

# Transitional Shock Waves Using Overlapping Grids Methods

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**Abstract**—We consider initial value problems for several systems of two conservation laws in one spatial dimension, where the initial data result in shocks, rarefaction waves, and transitional shock waves. We compute the approximate solutions using a recently proposed overlapping grids finite volume numerical method. We confirm the effectiveness of this method by comparing the obtained numerical results to the approximate solutions obtained using a regular finite volume method with the Lax–Friedrich numerical flux.

**Index Terms**—conservation laws, finite volume methods, Riemann problem, shocks, rarefaction waves.

## I. INTRODUCTION AND SUMMARY

CONSERVATION laws are time-dependent partial differential equations that describe many fundamental problems in science and engineering, including gas dynamics and aerospace engineering, multiphase flow applications in secondary oil recovery in petroleum industry, applications to traffic flow, supply-chain systems, chemistry, biology, etc. Since such problems are often very complex and nonlinear and, as a consequence, it is not possible to solve them analytically, there is a need to develop numerical methods for their approximate solving. These numerical methods have to be carefully designed in order that the numerical approximations converge to a physically correct solution of the conservation law. In this paper we consider a recently proposed overlapping grids finite volume method by Jegdić [10] and we test its effectiveness on several examples of conservation laws. The significance of the overlapping grids methods in general is in the applications when the topology of the object considered is very complex that it might be very difficult, if not impossible, to create a grid around that object.

A system of conservation laws in one space dimension is given by

$$\partial_t u + \partial_x f(u) = 0, \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

where  $u = (u^1, \dots, u^m)$  is the unknown vector of densities of conserved quantities and  $f = (f^1, \dots, f^m)$  is the known spatial flux density field defined on a domain of conservation states. The system is supplemented by an initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

where  $u_0$  is a bounded and measurable function on  $\mathbb{R}$ . The system is hyperbolic if the Jacobian matrix

$$[\partial f^i / \partial u^j]_{m \times m}$$

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has real eigenvalues. If, in addition, the eigenvalues are distinct, the system is strictly hyperbolic. Let  $\lambda_i$  and  $r_i$  denote the  $i$ th eigenvalue and the  $i$ th right eigenvector, respectively. The  $i$ -characteristic family is said to be genuinely nonlinear if

$$\nabla \lambda_i(u) r_i(u) \neq 0,$$

and linearly degenerate if

$$\nabla \lambda_i(u) r_i(u) = 0,$$

for all conservation states  $u$ . In this paper we consider a piecewise constant Riemann initial data

$$u(x, 0) = \begin{cases} u_l, & x < 0, \\ u_r, & x \geq 0. \end{cases}$$

It is well known that if each characteristic family is either genuinely nonlinear or linearly degenerate, the above Riemann problem with  $u_l$  sufficiently close to  $u_r$ , has a solution consisting of the superposition of  $m$  elementary waves (shocks, rarefaction waves, or contact discontinuities). For theoretical results, we refer to Bressan [3], Dafermos [4], Godlewski and Raviart [6], Holder and Risebro [8], Kružkov [12], Lax [13], and Smoller [19].

However, for non strictly hyperbolic and mixed type conservation laws, there are examples of systems where the solutions have much more complex structure and the main focus of this paper is on the numerical study of transitional shock waves. We consider three such systems studied by Azevedo, Marchesin, Plohr, and Zumburn in [1], Schecter, Plohr, and Marchesin in [18], and Hwang in [9], using a recently proposed overlapping grids finite volume method by Jegdić [10].

In §2 we recall the definition of the overlapping grids finite volume method and the main results from [10]. In §3 we use the overlapping grids method to compute solutions to a variety of Riemann problems illustrating transitional shock waves. We test the effectiveness of the overlapping grids method by comparing the obtained numerical results with the results obtained using a regular finite volume method. In both cases we use the well-known Lax–Friedrich numerical flux.

## II. OVERLAPPING GRIDS FINITE VOLUME METHODS

The application problems in conservation laws usually deal with the nonlinear spatial flux function  $f$  leading to nonlinear systems of equations which cannot be solved exactly and there is a need to develop numerical schemes.

In this section we first briefly recall derivation of the finite volume method. One of the assumptions in the finite volume methods is that the approximate solution is given as

a piecewise constant function. Let us assume that the spatial domain  $\mathbb{R}$  has a partition consisting of cells

$$\Delta = \{\Omega_i \mid i \in I\},$$

where  $|\Omega_i|$  denotes the size of the  $i$ th cell. The numerical solution obtained via the finite volume method is a piecewise constant function at every time step and we denote it by

$$u^n(x) = \sum_I u_i^n \chi_{\Omega_i}(x),$$

where  $\chi$  represents characteristic function and  $u_i^n$  represents the cell average of the cell  $i$  at the time step  $n$ . Clearly, we have

$$u^0(x) = \sum_I u_i^0 \chi_{\Omega_i}(x),$$

where  $u_i^0$  is calculated using the initial data by

$$u_i^0 = \frac{1}{|\Omega_i|} \int_{\Omega_i} u_0(x) dx.$$

Besides the spatial direction, the time direction is also discretized, meaning that if we need to find the approximate solution at a time level  $T$ , then we will be required to do a series of time steps of size  $\Delta t$  until we reach  $T$ . The time steps are defined by

$$\Delta t M \leq \min_i \frac{|\Omega_i|}{|\partial\Omega_i|} CFL,$$

where the  $CFL$  constant is given by the Courant – Friedrichs – Lewy condition,  $|\partial\Omega_i|$  is the size of  $\Omega_i$ 's boundary, and  $M$  is a constant proportional to fastest wave speed. In scalar case we take

$$M = \max_u |f'(u)|.$$

It is clear that the time step depends directly on the size of the smallest cell, meaning that, if we have at least one very small cell, our time step will be very small. Besides the problems with the time step, the small cells being next to the big cells could cause the finite volume method not to converge to a weak solution anymore, i.e., the Lax-Wendroff Theorem [14] does not hold. Then for the finite volume method we have

$$0 = |\Omega_i| (u_i^{n+1} - u_i^n) + \Delta t \sum_k \int_{S_{i,k}} h_{n_k \cdot F} ds,$$

where  $\Delta t = t_{n+1} - t_n$  represents the time step from time  $t_n$  to time  $t_{n+1}$ ,  $S_{i,k}$  is the edge between cells  $i$  and  $k$ ,  $n_k$  is the corresponding outward normal and  $h_{n_k \cdot F}$  is the numerical flux function.

Let us now consider the scalar case ( $m = 1$ ). Then the finite volume method is given by

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i} (h_{i+1} - h_i),$$

where by  $h_{i+1}$  we denote the numerical flux between the cells  $i$  and  $i+1$ , which usually depends only on  $u_i$  and  $u_{i+1}$ , and  $\Delta x_i$  is the size of cell  $i$ . Some of the most well known numerical fluxes are Godunov's, Roe's, Engquist-Osher's and Lax-Friedrichs's fluxes [5], [6], [7], [8], [15], [16], [17].

Depending on the topology of the object that is considered, sometimes it may be very difficult, if not impossible, to design a grid around that object. The idea of so-called overlapping grids is used in such cases. Basically, a different

grid is designed for each piece of the object and some of those grids may overlap in certain regions. In our case we study the problem of two overlapping grids.

Let us consider two overlapping grids; the "bottom" grid on the interval  $(-\infty, b]$  (with cell size  $\Delta x_B$ ) and the "top" grid on  $[a, \infty)$  (with cell size  $\Delta x_T$ ), such that  $a < b$ . The numerical method that we suggest is to extend both grids to cover the whole space  $\mathbb{R}$ . The *extended bottom grid* has the same partition as earlier on  $(-\infty, b]$ , while on  $(b, \infty)$  it is defined using the partition from the top grid. Similarly, we define the *extended top grid*.

We also define the *union grid* on  $\mathbb{R}$  as the union of partitions of the bottom and the top grids. We denote by  $u^n(x)$ ,  $v^n(x)$  and  $w^n(x)$  the numerical solutions on the bottom, the top, and the union grids, respectively, as depicted in Figure 1.

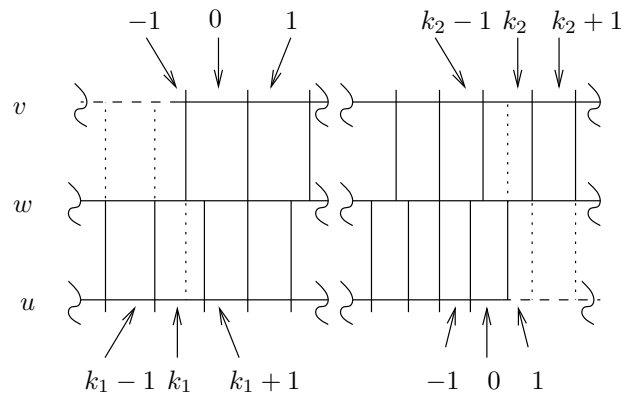


Fig. 1. Extended bottom and top grids with the union grid

Our idea is to first compute the mid solutions  $\tilde{u}^n(x)$  and  $\tilde{v}^n(x)$  by performing a regular finite volume method, where we modify the fluxes  $h_{b_0}$  and  $h_{b_1}$  for the bottom grid and  $h_{t_{-1}}$  and  $h_{t_0}$  for the top grid. The modified fluxes are computed as

$$\begin{aligned} h_{b_0} &= \theta_B h(u_0, u_1) + (1 - \theta_B) h(u_0, u_2), \\ h_{b_1} &= (1 - \theta_B) h(u_0, u_2) + \theta_B h(u_1, u_2), \\ h_{t_{-1}} &= \theta_T h(v_{-2}, v_{-1}) + (1 - \theta_T) h(v_{-2}, v_0), \\ h_{t_0} &= (1 - \theta_T) h(v_{-2}, v_0) + \theta_T h(v_1, v_0), \end{aligned}$$

where

$$\theta_B = \Delta x_{-1} / \min\{\Delta x_B, \Delta x_T\}$$

and

$$\theta_T = \Delta x_1 / \min\{\Delta x_B, \Delta x_T\},$$

as illustrated in Figure 2.

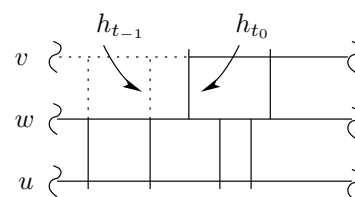


Fig. 2. Numerical fluxes  $h_{t_{-1}}$  and  $h_{t_0}$  for the top grid

Next, we define  $w^n(x)$  by projecting those mid solutions over the union grid and, finally, we get  $u^{n+1}(x)$  and  $v^{n+1}(x)$

by projecting  $w^n(x)$  over the bottom and the top grids, respectively. We note that the time step is defined in the same way as in a regular finite volume method by

$$\Delta t \max_u |f'(u)| \leq \min\{\Delta x_B, \Delta x_T\} CFL.$$

A similar idea for computing the fluxes is presented by Berger, Helzel, and LeVeque in [2]. One of the main differences between their and our work is that we average fluxes while they average cell averages. Also, we are able to use any numerical flux, including the Godunov numerical flux.

For the overlapping grids methods in [10], in the scalar case  $m = 1$ , we prove that if we have bounded convergence of approximate solutions, then the approximate solutions converge to the weak solution which is also the entropy solution. It is important to emphasize that we prove convergence to the entropy solution regardless of the cell sizes in both grids. The main contribution of this paper is in confirming the effectiveness of the overlapping grids method proposed in [10] numerically on several examples of systems of conservation laws ( $m > 1$ ).

### III. NUMERICAL EXAMPLES

We consider three one-dimensional systems of two conservation laws ( $m = 2$ ) with quadratic flux functions. These systems are examples of conservation laws for which there are Riemann initial data where the solutions are not superpositions of shocks and rarefaction waves. We illustrate the effectiveness of the overlapping grids method on several Riemann problems by comparing our solutions to the numerical solutions obtained using a regular finite volume method. In both cases we use the Lax–Friedrich numerical flux.

#### A. The first system

This system of conservation laws was studied in [1] by Azevedo, Marchesin, Plohr, and Zumburn, and in [18] by Schecter, Plohr, and Marchesin, and is given by

$$\begin{aligned} u_t + (-0.5u^2 + 0.5v^2 - 0.12u + 0.23v)_x &= 0, \\ v_t + (uv - 0.23u - 0.12v)_x &= 0. \end{aligned}$$

The eigenvalues are

$$\lambda_{1,2} = -0.12 \pm \sqrt{u^2 + v^2 - 0.0529},$$

implying that the system is strictly hyperbolic only if

$$u^2 + v^2 > 0.0529.$$

We consider several one-parameter Riemann problems for this system that were studied in [18] using a numerical method based on the Dafermos regularization and a continuation method. We compute numerical solutions to these problems using the overlapping grids finite volume method.

We solve the Riemann problems approximately on the interval  $[-3, 3]$ . The bottom grid is taken to be on the interval

$$[-3, 0.99854852],$$

while the top grid is taken to be on the interval

$$[-1.033665588, 3],$$

so that the overlap is not trivial. Both the bottom and the top grids contain 1000 cells each. The overlap is on approximately 500 cells. The CFL constant is taken to be 0.8 relative to  $\min\{\Delta x_B, \Delta x_T\}$ . For the comparison, we use the regular finite volume method with the grid consisting of 1500 cells.

As in [18], we fix

$$(u_l, v_l) = (0.366078, 0.308156)$$

and  $v_r = 0.1$ . The solutions of the Riemann problems in Examples III.1 and III.2 have a classical structure consisting of shocks and/or rarefaction waves, while the solution in Example III.3 is much more complicated and, in particular, it contains transitional shocks.

We note that the numerical results obtained using our overlapping grids method from [10] do not differ from the numerical results obtained using a regular finite volume method.

**Example III.1.** Assume  $u_r = 0.3$  (this problem corresponds to problem 1 in [18]). The solution of the Riemann problem consists of a 1-shock followed by a 2-shock. We plot numerical solutions for  $u$  and  $v$  obtained using the overlapping grids method and the finite volume method in Figures 3 and 4, respectively, at three different time levels.

**Example III.2.** Let  $u_r = 0.15$  (this problem corresponds to problem 3 in [18]). The solution of the Riemann problem consists of a 1-rarefaction wave followed by a 2-shock, and is plotted in Figures 5 and 6, using the overlapping grids method and the finite method, respectively, at three different time levels.

**Example III.3.** Assume  $u_r = -0.61$  (this problem corresponds to problem 17 in [18]). In this case the solution of the Riemann problem is much more complex and it consists of a 1-shock, 1-transitional shock, and a composite 2-wave (2-transitional shock and a 2-rarefaction wave). The numerical approximations for  $u$  and  $v$  are plotted in Figures 7 and 8, using the overlapping grids method and the finite volume method, respectively, at three different time levels.

#### B. The second system

This system of two conservation laws with quadratic flux functions was studied by Hwang in [9] using a front-tracking method, and is given by

$$\begin{aligned} u_t + (0.25u^2 + 0.2uv + 0.5v^2)_x &= 0, \\ v_t + (0.1u^2 + uv)_x &= 0, \end{aligned}$$

with eigenvalues

$$\lambda_{1,2} = 0.75u + 0.1v \pm \sqrt{0.1025u^2 + 0.35uv + 1.01v^2}.$$

It is easy to show that

$$0.1025u^2 + 0.35uv + 1.01v^2 > 0$$

for all states  $(u, v) \in \mathbb{R} \setminus \{(0, 0)\}$ , implying that this system fails to be strictly hyperbolic if  $(u, v) = (0, 0)$ . We illustrate the effectiveness of the overlapping grids method from [10] in the next example modeling a transitional wave.

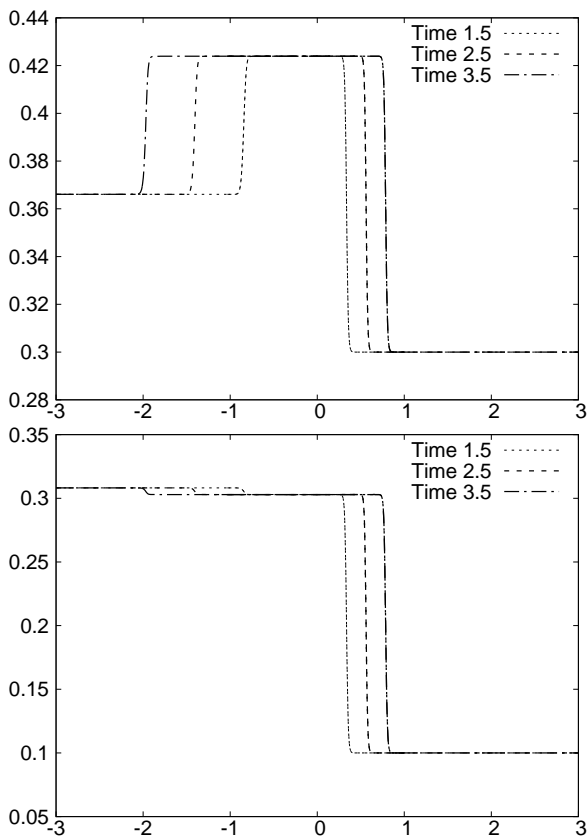


Fig. 3. Example III.1: the solutions  $u$  and  $v$  at times 1.5, 2.5, and 3.5 using the overlapping grids method.

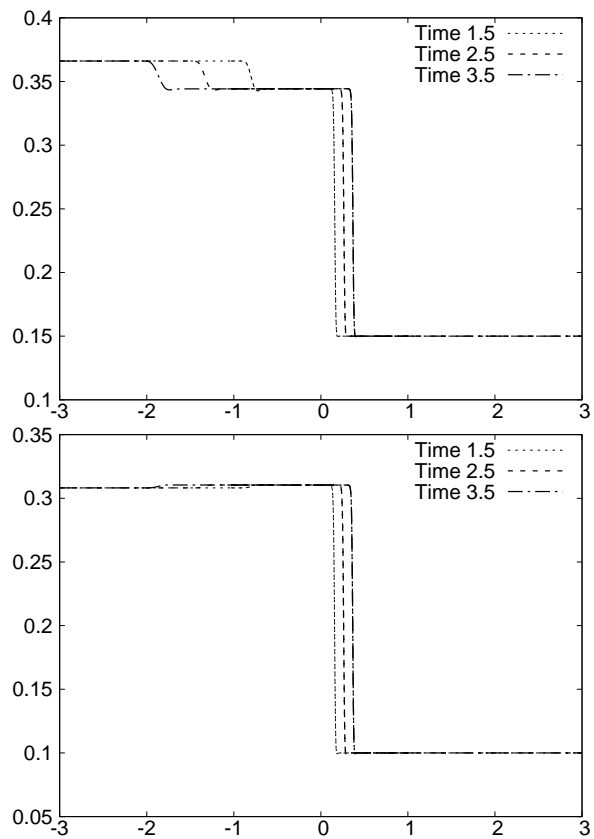


Fig. 5. Example III.2: the solution  $u$  and  $v$  at times 1.5, 2.5, and 3.5 using the overlapping grids method.

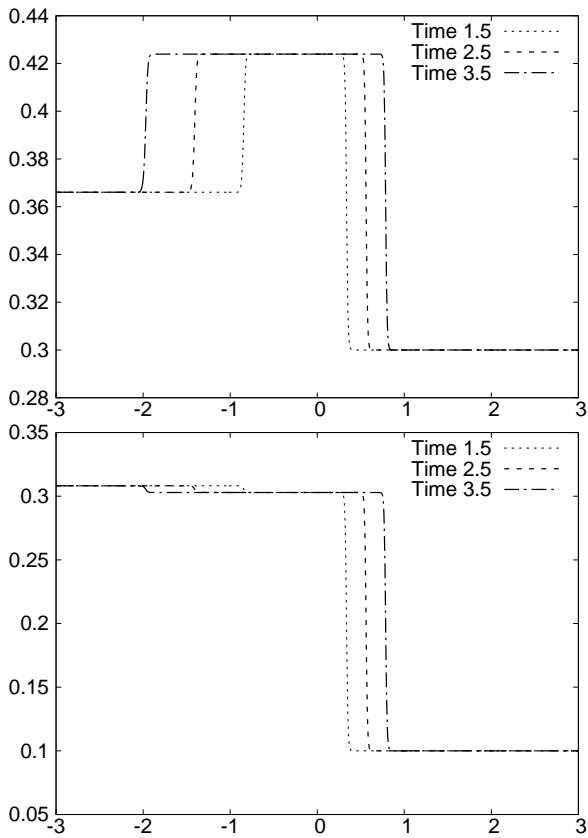


Fig. 4. Example III.1: the solutions  $u$  and  $v$  at times 1.5, 2.5, and 3.5 using the finite volume method.

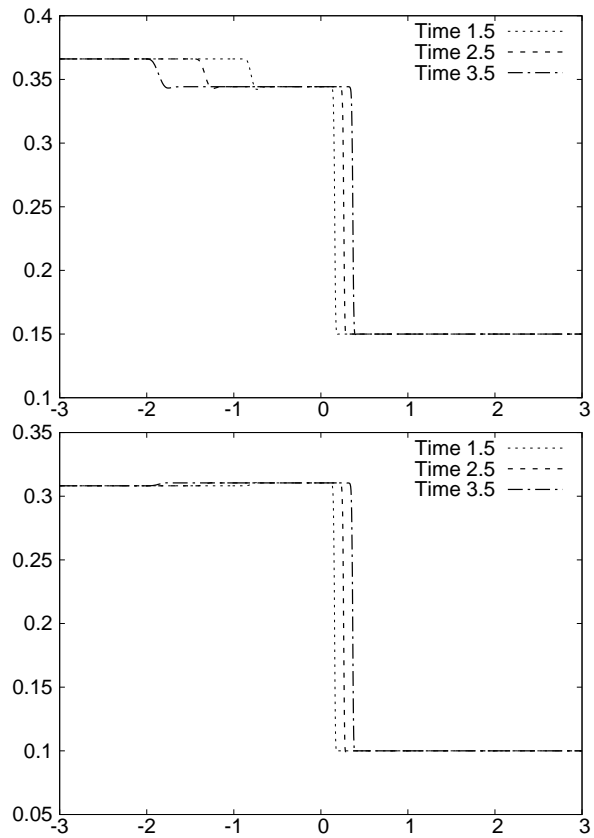


Fig. 6. Example III.2: the solution  $u$  and  $v$  at times 1.5, 2.5, and 3.5 using the finite volume method.

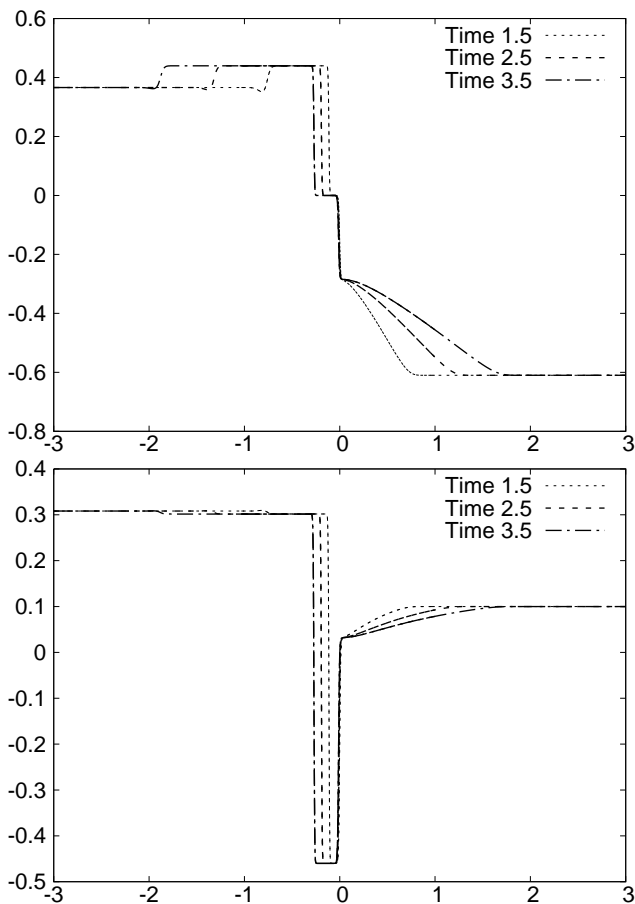


Fig. 7. Example III.3: the solution  $u$  and  $v$  at times 1.5, 2.5, and 3.5 using the overlapping grids method.

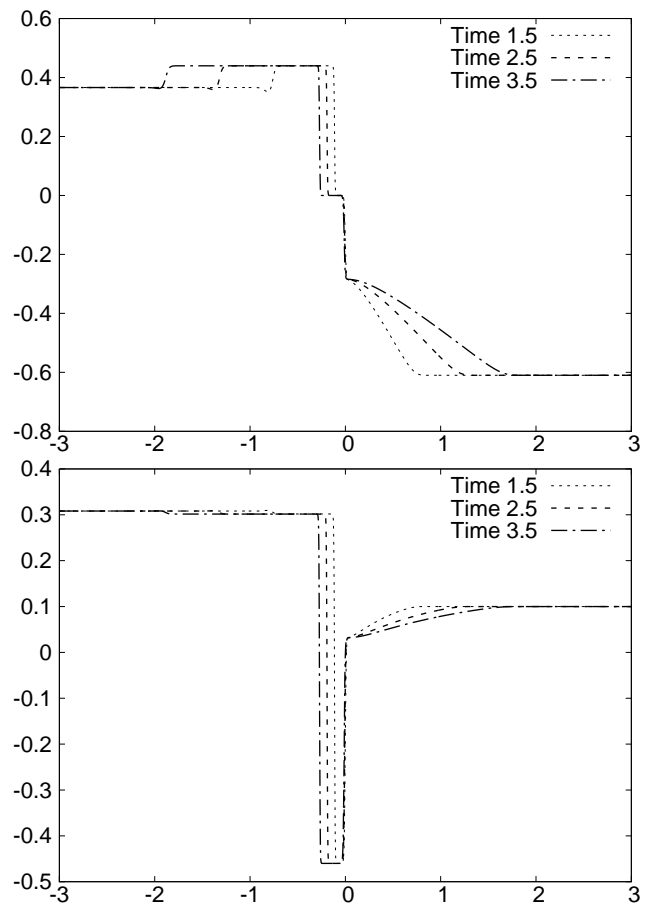


Fig. 8. Example III.3: the solution  $u$  and  $v$  at times 1.5, 2.5, and 3.5 using the finite volume method.

**Example III.4.** As in [9], we assume the following Riemann initial data

$$\begin{aligned} (u_l, v_l) &= (-1.632, 1.278), \\ (u_r, v_r) &= (-1.658, -1.269). \end{aligned}$$

The solution of the Riemann problem consists of a 1-shock, transitional wave, and a 2-shock. We solve this Riemann problem approximately on the interval  $[-2, 2]$ , using the overlapping grids method and the regular finite volume method. The numerical solutions are plotted in Figures 9 and 10, respectively, at three different time levels.

For the overlapping grids method, the bottom grid is taken to be on the interval

$$[-2, 1.0101010101],$$

while the top grid is taken to be on the interval

$$[-0.59999, 2],$$

so that the overlap is not trivial. The bottom grid consists of 1000 cells, while the top grid consists of 1200 cells. The CFL constant is taken to be 0.8 relative to  $\min\{\Delta x_B, \Delta x_T\}$ . For the comparison, we use the regular finite volume method on a grid with 1300 cells.

### C. The third system

This system was considered by Hwang in [9] using the front-tracking method. It is given by

$$\begin{aligned} u_t + (-0.5u^2 + 0.5v^2 + 0.23v)_x &= 0, \\ v_t + (uv - 0.23u)_x &= 0, \end{aligned}$$

with eigenvalues

$$\lambda_{1,2} = \pm \sqrt{u^2 + v^2 - 0.0529}.$$

The system is strictly hyperbolic only if

$$u^2 + v^2 - 0.0529 > 0.$$

We consider one example of initial data where the Riemann problem results in a transitional wave and we compare the numerical results obtained using the overlapping grids method and the regular finite volume method.

**Example III.5.** As in [9], let us assume

$$\begin{aligned} (u_l, v_l) &= (0.273, 0.182), \\ (u_r, v_r) &= (-0.246, 0.31). \end{aligned}$$

The solution of the Riemann problem consists of a 1-rarefaction, a transitional wave, and a 2-shock. We solve the Riemann problems approximately on the interval  $[-2, 2]$  using the overlapping grids method and the regular finite volume methods, in Figures 11 and 12, respectively, at three different time levels.

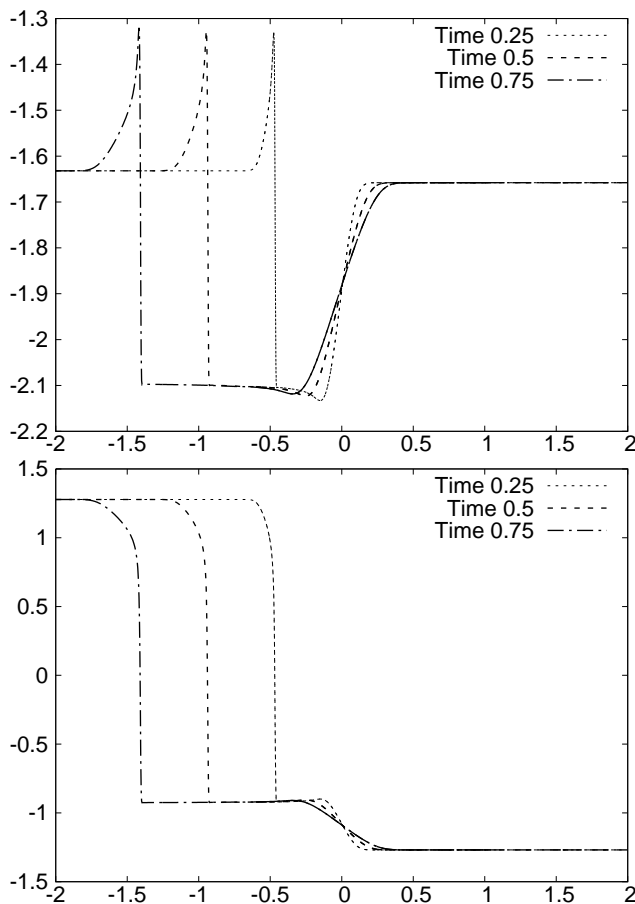


Fig. 9. Example III.4: the solution  $u$  and  $v$  at times 0.25, 0.5, and 0.75 using the overlapping grids method.

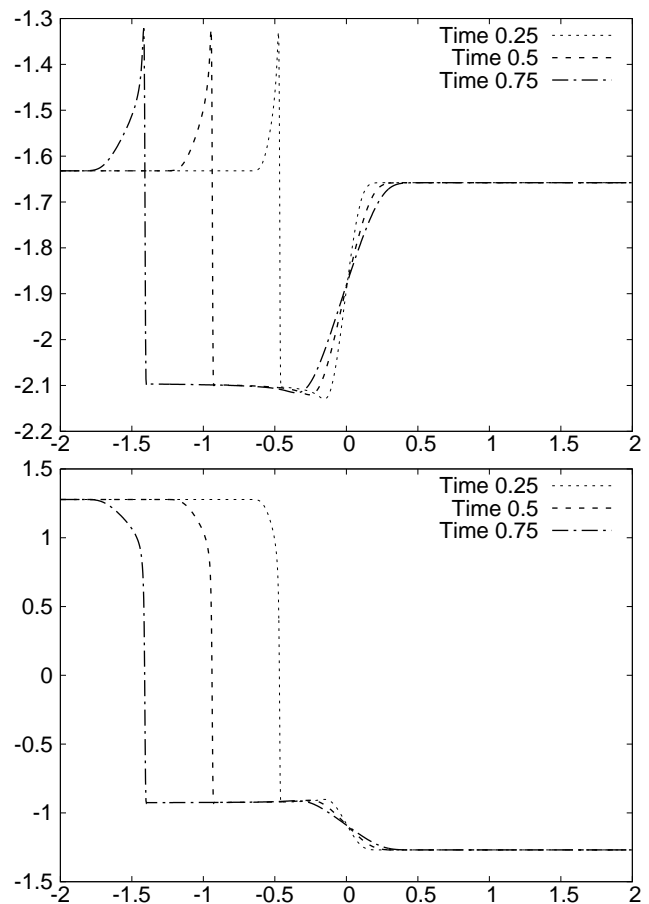


Fig. 10. Example III.4: the solution  $u$  and  $v$  at times 0.25, 0.5, and 0.75 using the finite volume method.

For the overlapping grids method, the bottom grid is taken to be on the interval

$$[-2, 1.0101010101],$$

while the top grid is taken to be on the interval

$$[-0.59999, 2],$$

so that the overlap is not trivial. Both the bottom and the top grid contain 100 cells each. The CFL constant is taken to be 0.8 relative to  $\min\{\Delta x_B, \Delta x_T\}$ . For the comparison, we use the regular finite volume method consisting of 150 cells.

#### IV. CONCLUSION

The overlapping grids are important when the topology of the object considered is very complex and it is difficult, or impossible, to create a grid around it. In [10] we define an overlapping grids method and we prove several theoretical results including that, in the scalar case ( $m = 1$ ), if we have bounded convergence of approximate solutions, then the approximate solutions converge to the weak solution which is also the entropy solution regardless of the cell sizes in overlapping grids. In this paper we numerically test and confirm the effectiveness of this overlapping grids method by computing approximate solutions to Riemann problems for several systems of conservation laws ( $m > 1$ ).

Besides computing the approximate solutions to problems whose solutions possess the classical structure containing

rarefaction and shock waves, we also consider problems resulting in a much more complicated structure and which contain transitional shock waves.

We compare our numerical results to those obtained using the regular finite volume method with the Lax–Friedrich numerical flux. We note that the approximate solutions for the overlapping grids method in all of the above examples have graphs with thicker curves because the approximate solution is composed of solutions on both the bottom and the top grids.

In all of the above examples, the numerical solutions obtained using the overlapping grids method do not differ from the numerical solutions obtained using the regular finite volume method. Even though, the theoretical results regarding convergence of the approximate solutions of this overlapping grids method for systems of conservation laws are still not known, the numerical results of this paper show that it could be used effectively also for one-dimensional systems.

#### REFERENCES

- [1] A. Azevedo, D. Marchesin, B. J. Plohr, and K. Zumbrun, "Nonuniqueness of solutions of Riemann problems", *Zeit. angew. Math. Phys.*, vol. 47, pp. 977-998, 1996.
- [2] M. J. Berger, C. Helzel, and R. J. LeVeque, "h-box Methods for the Approximation of Hyperbolic Conservation Laws on Irregular Grids", *SIAM J. Numer. Anal.*, vol. 41, no. 3, pp. 893-918, 2003.
- [3] A. Bressan, "Lecture Notes on Hyperbolic Conservation Laws", 2009. <http://php.math.unifi.it/users/cimeCourses/2009/01/200912-Notes.pdf>

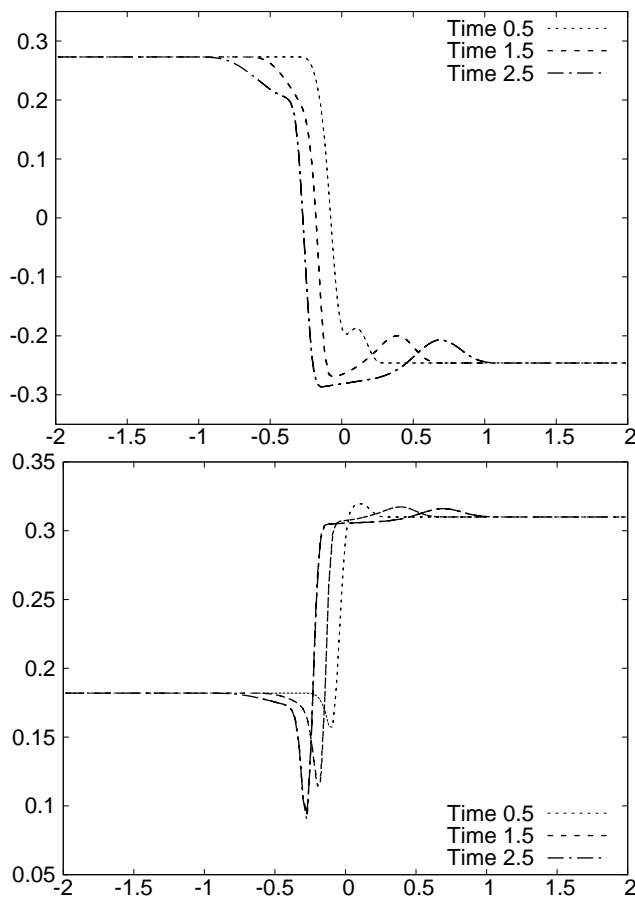


Fig. 11. Example III.5: the solution  $u$  and  $v$  at times 0.5, 1.5, and 2.5 using the overlapping grids method.

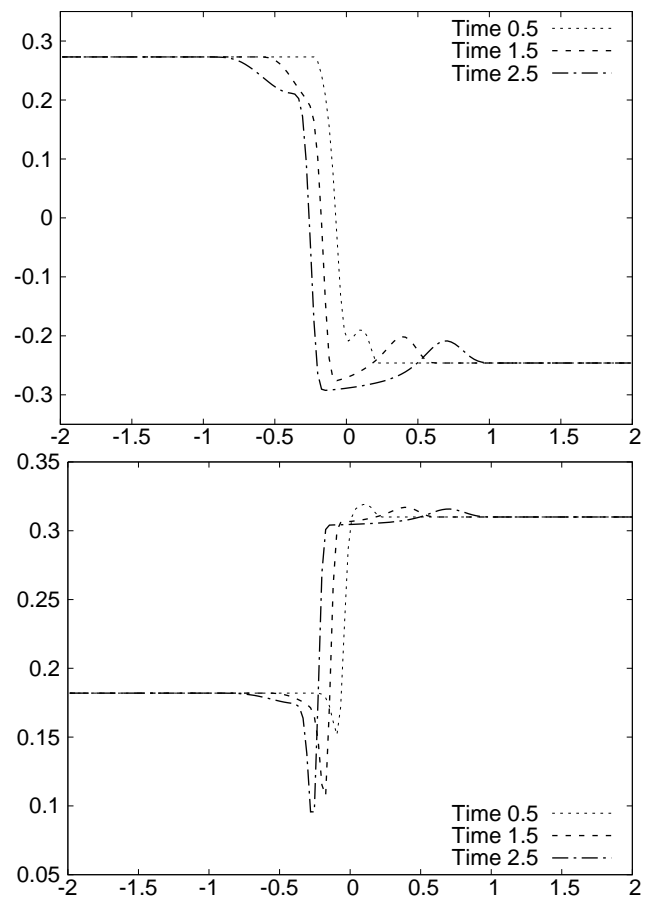


Fig. 12. Example III.5: the solution  $u$  and  $v$  at times 0.5, 1.5, and 2.5 using the regular finite volume method.

- [4] C. Defermos, "Hyperbolic Conservation laws in Continuum Physics", Springer-Verlag, 2016.
- [5] B. Engquist and S. Osher, "One sided difference approximation for nonlinear conservation laws", *Math. of Comp.*, vol. 36, pp. 321-351, 1981.
- [6] E. Godlewski, P. A. Raviart, "Hyperbolic Systems of Conservation Laws", *Mathematiques & Applications*, 1991.
- [7] S. K. Godunov, "A Finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics", *Math. Sb.*, vol. 47, pp. 271-290, 1959.
- [8] H. Holden and N. H. Risebro, "Front Tracking for Hyperbolic Systems of Conservation Laws", Springer-Verlag, New York, 2002.
- [9] H.-C. Hwang, "A front tracking method for transitional shock waves", *Mat. Contemp.*, vol. 10, pp. 149-167, 1996.
- [10] I. Jegdić, "Large time step and overlapping grids for conservation laws", *Ph.D. Thesis*, University of Houston, 2014.
- [11] I. Jegdić, "Numerical study of singular and delta shock solutions using a large time step method", *Applications and Applied Mathematics: An International Journal*, vol. 13, issue 2, pp. 1110-1122, 2018.
- [12] S. N. Kružkov, "First order quasilinear equations in several independent variables", *Math. USSR Sb.*, vol. 10, pp. 217-243, 1970.
- [13] P. D. Lax, "Hyperbolic systems of conservation laws and the mathematical theory of shock waves", *CBMS Regional Conference Series in Mathematics* No. 11. Philadelphia: SIAM, 1973.
- [14] P. Lax and B. Wendroff, "Systems of conservation laws", *Comm. Pure and Appl. Math.*, vol. 13, pp. 217-237, 1960.
- [15] R. LeVeque, "Numerical Methods for Conservation Laws", Birkhauser, 1990.
- [16] P. L. Roe, "Approximate Riemann solvers, parameter vectors and difference schemes", *Journal of Computational Physics*, vol. 43, pp. 357-372, 1981.
- [17] R. Sanders, "An Introduction to the Finite Volume Method for the Equations of Gas Dynamics", *Boletín de la Sociedad Española de Matemática Aplicada*, vol. 38, pp. 39-68, 2007.
- [18] S. Schester, B. J. Plohr, and D. Marchesin, "Computation on Riemann solutions using the Dafermos regularization and continuation", *Discrete and Continuous Dynamical Systems*, vol. 10, pp. 965-986, 2004.
- [19] J. Smoller, "Shock waves and reaction-diffusion equations", Springer Science + Business Media, LLC, 1994.