Critical Points of Solutions to Semilinear Elliptic Problems in 3D Space

Jaime Arango, Jairo Delgado

Abstract—**We describe the critical set (points of vanishing gradient) of solutions to certain semilinear elliptic boundary value problem in a solid of revolution in 3D Euclidean space. For a torus like regions we show that the critical set is made up of exactly one circle obtained by rotating a point around the axis of rotation.**

Index Terms—**Critical points, elliptic problems, Dirichlet problem, solid of revolutions**

I. Introduction

In this paper we investigate the critical set of a solution to the semilinear Dirichlet problem

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the domain $\Omega \subset \mathbb{R}^3$ is a solid of revolution. We determine the nature of the critical set of the solution u to problem (1) in the case that Ω is generated by rotating a smooth planar region D around an axis non intersecting D. Further, we establish general conditions on the nonlinear term f so that the critical set of u is made up of finitely many circles in 3D space. By imposing additional symmetry conditions on the region D we are able to show that the critical set is made up of exactly one circle.

Semilinear elliptic equations are key building blocks of several important models in science and engineering, and as a consequence, there is a considerable amount of mathematical research on the subject dating back well before Pierre Laplace. Some of the topics of elliptic equations are relatively well understood. For instance, there is a established theory accounting for the existence, uniqueness and regularity of the solution to fully nonlinear elliptic equations in which problem (1) is a merely a particular case. For example, if f is non increasing analytic and satisfies $f(0) > 0$, then there exits a unique solution to problem (1) (see [12, Theorem 6.13]). On the other side, geometrical properties of the solution to elliptic equations are less investigated. Despite of more than two hundred years of research, there does not exist as today a reasonable complete description of the critical set of the solutions u to (1), not even for its linear version, namely the torsion problem. The critical set of an elliptic equation may be relevant for applications since these equations model a variety of physical phenomena, say for instance the model for a steady viscous flow in a horizontal pipe. Moreover, several open questions, as to determine the Gaussian curvature of the level set of solution to semi linear elliptic equations, are deeply related to the structure of the critical set. We refer the reader to [18] and the references cited there.

It is not easy to trace and pinpoint the main sources of the research of the critical set of elliptic equations and we do not attempt to review this vast subject. We mention the the seminal work of Makar–Limanov [15] addressing the uniqueness of the critical point of the torsion problem in a planar convex region. Makar–Limanov's result was latter generalized for semilinear equations by paper Cabré and Chanilllo [9]. See also [11] and [13].

A closer look at the literature reveals a gap when the convexity assumption is removed or when the elliptic problem is considered in $\mathbb{R}^3$. For arbitrary solids of revolution, we are not aware of examples with explicit solutions that could provide clues for the critical set of u. To the author's knowledge, there are very few results concerning the nature a of the critical set of a solution to semilinear elliptic equations in 3d Euclidean space. Indeed, most of the results in the current literature are restricted to planar domains. See for example [3], [4], [5]. In 3D space we only are aware of a result that guarantee the existence of a unique critical point in a convex solid of revolution (see [9, Theorem 2]).

II. Critical set in 3D space

Since Ω is a solid of revolution, the solution u to problem (1) does not depend on the angular component of the cylindrical coordinates describing Ω. Let us denote by r the radial coordinate and by z the axis of rotation of Ω. It will be convenient to write

$$L[v] = -\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right),$$

so that if u solves (1), then $v(r,z) = u(x,y,z)$, with $x^2 + y^2 = r^2$, satisfies

$$\begin{aligned} L[v] &= f(v) && \text{in } D, \\ v &= 0 && \text{on } \partial D, \end{aligned} \tag{2}$$

Notice that $r > 0$ for all $(r,z) \in D$. The following result issues the general structure of the critical set of problem (1).

Theorem II.1. *Let D be a planar smooth region and denote by Ω the solid of revolution obtained by rotating D around an axis non intersecting D. If f is a non increasing real analytic function such that $f(0) > 0$, then the critical set of the solution u to problem* (1) *is made up of finitely many circles in Ω obtained by revolving finitely many points of D around the axis of rotation.*

Manuscript received September 17, 2018; revised August 2, 2019. This work was supported by Universidad del Valle, Cali, Colombia and Universidad Nacional de Colombia Sede Palmira, Palmira, Colombia.

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(Revised online publication: 25 February 2020)

Proof: It is seen that critical circles of the solution u in the 3D problem (1) correspond to critical points of solutions v of the planar problem (2). As a consequence, the theorem follows if we manage to prove that the solution v possesses finitely many critical points in D.

First we claim that $L[v] > 0$ in D. If $f > 0$, then $L[v] > 0$, and the claim follows immediately. Now, let $x_m \in D$ be a point where v reaches its maximum value $v_m = v(x_m)$. Since L is elliptical, it follows that $L[v](x_m) \geq 0$. If $0 = L[v](x_m)$ then $f(v(x_m)) = 0$ so the constant function defined in D, $w(x) = v(x_m)$ satisfies $L[v] = f(v)$ in D. Then, by the tangency principle (see [17],Thm. 2.1.3), we have that $v = w$, that is v is constant, which is a contradiction. Whereby $0 < L[v](x_m)$. As f is non increasing, we get $0 < f(w) \leq f(v) = L[v]$ then $0 < L[v]$. Compare [4, Lemma 2.2] and [6, Lemma 1]. As a result of the claim it follows by the Maximum Principle that v is positive in D.

Since $L[v] > 0$, v is semi-Morse function in the sense that its Hessian matrix is non vanishing (see [7]). Moreover, by the Hopf's boundary point Lemma we see that v has no critical points at the border ∂D. By a result of Arango y Perdomo [7], the critical set of v is made up of finitely many isolated critical points and finitely many critical Jordan curves in D. Now, if β is a critical curve in D of a solution v of (2), it is easy to see that β must be a curve of maxima enclosing a sub domain $D_\beta \subset D$. As a consequence, v must have a minimum inside D_β; and this contradicts the Maximum principle since $L[v] > 0$ in D_β. As any Jordan critical is ruled out, then the critical set of v is made up of finitely many isolated points ■

III. Bifurcation of nodal lines

Theorem II.1 is the best result we can obtain for a general solid of revolution in 3D space. To obtain a finer description of the critical set of the solution u to (1), we have to impose additional conditions on D and to investigate the critical set of the solution v to the planar boundary value problem (2). Now, a technique to study the critical set of a real vale function consists in describing the nodal set of its directional derivatives. It happens that the directional derivatives of v satisfy a relatively simple linear elliptic equations. Moreover, a classical result due to L. Bers ([8, Theorem I]) precisely describes the nodal set of the directional derivatives of v.

Theorem III.1. *Let M a linear elliptic operator given by*

$$M[w] = \sum_{i+j=2} a_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x)\frac{\partial w}{\partial x_i} + c(x)\, w,$$

where the coefficients a_{ij} and b_i and c are smooth. Assume that v solves $M[w] = 0$ in a neighborhood of the origin and denote by

$$M_0[w] = \sum_{i+j=2} a_{ij}(0)\frac{\partial^2 w}{\partial x_i \partial x_j}$$

the osculating operator (with constant coefficients). If $w(x) = O(|x|^N)$, with $1 \leq N < \infty$, then there exists a homogeneous polynomial of degree N, $p_N(x) \not\equiv 0$ such that $M_0[p_N] = 0$ and

$$w(x) \sim p_N(x),\ \frac{\partial w}{\partial x_i}(x) \sim \frac{\partial p_N}{\partial x_i}(x),$$

$$\frac{\partial^2 w}{\partial x_i \partial x_j}(x) \sim \frac{\partial^2 p_N}{\partial x_i \partial x_j}(x),$$

where the error terms are $O(|x|^{N+1-\delta})$, $O(|x|^{N-\delta})$, and $O(|x|^{N-1-\delta})$ respectively and $0 < \delta$ arbitrary small.

Let us assume that the planar border ∂D is analytic. Then v as well as its directional derivatives can be analytically extended to $\overline{D}$ (see [16]). We write v_θ to denote the directional derivative of v in the θ direction. The nodal set of v_θ has an important role in this investigation, therefore we highlight its definition:

$$N_\theta = \left\{ x \in \overline{D} : v_\theta(x) = 0 \right\}. \tag{3}$$

Notice that the critical set K of a solution u to problem in D can be described as $K = N_\theta \cap N_\phi$, where θ and ϕ are two non collinear directions. Moreover, a boundary point $q \in \partial D$ belongs to N_θ if and only if a unitary tangent vector of ∂D at q is parallel to θ. Now, the key observation is the following: if the curvature of ∂D never vanishes, then N_θ has exactly two points in ∂D. If we could ruled out Jordan curves in N_θ, then N_θ is an analytic curve connecting two boundary points.

Figure 1 pictures several nodal lines N_θ of the solution v to problem 2 where $f \equiv 5$ and D is a disk of radius 1 centered at $3/2$. Notice that the nodal lines meet at the unique critical point of v (see Lemma III.2 and Example 1). The corresponding θ values of each line are easy to spot since the nodal lines meet ∂D at a point with a tangent vector parallel to θ.

Lemma III.1. *Assume that f satisfies the hypothesis of Theorem II.1 and that D is a planar convex region with analytic border ∂D such that the curvature of ∂D never vanishes, and let v be a solution to* (2)*. If $\theta = (1, 0)$ or $\theta = (0, 1)$, then the nodal set N_θ corresponding to v possesses no Jordan curves and v_θ has no critical points on N_θ.*

Proof: A straightforward calculation shows that for $\theta = (\theta_1, \theta_2)$, v_θ satisfies

$$L[v_\theta] - f'(v)v_\theta + \frac{\theta_1}{r^2}\frac{\partial v}{\partial r} = 0. \tag{4}$$

Set $\theta = (1, 0)$ and write

$$M[w] = L[w] + \left(\frac{1}{r^2} - f'(v)\right) w.$$

For $\theta = (1, 0)$ equation (4) reads $M[v_\theta] = 0$. Let q be a $N-$ th order zero of v_θ. By Theorem III.1 there exists an homogeneous polynomial P_N of degree N such that for $l = 0, 1, 2$ and $j + k = l$

$$\frac{\partial^l v_\theta(x)}{\partial x_1^j \partial x_2^k} = \frac{\partial^l p_N(x-q)}{\partial x_1^j \partial x_2^k} + O(|x-q|^{N-l+1}), \quad x = (x_1, x_2), \tag{5}$$

where p_N satisfies $M_0[p_N] = \Delta p_N = 0$, so that p_N is a harmonic polynomial with a $N-$th order zero at the origin. As a consequence, the nodal lines of v_θ are homeomorph to the nodal lines of p_N near the origin. Therefore the set N_θ is locally made up of N rays crossing at q (see [10]). The above implies that for the mentioned directions the critical points of v_θ on the nodal line N_θ are isolated. Certainly at regular points of v_θ, the nodal set N_θ is an analytics curve. Further, since v_θ is approximated by a homogeneous harmonic polynomial, then at each critical point of u_θ, the nodal set N_θ is a system of at least two curves meeting at the critical point.

Set $\theta = (1,0)$ and assume that p is a critical point of v_θ on N_θ, so that N_θ bifurcates in p systems of at least two curves crossing at p. Since ∂D contains exactly two points having a unitary tangents parallel to θ, then one of the curves meeting at p must be a Jordan curve enclosing a domain $D_\theta \subset D$. By (4) it follows that v_θ, with $\theta = (1,0)$, satisfies

$$\begin{aligned} L[v_\theta] + \left(\frac{1}{r^2} - f'(v)\right) v_\theta &= 0 \quad \text{in} \quad D_\theta, \\ v_\theta &= 0 \quad \text{on} \quad \partial D_\theta, \end{aligned}$$

By the Maximum Principle [17, Theorem 2.1.1] for u_θ, $c(x) = \frac{1}{r^2} - f'(v) \geq 0$, as $f' \leq 0$, we get $v_\theta \equiv 0$ en D_θ, and by the analyticity of v we can get $v \equiv 0$, to arrive at a contradiction. For $\theta = (0,1)$ we proceed in an analog way. Therefore, for the directions $\theta = (0,1)$ and $\theta = (1,0)$, the corresponding nodal sets N_θ are smooth curves, each of one connects exactly two points on ∂D. ■

Next, we impose a symmetry condition on D to obtain a more precise description of the critical set of the solutions to problem (1). Certainly, the symmetries of D are inherit by the solution v and the nodal set of its directional derivatives. The proof of the following lemma is straightforward.

Lemma III.2. *Assume the hypothesis of Lemma III.1. If D is region of the rz plane symmetric with respect to $z = 0$, then $N_{\theta^*} = (N_\theta)^*$ and $v(x) = v(x^*)$ for all $x \in D$, where * stands for the reflection with respect to $z = 0$.*

Lemma III.3. *Let D be a planar convex smooth region such that the curvature of ∂D never vanishes. If D is symmetric with respect an axis of symmetry and f satisfies the assumptions of Theorem II.1, then the solution v to* (2) *has a unique critical point which is non degenerated.*

Proof: Without lost of generality let us suppose that $z = 0$ is the axis of symmetry of D. By Lemma III.2, v is symmetric with respect to $z = 0$, therefore $\{z = 0\} \cap D \subset N_{(0,1)}$. Next, by Lemma III.1 we have

$$N_{(0,1)} = \left\{(r,0) : (r,0) \in \overline{D}\right\}. \tag{6}$$

Notice now that Lemma III.2 guarantees that $N_{(1,0)}$ is symmetric with respect to $z = 0$. Further, by Lemma III.1 $N_{(1,0)}$ does not contain any Jordan curve. Since $N_{(1,0)}$ is a smooth curve connecting exactly two points of ∂D, which is symmetric with respect to $z = 0$, then the critical set of v, which is given by $N_{(1,0)} \cap N_{(0,1)}$, is made up of exactly one point.

For a direction $\theta = (\theta_1, \theta_2)$ let us write $v_\theta(x) = \nabla v(x) \cdot \theta$ and see that

$$\nabla v_\theta(x) = H_v(x)\,\theta,$$

where $H_v(x)$ stands for the Hessian matrix of the solution v to (2). Furthermore, if $q \in N_\theta$ and $H_v(q)\,\theta$ does not vanish, then N_θ can be locally parametrized by the ODE

$$z' = J\,H_v(z)\,\theta, \tag{7}$$

where J is the $-\frac{\pi}{2}$ rotation matrix. Let p the unique critical point of v and see that p belongs to $N_{(1,0)} \cap N_{(0,1)}$. By Lemma III.1, $H_v(p)\,\theta \neq 0$ for $\theta = (1,0)$ and for $\theta = (0,1)$. Moreover, by (6) and (7), for some $\lambda \neq 0$ we have $H_v(p)\,(0,1) = \lambda\,(0,1)$. By contradiction, suppose that $H_v(p)$ is singular. That is to say that 0 must be an eigenvalue of $H_v(p)$ with an eigen space generated by $(1,0)$, and fortiori $H_v(p)\,(1,0) = 0$ which is a contradiction. ■

IV. Uniqueness and stability of the critical circle

We are in a position to come back to the original problem (1) to show that the critical set of the solution u to (1) is made up of a unique circle in the domain Ω. Moreover, this circle is stable under small perturbations of the generating domain D.

Theorem IV.1. *Suppose the region D satisfies the hypothesis of Lemma III.3. If Ω is the solid of revolution obtained by rotating D with respect to an axis which is orthogonal to the axis of symmetry, then the critical set of the solution u to* (1) *is a circle obtained by revolving a unique point of D around the axis of rotation.*

Proof: Denote by z the axis of rotation and by r the axis of symmetry of the region D. Since to any critical point of the solution u to problem (2) corresponds a critical Jordan curve of problem (1), it only remains to prove that problem (2) has exactly one critical point, but this is precisely the claim of Lemma III.3. ■

Concerning Lemma III.3, it is worth noticing that the critical circle in Theorem IV.1 corresponding to the solution u to problem (1) is degenerated. Indeed, if $w = w(t)$ parametrizes the critical circle, then $\nabla u(w(t))$ vanishes for all t, and as a consequence for al t we have

$$H_u\left(w\left(t\right)\right)\,w'\left(t\right) = 0.$$

Hence, the Hessian matrix $H_u(w)$ is singular for all w in the critical set.

Example 1. *Let $a > 0$. The 3D torus $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (a - \sqrt{x_1^2 + x_2^2})^2 + x_3^2 = 1\}$ is obtained by rotation of*

$$D = \left\{(r,z) : (a - r)^2 + z^2 < 1\right\}.$$

Certainly, D satisfies the assumptions of Theorem IV.1. As a consequence, If f satisfies the assumptions of Theorem II.1, then the solution of problem (1) *has a critical set made up of exactly a critical circle as described in Theorem IV.1. This critical circle is degenerated.*

To study the stability of the critical circle of problem (1) we consider first a family of diffeomorphisms

$$T : H \times (-\epsilon, \epsilon) \to H,$$

where $\epsilon > 0$ and H is a open neighborhood of D and $T(\cdot, 0)$ is the identity map. Let us write $T(D, \epsilon) = D_\epsilon$ and think of D_ϵ as a small perturbation of $D \equiv D_0$. Furthermore, denote by $v(\cdot; \epsilon)$ the solution to

$$\begin{aligned} L[v] &= f(v) \quad && \text{in} \quad D_\epsilon, \\ v &= 0 \quad && \text{on} \quad \partial D_\epsilon. \end{aligned} \tag{8}$$

Theorem II.1 remains true for D_ϵ instead of D. Moreover, the solution family $v(\cdot, \epsilon)$ smoothly depend on ϵ and it make sense to study the dependence of the critical set on the parameter ϵ. By Lemma III.2, if p is the unique critical point of $v \equiv v(\cdot; 0)$, then $H_v(p)$ is a nonsingular matrix. As a consequence, the implicit function theorem guarantees the existence of a smooth curve $z : (-\epsilon_0, \epsilon_0) \to D$ such that $z(0) = p$ and

$$\nabla v(z(s); s) = 0, \quad s \in (-\epsilon_0, \epsilon_0).$$

Furthermore, dz/ds is implicitly given by

$$H_{v(\cdot;,s)}\,(z(s), s)\,\frac{dz}{ds} + \frac{\partial}{\partial s}\nabla v(z(s); s) = 0, \quad s \in (-\epsilon_0, \epsilon_0).$$

Now, the critical points $z(s)$ of the planar problem (8) are associated to critical circles of the corresponding domains Ω_s, so that the critical circle to which Theorem IV.1 refers remains a circle under small perturbations of the generating 2D domain D.

The stability of the the critical circle under small perturbations of D is by no means obvious. For the sake of comparison consider the 2D torsion problem $-\Delta u = 1$ with null Dirichlet condition on a circular concentric 2D annuls. It is easy to explicitly calculate the solution u (see Example 1 in [4]) and to realize that the critical set of u is a also a 2D circle inside the annuls. It is easily seen (perturbing the explicit solution u with a small harmonic term with no radial symmetry) that this critical circle is unstable. It collapse to a finite number of critical points under almost any imaginable non radially symmetric harmonic perturbation.

V. Numerical experiments

Explicit solutions u to problem (1) are not known to the authors. Fortunately, there is an ample choice of numerical methods to approximate solutions of elliptic PDE's (see for instance [14]). The following examples are calculated with the public domain software FENICS ([1], [2]). With a modicum of coding, FENICS easily handles the meshes, FEM spaces and the boundary conditions to obtain a FEM approximation of the PDE's of this paper. Since $v(r, z) = u(x, y, z)$, $x^2 + y^2 = r^2$, the solution v to the 2D problem (2) on the planar region D contains all the information to the solution u.

For starters, let us consider the disk D in Example 1 with constant right hand $f \equiv 5$. Figure 1 pictures several uninterrupted lines corresponding to the level sets $v(r, z) = a$, where the bold thick black curve corresponds to $a = 0$ and coincides with boundary of D, whereas the shaded circle–like curves (blue in the online edition) correspond to $a = 0.63, a = 1.0$ and $a = 1.23$. With a second order interpolation FEM space we are able to calculate the nodal lines of v. Figure 1 shows also in dotted lines the nodal lines N_θ of v for $\theta = (1, 0), (0, 1), (1/\sqrt{2}, 1/\sqrt{2})$. Notice that the nodal lines cross the region D and meet at the unique critical point of v (lying to the left of the center of D). Notice also the symmetry of v with respect to the horizontal line passing through the (unique) critical point of v.

If the curvature of ∂D changes sign, the solution u to (1) might have several critical circles as it is suggested by the FEM approximation of v in a suitable region D. Figure 2 shows ∂D (black bold thick line) and the level sets $v(r, z) = a$, for $a = 0.6$ and $a = 1.2$ (blue and red line respectively in the online version). The region D was obtained by union

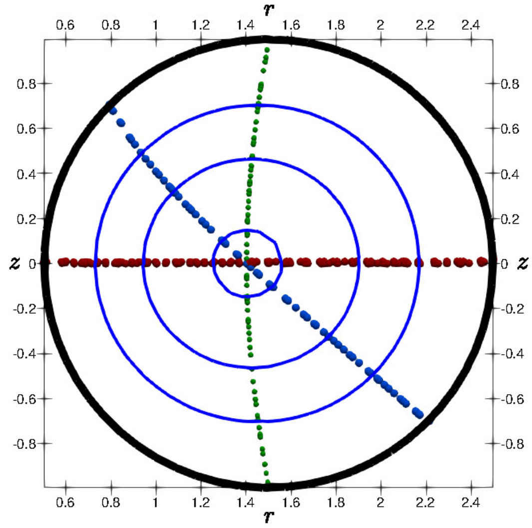

Figure 1: FEM approximation of the solution to Example 1 on a disk D with $f \equiv 5$. The level sets of v are drawn in uninterrupted lines whereas the nodal lines N_θ are pictured in dotted lines for $\theta = (1, 0), (0, 1), (1/\sqrt{2}, 1/\sqrt{2})$.

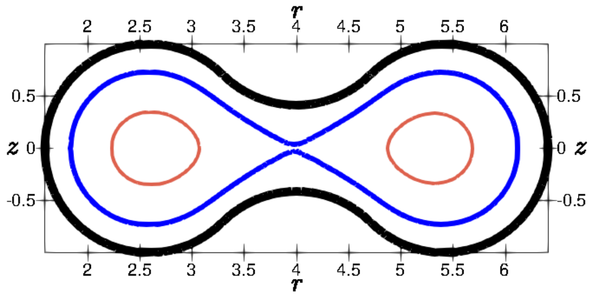

Figure 2: Level sets of the solution v to problem 2, $f \equiv 5$, on a region D such that the curvature of ∂D changes sign. Notice that v possesses 3 critical points

and differences of elementary shapes (disks and rectangles). The numerical calculations were implemented using $f \equiv 5$ and the "mshr" library of FENICS.

A Jupyter Notebook (https://jupyter.org/) containing the Python code to handle the FEM approximation using FENICS can be downloaded at https://github.com/arangogithub/Critical-points-toroidal

VI. Conclusions

This paper discusses the nature of the critical set of a class of semilinear elliptic equations along with a null Dirichlet condition on the border of a solid of revolution Ω in 3D space. We established necessary conditions to guarantee that the critical set is made up of finitely many circles (Theorem II.1), and imposing additional symmetry conditions on Ω, we are able to show that the critical set is made up of exactly one critical circle (Theorem IV.1). We also address the question of the stability of the critical set under small perturbations of the domain. If the domain Ω is obtained by rotating a planar symmetric region D, we show that the critical circle is stable under small perturbations of D. Yet, this result likely do not carry over small perturbations of the whole domain Ω.

Our results might be of relevance in some applications, as for example to get a qualitatively description of the steady state of a viscous laminar flow in a toroidal pipe.

ACKNOWLEDGEMENTS

The authors thank their universities, Universidad del Valle and Universidad Nacional de Colombia, sede Palmira for supporting this investigation. J. Delgado thanks the Universidad del Valle, part of the results in this manuscript are the product of the research work of the doctoral thesis developed in this institution.

REFERENCES

[1] Martin S. Alnæs, Jan Blechta, Johan Hake, August Johansson, Benjamin Kehlet, Anders Logg, Chris Richardson, Johannes Ring, Marie E. Rognes, and Garth N. Wells. The fenics project version 1.5. *Archive of Numerical Software*, 3(100), 2015.

[2] Martin S. Alnæs, Anders Logg, Kent-Andre Mardal, Ola Skavhaug, and Hans Petter Langtangen. Unified framework for finite element assembly. *International Journal of Computational Science and Engineering*, 4(4):231–244, 2009.

[3] J. Arango. Uniqueness of critical points for semi-linear elliptic problems in convex domains. *EJDE*, 43:1–5, 2005.

[4] J. Arango and A. Gómez. Critical points of solutions to elliptic problems in planar domains. *Comm. Pure Appl. Anal.*, 10:327–338, 2011.

[5] J. Arango and A. Gómez. Critical points of solutions to quasilinear elliptic problems. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 75(11):4375–4381, 2012.

[6] J. Arango and A. Gómez. Critical points of solutions to quasilinear elliptic problems. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 75(11):4375–4381, 2012.

[7] J. Arango and O. Perdomo. Morse theory for analytic functions on surfaces. *J. Geom.*, 84(1-2):13–22, 2005.

[8] L. Bers. Local behavior of solutions of general linear elliptic equations. *Comm. Pure Appl. Math.*, 8:473–496, 1955.

[9] X. Cabré and S. Chanillo. Stable solutions of semilinear elliptic problems in convex domains. *Sel. Math., New ser.*, 4(1):1–10, 1998.

[10] Shiu-Yuen Cheng. Eigenfunctions and nodal sets. *Comment. Math. Helv.*, 51:43–55, 1976.

[11] David L. Finn. Convexity of level curves for solutions to semilinear elliptic equations. *Commun. Pure Appl. Anal.*, 7(6):1335–1343, 2008.

[12] G. Gilbarg y N. Trudinger. *Elliptic partial differential equations of second order*. Springer, 1983.

[13] B. Kawohl. When are solutions to nonlinear elliptic boundary value problems convex? *Commun. Part. Diff. Eq.*, 10(10):1213–1225, 1985.

[14] H. Guan P. Yin M., Zhao. A stable mixed finite element scheme for the second order elliptic problem. *International Journal of Applied Mathematics*, 46(4):545–549, 2016.

[15] L. Makar-Limanov. Solutions for the dirichlet's problem for the equation $\Delta u = -1$ in a convex region. *Notes Acad. Sci. USSR.*, 19:52–53, 1971.

[16] F. Müller. On the continuation of solutions for elliptic equations in two variables. *Annales de I'Institut Henri Poincare (C) Non Linear Analysis*, 19(6):745–776, 2002.

[17] P. Pucci and J. Serrin. *The maximum principle*. Birkhauser, Bassel-Boston-Berlin, 1rd edition, 2007.

[18] Paul Yang Sun-Yung Alice Chang, Xi-Nan Ma. Principal curvature estimates for the convex level sets of semilinear elliptic equations. *Discrete & Continuous Dynamical Systems*, 28(3):1151–1164, 2010.

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(Revised online publication: 25 February 2020)