i-1}\rho_K(u)^{m-i}c(u)$$
$$\times\frac{\rho_{K_\varepsilon}(u)-\rho_K(u)}{\varepsilon}dS(u).$$

This together with definition (1.8), (3.1) and (3.2), yields

$$\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})$$
$$=\frac{\phi_r'(1)}{m}\lim_{\varepsilon\to0^+}\frac{\widetilde{V}_{(0)}(K_\varepsilon)-\widetilde{V}_{(0)}(K)}{\varepsilon}$$
$$=\frac{\phi_r'(1)}{m}\lim_{\varepsilon\to0^+}\Bigg[\sum_{i=1}^{m}\frac{1}{n}\int_{S^{n-1}}\rho_{K_\varepsilon}(u)^{i-1}\rho_K(u)^{m-i}$$
$$\times c(u)\frac{\rho_{K_\varepsilon}(u)-\rho_K(u)}{\varepsilon}dS(u)\Bigg]$$
$$=\frac{\phi_r'(1)}{m}\sum_{i=1}^{m}\Bigg[\lim_{\varepsilon\to0^+}\frac{1}{n}\int_{S^{n-1}}\rho_{K_\varepsilon}(u)^{i-1}\rho_K(u)^{m-i}$$
$$\times c(u)\frac{\rho_{K_\varepsilon}(u)-\rho_K(u)}{\varepsilon}dS(u)\Bigg]$$
$$=\frac{\phi_r'(1)}{m}\sum_{i=1}^{m}\Bigg[\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{m-1}c(u)$$
$$\times\frac{\rho_K(u)}{\phi_r'(1)}\phi\Big(\frac{\rho(L,u)}{\rho(K,u)}\Big)dS(u)\Bigg]$$
$$=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^m c(u)\phi\Big(\frac{\rho(L,u)}{\rho(K,u)}\Big)dS(u).$$

Hence, we obtain (1.9).

Let $\phi_i,\phi\in\Phi$ and $i\in N$, we say that $\phi_i\to\phi$ implies

$$\lim_{i\to+\infty}\max_{t\in I}\mid\phi_i(t)-\phi(t)\mid=0,$$

for each compact interval $I\subset(0,+\infty)$.

The following results can be immediately obtained from Theorem 1.1.

Proposition 3.1 *Let $K_i,L_i,K,L\in\mathcal{S}_o^n$, $\phi_i,\phi\in\Phi$ $(i\in N)$, $1\le m\le n-1$ and $\mathbf{C}=(K_{m+1},\cdots,K_n)$, we have*
(i) *If $L_1\subseteq L_2$, then*

$$\widetilde{V}_{(\phi,m,1)}(K,L_1,\mathbf{C})\ge\widetilde{V}_{(\phi,m,1)}(K,L_2,\mathbf{C}).$$

(ii) *If $L_i\to L$, $K_i\to K$, then*

$$\widetilde{V}_{(\phi,m,1)}(K_i,L_i,\mathbf{C})\to\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C}).$$

(iii) *If $\phi_i\to\phi$, then*

$$\widetilde{V}_{(\phi_i,m,1)}(K,L,\mathbf{C})\to\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C}).$$

(iv) *For any general linear transformation $\Lambda\in GL(n)$, one has*

$$\widetilde{V}_{(\phi,m,1)}(\Lambda K,\Lambda L,\Lambda\mathbf{C})=\mid det\Lambda\mid\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C}).$$

Proof. The conclusions (i)-(iii) are obvious, we only prove the case (iv). For convenience, we write $c'(u)=\prod_{i=m+1}^{n}\rho(\Lambda K_i,u)$, $u\in S^{n-1}$ and $v=\frac{\Lambda^{-1}(u)}{\|\Lambda^{-1}(u)\|}$, then

$$c'(u)=\|\Lambda^{-1}(u)\|^{-(n-m)}\rho(K_{m+1},v)\cdots\rho(K_n,v)$$
$$=\|\Lambda^{-1}(u)\|^{-(n-m)}c(v).$$

By (1.9), one has

$$\widetilde{V}_{(\phi,m,1)}(\Lambda K,\Lambda L,\Lambda\mathbf{C})$$
$$=\frac{1}{n}\int_{S^{n-1}}\phi\Big(\frac{\rho(\Lambda L,u)}{\rho(\Lambda K,u)}\Big)\rho(\Lambda K,u)^m c'(u)dS(u)$$
$$=\frac{1}{n}\int_{S^{n-1}}\phi\Big(\frac{\rho(L,\Lambda^{-1}u)}{\rho(K,\Lambda^{-1}u)}\Big)\rho(K,\Lambda^{-1}u)^m c'(u)dS(\Lambda\Lambda^{-1}u)$$
$$=\mid det\Lambda\mid\frac{1}{n}\int_{S^{n-1}}\phi\Big(\frac{\rho(L,v)}{\rho(K,v)}\Big)\rho(K,v)^m c(v)dS(v)$$
$$=\mid det\Lambda\mid\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C}),$$

this gets the desired result.

The following dual Aleksandrov-Fenchel inequality [7] is needed for the proof of Theorem 1.2.

Lemma 3.3 *If $K_1,K_2,\cdots,K_n\in\mathcal{S}_o^n$, $m=1,2,\cdots,n$, then*

$$\widetilde{V}(K_1,K_2,\cdots,K_n)^m\le\prod_{i=1}^{m}\widetilde{V}(K_i[m],K_{m+1},\cdots,K_n),\tag{3.3}$$

equality holds if and only if $K_1,\cdots,K_m$ are dilates of each other.

Proof of Theorem 1.2. From (1.6), (1.7) and (3.3), we see that

$$\widetilde{V}_{(1)}(K,L)\le\widetilde{V}_{(0)}(K)^{\frac{m-1}{m}}\widetilde{V}_{(0)}(L)^{\frac{1}{m}}.\tag{3.4}$$

Note that ϕ is convex and strictly decreasing, and $\frac{\rho(K,u)^m c(u)dS(u)}{n\widetilde{V}_{(0)}(K)}$ is a probability measure on S^{n-1}, these combined with (1.7), (1.9), (3.4) and Jensen's inequality, yield

$$\frac{\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})}{\widetilde{V}_{(0)}(K)}$$
$$=\int_{S^{n-1}}\phi\Big(\frac{\rho(L,u)}{\rho(K,u)}\Big)\frac{\rho(K,u)^m c(u)dS(u)}{n\widetilde{V}_{(0)}(K)}$$
$$\ge\phi\Big(\int_{S^{n-1}}\frac{\rho(K,u)^{m-1}\rho(L,u)c(u)}{n\widetilde{V}_{(0)}(K)}dS(u)\Big)$$
$$=\phi\Big(\frac{\widetilde{V}_{(1)}(K,L)}{\widetilde{V}_{(0)}(K)}\Big)$$
$$\ge\phi\Big(\frac{\widetilde{V}_{(0)}(K)^{\frac{m-1}{m}}\widetilde{V}_{(0)}(L)^{\frac{1}{m}}}{\widetilde{V}_{(0)}(K)}\Big)$$
$$=\phi\Big(\Big(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\Big)^{\frac{1}{m}}\Big),$$

this establishes (1.10), with equality if and only if K and L are dilates.

Remark 3.1 If $m=n$, then (1.6) implies $\widetilde{V}_{(0)}(K)=V(K)$, combining (1.3) and (1.9), one gets $\widetilde{V}_{(\phi,n,1)}(K,L,\mathbf{C})=\widetilde{V}_\phi(K,L)$ and (1.10) just is (1.4).

Remark 3.2 If $m = n - i$, $\phi(t) = t^{-p}$ with $p \geq 1$ and the tuple $\mathbf{C}$ consists only of unit ball B, then $c(u) = 1$, $\widetilde{V}_{(0)}(K) = \widetilde{V}(K[n-i], B[i]) = \tilde{W}_i(K)$. In such case, (1.9) means

$$\begin{aligned} &\widetilde{V}_{(\phi,n-i,1)}(K, L, \mathbf{C}) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p-i}\rho(L,u)^{-p}dS(u) \\ &= \tilde{W}_{-p,i}(K, L), \end{aligned}$$

where $\tilde{W}_i$ and $\tilde{W}_{-p,i}$ are called dual quermassintegral and L_p dual mixed quermassintegral, respectively. Thus, (1.10) yields

$$\tilde{W}_{-p,i}(K, L) \geq \tilde{W}_i(K)^{\frac{n+p-i}{n-i}}\tilde{W}_i(L)^{\frac{-p}{n-i}},$$

which was just established by Wang and Leng [31].

Proposition 3.2 *For $1 \leq m \leq n-1$ and $K_{m+1}, \cdots, K_n, K, L \in \mathcal{S}_o^n$, let $\mathbf{C} = (K_{m+1}, \cdots, K_n)$. If $\phi \in \Phi$ and for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_{(\phi,m,1)}(Q, K, \mathbf{C}) = \widetilde{V}_{(\phi,m,1)}(Q, L, \mathbf{C}), \qquad (3.5)$$

or

$$\frac{\widetilde{V}_{(\phi,m,1)}(K, Q, \mathbf{C})}{\widetilde{V}_{(0)}(K)} = \frac{\widetilde{V}_{(\phi,m,1)}(L, Q, \mathbf{C})}{\widetilde{V}_{(0)}(L)}, \qquad (3.6)$$

then $K = L$.

Proof. Suppose (3.5) holds. If we take K for Q, then from (1.9), (1.6) and (1.10), we get

$$\begin{aligned} \phi(1)\widetilde{V}_{(0)}(K) &= \widetilde{V}_{(\phi,m,1)}(K, L, \mathbf{C}) \\ &\geq \phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\right)^{\frac{1}{m}}\right)\widetilde{V}_{(0)}(K). \end{aligned}$$

That is to say,

$$\phi(1) \geq \phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\right)^{\frac{1}{m}}\right),$$

by the monotonicity of ϕ, we know $\widetilde{V}_{(0)}(K) \leq \widetilde{V}_{(0)}(L)$. Similarly, If we take L for Q, we have $\widetilde{V}_{(0)}(K) \geq \widetilde{V}_{(0)}(L)$. Hence, $\widetilde{V}_{(0)}(K) = \widetilde{V}_{(0)}(L)$, i.e., $K = L$.

By the same way, we can state the other case (3.6).

Proof of Theorem 1.3. Since definition (1.2) can be equivalently transformed into

$$a\phi\left(\frac{\rho_K(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)}\right) + b\phi\left(\frac{\rho_L(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)}\right) = \phi(1), \quad (3.7)$$

this together with (1.6), (1.9) and (1.10), yields

$$\begin{aligned} &\phi(1)\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L) \\ &= \phi(1)\frac{1}{n}\int_{S^{n-1}} \rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)^m c(u)dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} a\phi\left(\frac{\rho_K(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)}\right) \\ &\quad\times\rho(a\cdot K\tilde{+}_\phi b\cdot L, u)^m c(u)dS(u) \\ &\quad+\frac{1}{n}\int_{S^{n-1}} b\phi\left(\frac{\rho_L(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)}\right) \\ &\quad\times\rho(a\cdot K\tilde{+}_\phi b\cdot L, u)^m c(u)dS(u) \\ &= a\widetilde{V}_{(\phi,m,1)}(a\cdot K\tilde{+}_\phi b\cdot L, K, \mathbf{C}) \\ &\quad+b\widetilde{V}_{(\phi,m,1)}(a\cdot K\tilde{+}_\phi b\cdot L, L, \mathbf{C}) \\ &\geq a\phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{m}}\right)\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L) \\ &\quad+b\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{m}}\right)\widetilde{V}_{(0)}(a\cdot K\tilde{+}_\phi b\cdot L). \end{aligned}$$

This obtains (1.11), the equality condition of (1.10) shows that equality holds in (1.11) if and only if K and L are dilates.

Remark 3.3 If $m = n$, then (1.11) reduces to (1.5).

Remark 3.4 If $m = n - i$ and the tuple $\mathbf{C}$ consists only of unit ball B, then $\widetilde{V}_{(0)}(K) = \tilde{W}_i(K)$, and (1.11) implies

$$\begin{aligned} \phi(1) \geq a\phi&\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{n-i}}\right) \\ &+b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{n-i}}\right). \qquad (3.8) \end{aligned}$$

Specially, the case of $\phi(t) = t^{-p}$ with $p \geq 1$ reduces to

$$\tilde{W}_i(a\cdot K\tilde{+}_{-p} b\cdot L)^{-\frac{p}{n-i}} \geq a\tilde{W}_i(K)^{-\frac{p}{n-i}} + b\tilde{W}_i(L)^{-\frac{p}{n-i}},$$

this inequality was previously proved in [31].

Finally, we demonstrate the equivalence between two types of fundamental inequalities.

Proof of Theorem 1.4. The proof of Theorem 1.3 shows that (1.11) can be deduced from (1.10), here we only need to prove (1.10) by (1.11).

For $\varepsilon \geq 0$, let $K_\varepsilon = K\tilde{+}_\phi\varepsilon\cdot L$ and construct a new function

$$F(\varepsilon) = \phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) + \varepsilon\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) - \phi(1),$$

it is easy to check that $F(\varepsilon) \leq 0$, $F(0) = 0$ and

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\frac{F(\varepsilon) - F(0)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) + \varepsilon\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) - \phi(1)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) - \phi(1)}{\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}} - 1}\lim_{\varepsilon\to 0^+}\frac{\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}} - 1}{\varepsilon} \\ &\quad+\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\right)^{\frac{1}{m}}\right). \qquad (3.9) \end{aligned}$$

Since ϕ is decreasing and (3.7) implies $K_\varepsilon \subseteq K$ for a given fully small positive ε. Thus, $\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}} \to 1^+$ as $\varepsilon \to 0^+$ and

$$\lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) - \phi(1)}{\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}} - 1}$$

$$= \lim_{t\to 1^+} \frac{\phi(t)-\phi(1)}{t-1}$$
$$= \phi_r'(1). \tag{3.10}$$

From (1.6) and (1.8), we know

$$\lim_{\varepsilon\to 0^+} \frac{\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}} - 1}{\varepsilon}$$
$$= -\lim_{\varepsilon\to 0^+} \widetilde{V}_{(0)}(K_\varepsilon)^{-\frac{1}{m}} \lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_{(0)}(K_\varepsilon)^{\frac{1}{m}} - \widetilde{V}_{(0)}(K)^{\frac{1}{m}}}{\varepsilon}$$
$$= -\widetilde{V}_{(0)}(K)^{-\frac{1}{m}} \frac{1}{m} \widetilde{V}_{(0)}(K)^{\frac{1}{m}-1} \lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_{(0)}(K_\varepsilon) - \widetilde{V}_{(0)}(K)}{\varepsilon}$$
$$= -\frac{1}{\phi_r'(1)} \frac{\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})}{\widetilde{V}_{(0)}(K)}. \tag{3.11}$$

It follows from (3.9), (3.10) and (3.11) that

$$\lim_{\varepsilon\to 0^+} \frac{F(\varepsilon)-F(0)}{\varepsilon}$$
$$= -\frac{\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})}{\widetilde{V}_{(0)}(K)} + \phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\right)^{\frac{1}{m}}\right) \le 0,$$

this gets (1.10). Equality holds in (1.10) if and only if $F(\varepsilon) = F(0) = 0$, which shows that equality condition of (1.10) can be obtained from (1.11).

Remark 3.5 In fact, by (1.11) we can give another proof of (1.10), however, the equality condition is hard to obtain.

Second proof of Theorem 1.2. Since ϕ is convex and strictly decreasing, we know

$$\phi_r'(1)(x-1) \le \phi(x) - \phi(1), \ for\ x \ge 1. \tag{3.12}$$

By (1.8), (3.11), (3.12) and (1.11), we have

$$\widetilde{V}_{(\phi,m,1)}(K,L,\mathbf{C})$$
$$= \frac{\phi_r'(1)}{m} \lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_{(0)}(K_\varepsilon) - \widetilde{V}_{(0)}(K)}{\varepsilon}$$
$$= \frac{\phi_r'(1)}{m} \cdot m\widetilde{V}_{(0)}(K)^{\frac{m-1}{m}} \lim_{\varepsilon\to 0^+} \frac{\widetilde{V}_{(0)}(K_\varepsilon)^{\frac{1}{m}} - \widetilde{V}_{(0)}(K)^{\frac{1}{m}}}{\varepsilon}$$
$$= -\phi_r'(1)\widetilde{V}_{(0)}(K) \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left(\frac{\widetilde{V}_{(0)}(K)^{\frac{1}{m}}}{\widetilde{V}_{(0)}(K_\varepsilon)^{\frac{1}{m}}} - 1\right)$$
$$\ge -\widetilde{V}_{(0)}(K) \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left(\phi\left(\left(\frac{\widetilde{V}_{(0)}(K)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right) - \phi(1)\right)$$
$$\ge \widetilde{V}_{(0)}(K) \lim_{\varepsilon\to 0^+} \phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K_\varepsilon)}\right)^{\frac{1}{m}}\right)$$
$$= \widetilde{V}_{(0)}(K)\phi\left(\left(\frac{\widetilde{V}_{(0)}(L)}{\widetilde{V}_{(0)}(K)}\right)^{\frac{1}{m}}\right).$$

Thus, we have concluded (1.10).

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