Multiplication and Translation Formulas for the Generalized Hypergeometric Functions and Their Applications

Zhongfeng Sun and Huizeng Qin

***Abstract*—In his paper, the following multiplication and translation formulae of the generalized hypergeometric functions**

$$\begin{aligned} & {}_{r+1}F_s\left(\begin{array}{c} -n, a_R \\ b_S \end{array} \middle| MT(\theta,x)\right) \\ = & \sum_{k=0}^{n} C_k^{r,s}(n,\theta)\ {}_{r+1}F_s\left(\begin{array}{c} -k, a_R \\ b_S \end{array} \middle| x\right), \end{aligned}$$

and

$$\begin{aligned} & {}_{r+2}F_s\left(\begin{array}{c} -n, MT(\theta,x), a_R \\ b_S \end{array} \middle| y\right) \\ = & \sum_{k=0}^{n} D_k^{r,s}(n,\theta,y)\ {}_{r+2}F_s\left(\begin{array}{c} -k, x, a_R \\ b_S \end{array} \middle| y\right), \end{aligned}$$

are discussed for positive integers n, r, s, where $MT(\theta,x)=\theta x$ or $\theta + x$ and the notation a_R, b_S are r,s-dimensional vector, respectively. Similarly, the multiplication and translation formulas for the basic hypergeometric series ${}_r\phi_s$ can also be derived. Based on these results, the multiplication and translation formulas for the classical continuous, discrete orthogonal polynomials and the q-classical orthogonal polynomials which are associated with ${}_rF_s$ or ${}_r\phi_s$ can be established directly.

***Index Terms*—Generalized Hypergeometric Function, Basic Hypergeometric Series, Classical Orthogonal Polynomials, Inversion Formula.**

I. Introduction

RECENTLY, the generalized hypergeometric functions have found significant applications in various fields such as Klein-Gordon equation [1], exact renormalization group equations [2], the fractional integral and derivative operators [3], Hilbert transform [4], the urban wireless channel [5], the fractional Jacobi functions [6] and so on. Especially, the author [6] established a new formula between Jacobi polynomials and certain fractional Jacobi functions with the aid of the generalized hypergeometric functions with the nonnegative integer n.

Many orthogonal polynomials can be represented by the generalized hypergeometric functions or the basic hypergeometric series [7], [8], [9], [10]. In [7], the authors consider the multiplication problem

$$p_n(ax) = \sum_{m=0}^{n} D_m(n,a)p_m(x), \tag{1}$$

and the translation problem

$$p_n(a+x) = \sum_{m=0}^{n} D_m(n,a)p_m(x), \tag{2}$$

for the orthogonal polynomials $p_n(x)$, which are widely used in combinatorial analysis. To solve the multiplication problem (1) for some classical continuous and discrete orthogonal polynomials effectively, the generating functions are used in [11] .

We find that the inversion formula of the polynomial can be applied to solve the multiplication problem (1) and the translation problem (2) directly. For example, with the yields of the orthogonal polynomial $p_n(x)$

$$p_n(x) = \sum_{m=0}^{n} A_m(b)(b+x)^m, \tag{3}$$

and it's inversion formula

$$(c+x)^n = \sum_{m=0}^{n} B_m(c)p_m(x), \tag{4}$$

we can derive the following multiplication and translation formulae directly,

$$\begin{aligned} p_n(ax) &= \sum_{m=0}^{n} A_m(b)a^m \cdot \sum_{l=0}^{m} B_l\left(\frac{b}{a}\right)p_l(x), \\ p_n(a+x) &= \sum_{m=0}^{n} A_m(b) \cdot \sum_{l=0}^{m} B_l(a+b)p_l(x). \end{aligned} \tag{5}$$

Therefore, if we can determine the nature of coefficients of multiplication and translation formulas in the function transformation, it is easily to construct the multiplication and translation formulas for some orthogonal polynomials.

In this paper, we aim at constructing the multiplication and translation formulas for the generalized hypergeometric functions by using their inversion formula, and apply it to solve the multiplication problem and the translation problem of some classical continuous, discrete orthogonal polynomials and the q-classical orthogonal polynomials, which are associated with the generalized hypergeometric functions.

For convenience, we introduce the Pochhammer symbol $(x)_n$ and the q-shifted factorial $(x;q)_n$ [12]:

$$(x)_n = \prod_{j=0}^{n-1}(x+j), \quad (x;q)_n = \prod_{j=0}^{n-1}(1-q^j x), \quad n \in \mathbb{N}, \tag{6}$$

with $\mathbb{N} = \{1,2,\ldots\}$, $(x)_0 = (x;q)_0 := 1$, and their products

$$(a_R)_n := \prod_{k=1}^{r}(a_k)_n, \quad (a_R;q)_n := \prod_{k=1}^{r}(a_k;q)_n, \tag{7}$$

Manuscript received January 10, 2019; revised August 5, 2019. This work is supported by National Natural Science Foundation of China under Grant No. 61771010 and No. 61379009.

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where $r,n \in \mathbb{N}$, $0 < |q| < 1$ and the notation $a_R := (a_1, a_2, \ldots, a_r)$ is used.

The hypergeometric functions ${}_rF_s$ and the basic hypergeometric series ${}_r\phi_s$ [12], [13] are defined by

$$ {}_rF_s\left(\begin{array}{c} a_R \\ b_S \end{array}\middle| x\right) = \sum_{n=0}^{\infty} \frac{(a_R)_n}{(b_S)_n n!} x^n, \tag{8}$$

and

$$ \begin{aligned} {}_r\phi_s\left(\begin{array}{c} a_R \\ b_S \end{array}\middle| q; x\right) &= \sum_{n=0}^{\infty} \frac{(a_R;q)_n}{(b_S;q)_n (q;q)_n} \\ &\times \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} x^n, \end{aligned} \tag{9}$$

where $r, s \in \mathbb{N}$.

In particular, ${}_rF_s$ and ${}_r\phi_s$ can be reduced to the following finite sums,

$$ \begin{aligned} {}_{r+1}F_s\left(\begin{array}{c} -n, a_R \\ b_S \end{array}\middle| x\right) &= \sum_{l=0}^{n} \frac{(-n)_l (a_R)_l}{l! (b_S)_l} x^l \\ &= \sum_{l=0}^{n} (-1)^l C_n^l \frac{(a_R)_l}{(b_S)_l} x^l, \end{aligned} \tag{10}$$

and

$$ \begin{aligned} &{}_{r+1}\phi_s\left(\begin{array}{c} q^{-n}, a_R \\ b_S \end{array}\middle| q; x\right) \\ =& \sum_{j=0}^{n} \frac{(q^{-n};q)_j}{(b_S;q)_j} \frac{(a_R;q)_j}{(q;q)_j} \\ &\times \left((-1)^j q^{j(j-1)/2}\right)^{s-r} x^j, \end{aligned} \tag{11}$$

where $n \in \mathbb{N}$. According to Eqs. (3.3) and (3.4) in [12], the inversion formulas associated with (10) and (11) can be expressed as follows,

$$ x^n = \frac{(b_S)_n}{(a_R)_n} \sum_{k=0}^{n} C_n^k (-1)^k \; {}_{r+1}F_s\left(\begin{array}{c} -k, a_R \\ b_S \end{array}\middle| x\right), \tag{12}$$

and

$$ \begin{aligned} x^n &= \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{r-s} q^n \frac{(b_S;q)_n}{(a_R;q)_n} \sum_{k=0}^{n} (-1)^k \\ &\times \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{\frac{k(k-1)}{2}} \; {}_{r+1}\phi_s\left(\begin{array}{c} q^{-k}, a_R \\ b_S \end{array}\middle| q; x\right), \end{aligned} \tag{13}$$

where

$$ \left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{14}$$

Replacing x, a_R, r by $y, (x, a_R), r+1$ in (12) and (13), respectively, we have

$$ \begin{aligned} (x)_n = & \frac{(b_S)_n}{y^n (a_R)_n} \sum_{k=0}^{n} C_n^k (-1)^k \\ & \times {}_{r+2}F_s\left(\begin{array}{c} -k, x, a_R \\ b_S \end{array}\middle| y\right), \end{aligned} \tag{15}$$

and

$$ \begin{aligned} &(x;q)_n \\ =& \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{r+1-s} \frac{q^n (b_S;q)_n}{y^n (a_R;q)_n} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \\ &\times (-1)^k q^{\frac{k(k-1)}{2}} \; {}_{r+2}\phi_s\left(\begin{array}{c} q^{-k}, x, a_R \\ b_S \end{array}\middle| q; y\right). \end{aligned} \tag{16}$$

Basic relations between the Pochhammer symbol $(x)_n$ and x^n can be expressed as follows ([12] pp.157),

$$ \begin{aligned} (x)_n &= \sum_{m=0}^{n} (-1)^{n-m} S_1(n,m) x^m, \\ x^n &= \sum_{m=0}^{n} (-1)^{n-m} S_2(n,m) (x)_m, \end{aligned} \tag{17}$$

where $S_1(n,m), S_2(n,m)$ are the Stirling numbers of first and second kind, respectively.

Similarly, basic relations between the q-shifted factorial $(x;q)_n$ and x^n are given by

$$ (x;q)_n = \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array}\right]_q (-1)^m q^{m(m-1)/2} x^m, \tag{18}$$

and

$$ x^n = \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array}\right]_q (-1)^m q^{m(m+1-2n)/2} (x;q)_m, \tag{19}$$

which can be found in [12] (pp.157).

The structure of this paper is as follows. In Section II, the inversion formulas are used to establish the multiplication and translation formula of the generalized hypergeometric functions. In Section III, those results in Section II are applied to derive the multiplication and translation formulas for some classical continuous or discrete orthogonal polynomials and the q-classical orthogonal polynomials directly, which are associated with ${}_rF_s$ or ${}_r\phi_s$. A final conclusion is given in Section IV.

II. Multiplication and Translation Formulas of the Hypergeometric Functions and the Basic Hypergeometric Series

Theorem 2.1 The following multiplication and translation formulas of the hypergeometric functions are valid for $n \in \mathbb{N}$ and $\theta, x \in \mathbb{R}\backslash\{0\}$,

$$ \begin{aligned} &{}_{r+1}F_s\left(\begin{array}{c} -n, a_R \\ b_S \end{array}\middle| \theta x\right) \\ =& \sum_{k=0}^{n} C_n^k \theta^k (1-\theta)^{n-k} \; {}_{r+1}F_s\left(\begin{array}{c} -k, a_R \\ b_S \end{array}\middle| x\right), \end{aligned} \tag{20}$$

and

$$ \begin{aligned} &{}_{r+1}F_s\left(\begin{array}{c} -n, a_R \\ b_S \end{array}\middle| \theta + x\right) \\ =& \sum_{k=0}^{n} C_n^k (-1)^k \; {}_{r+1}F_s\left(\begin{array}{c} -k, a_R \\ b_S \end{array}\middle| x\right) \\ &\times \sum_{m=k}^{n} C_{n-k}^{n-m} (-1)^m \\ &\times {}_{r+1}F_s\left(\begin{array}{c} m-n, a_R + mI_R \\ b_S + mI_S \end{array}\middle| \theta\right), \end{aligned} \tag{21}$$

where $I_R := (1, 1, \ldots, 1)$ is r-dimensional vector.

Proof. 1) From (10) and (12), we have

$$\begin{aligned} & {}_{r+1}F_s\left(\begin{array}{c}-n, a_R\\ b_S\end{array}\middle|\,\theta x\right) \\ = & \sum_{l=0}^{n} C_n^l(-\theta)^l \\ & \times\sum_{k=0}^{l} C_l^k(-1)^k\ {}_{r+1}F_s\left(\begin{array}{c}-k, a_R\\ b_S\end{array}\middle|\,x\right) \\ = & \sum_{k=0}^{n} C_n^k\theta^k\ {}_{r+1}F_s\left(\begin{array}{c}-k, a_R\\ b_S\end{array}\middle|\,x\right) \\ & \times\sum_{l=k}^{n} C_{n-k}^{l-k}(-\theta)^{l-k}, \end{aligned} \tag{22}$$

which implies that (20) holds.

2) With the help of (10), we yield

$$\begin{aligned} & {}_{r+1}F_s\left(\begin{array}{c}-n, a_R\\ b_S\end{array}\middle|\,\theta+x\right) \\ = & \sum_{j=0}^{n}(-1)^j C_n^j\frac{(a_R)_j}{(b_S)_j}\sum_{m=0}^{j} C_j^m\theta^{j-m}x^m \\ = & \sum_{m=0}^{n} C_n^m(-1)^m x^m \\ & \times\sum_{j=m}^{n}(-1)^{j-m}C_{n-m}^{j-m}\frac{(a_R)_j}{(b_S)_j}\theta^{j-m}. \end{aligned} \tag{23}$$

By substituting $j=l+m$, (23) becomes

$$\begin{aligned} & {}_{r+1}F_s\left(\begin{array}{c}-n, a_R\\ b_S\end{array}\middle|\,\theta+x\right) \\ = & \sum_{m=0}^{n} C_n^m(-1)^m x^m \\ & \times\sum_{l=0}^{n-m} C_{n-m}^l(-1)^l\frac{(a_R)_{l+m}}{(b_S)_{l+m}}\theta^l. \end{aligned} \tag{24}$$

Combining (12) with (24), we obtain

$$\begin{aligned} & {}_{r+1}F_s\left(\begin{array}{c}-n, a_R\\ b_S\end{array}\middle|\,\theta+x\right) \\ = & \sum_{m=0}^{n} C_n^m(-1)^m\frac{(b_S)_m}{(a_R)_m} \\ & \times\sum_{k=0}^{m} C_m^k(-1)^k\ {}_{r+1}F_s\left(\begin{array}{c}-k, a_R\\ b_S\end{array}\middle|\,x\right) \\ & \times\sum_{l=0}^{n-m} C_{n-m}^l(-1)^l\frac{(a_R)_{l+m}}{(b_S)_{l+m}}\theta^l \\ = & \sum_{k=0}^{n} C_n^k(-1)^k\ {}_{r+1}F_s\left(\begin{array}{c}-k, a_R\\ b_S\end{array}\middle|\,x\right) \\ & \times\sum_{m=k}^{n} C_{n-k}^{m-k}(-1)^m \\ & \times\sum_{l=0}^{n-m} C_{n-m}^l(-1)^l\frac{(a_R+mI_R)_l}{(b_S+mI_S)_l}\theta^l. \end{aligned} \tag{25}$$

It follows by (10) that (21) holds. ■

For convenience sake, we introduce the following notation

$$\begin{aligned} & {}^{m}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;x\right) \\ := & \sum_{l=0}^{n-m} C_{m+l}^l\left((-1)^l q^{\frac{l(l-1)}{2}}\right)^{s-r}\left(xq^{m(s-r)}\right)^l \\ & \times\frac{(q^{m-n};q)_l(q^m a_R;q)_l}{(q^{m+1};q)_l(q^m b_S;q)_l}, \end{aligned} \tag{26}$$

where $m\in\mathbb{N}\cup\{0\}$. Specially, there is

$$\begin{aligned} & {}^{0}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;x\right) \\ = & \ {}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;x\right). \end{aligned} \tag{27}$$

Theorem 2.2 For $n\in\mathbb{N}$, $\theta, x\in\mathbb{R}\backslash\{0\}$ and $0<|q|<1$, the multiplication and translation formulas of ${}_r\phi_s$ can be expressed as follows,

$$\begin{aligned} & {}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;\theta x\right) \\ = & (-\theta)^n q^{\frac{n(1-n)}{2}}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(\theta^{-1};q)_{n-k}}{(q;q)_k} \\ & \times q^{kn}\cdot\ {}_{r+1}\phi_s\left(\begin{array}{c}q^{-k}, a_R\\ b_S\end{array}\middle|\,q;x\right), \end{aligned} \tag{28}$$

and

$$\begin{aligned} & {}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;\theta+x\right) \\ = & \sum_{k=0}^{n}(-1)^k q^{\frac{k(k+1)}{2}}\frac{(q^{-n};q)_k}{(q;q)_k} \\ & \times\ {}_{r+1}\phi_s\left(\begin{array}{c}q^{-k}, a_R\\ b_S\end{array}\middle|\,q;x\right) \\ & \times\sum_{l=0}^{n-k} q^l\frac{(q^{k-n};q)_l}{(q;q)_l}\ {}^{l+k}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;\theta\right). \end{aligned} \tag{29}$$

Proof. 1) From (11) and (13), it is easy to see that

$$\begin{aligned} & {}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;\theta x\right) \\ = & \sum_{j=0}^{n}\frac{(q^{-n};q)_j}{(q;q)_j}(q\theta)^j\sum_{k=0}^{j}\left[\begin{array}{c}j\\ k\end{array}\right]_q(-1)^k q^{\frac{k(k-1)}{2}} \\ & \times\ {}_{r+1}\phi_s\left(\begin{array}{c}q^{-k}, a_R\\ b_S\end{array}\middle|\,q;x\right) \\ = & \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\ {}_{r+1}\phi_s\left(\begin{array}{c}q^{-k}, a_R\\ b_S\end{array}\middle|\,q;x\right) \\ & \times(-1)^k q^{\frac{k(k-1)}{2}}\sum_{j=k}^{n}\frac{(q^{k-n};q)_{j-k}}{(q;q)_{j-k}}(q\theta)^j. \end{aligned} \tag{30}$$

It follows by variable substitution $t=j-k$ that

$$\begin{aligned} & {}_{r+1}\phi_s\left(\begin{array}{c}q^{-n}, a_R\\ b_S\end{array}\middle|\,q;\theta x\right) \\ = & \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\ {}_{r+1}\phi_s\left(\begin{array}{c}q^{-k}, a_R\\ b_S\end{array}\middle|\,q;x\right) \\ & \times(-1)^k q^{\frac{k(k-1)}{2}}(\theta q)^k\sum_{t=0}^{n-k}\frac{(q^{k-n};q)_t}{(q;q)_t}(\theta q)^t. \end{aligned} \tag{31}$$

For $m \geq n$, we have

$$(q^{-m};q)_n = (-1)^n \prod_{j=0}^{n-1} q^{j-m} \prod_{j=0}^{n-1}(1-q^{m-j}) = (-1)^n q^{-mn+n(n-1)/2} \frac{(q;q)_m}{(q;q)_{m-n}}, \tag{32}$$

which gives that

$$\sum_{t=0}^{n-k} \frac{(q^{k-n};q)_t}{(q;q)_t}(\theta q)^t = \sum_{t=0}^{n-k} \begin{bmatrix} n-k \\ t \end{bmatrix}_q q^{-(n-k)t} q^{t(t+1)/2}(-\theta)^t. \tag{33}$$

It follows by variable substitution $t = n-k-m$ that

$$\sum_{t=0}^{n-k} \frac{(q^{k-n};q)_t}{(q;q)_t}(\theta q)^t = \sum_{m=0}^{n-k} \begin{bmatrix} n-k \\ m \end{bmatrix}_q (-\theta)^{n-k-m} \times q^{-(n-k-m)(n-k+m-1)/2}. \tag{34}$$

Then, from (18) and (34), we derive

$$\sum_{t=0}^{n-k} \frac{(q^{k-n};q)_t}{(q;q)_t}(\theta q)^t = (-1)^{n-k}\theta^{n-k} q^{(n-k)(1-n+k)/2}(\theta^{-1};q)_{n-k}. \tag{35}$$

Inserting (35) into (31), we arrive at the required result (28).

2) In view of (11) and (13), we find

$$\begin{aligned} & {}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_R \\ b_S \end{matrix} \middle| q; \theta + x \right) \\ = & \sum_{j=0}^{n} \frac{(q^{-n};q)_j (a_R;q)_j}{(b_S;q)_j (q;q)_j} \left((-1)^j q^{j(j-1)/2}\right)^{s-r} \times \sum_{m=0}^{j} C_j^m \theta^m x^{j-m} \\ = & \sum_{j=0}^{n}\sum_{m=0}^{j} C_j^m \theta^m \frac{(q^{-n};q)_j (q^{j-m}a_R;q)_m}{(q;q)_j (q^{j-m}b_S;q)_m} q^{j-m} \times \left((-1)^m q^{mj-m(m+1)/2}\right)^{s-r} \\ & \times \left((-1)^{j-m} q^{(j-m)(j-m-1)/2}\right)^{s-r} \times (q^{-1}x)^{j-m} \frac{(a_R;q)_{j-m}}{(b_S;q)_{j-m}} \\ = & \sum_{j=0}^{n}\sum_{m=0}^{j} C_j^m \theta^m \frac{(q^{-n};q)_j (q^{j-m}a_R;q)_m}{(q;q)_j (q^{j-m}b_S;q)_m} q^{j-m} \times \left((-1)^m q^{mj-m(m+1)/2}\right)^{s-r} \\ & \times \sum_{k=0}^{j-m} \begin{bmatrix} j-m \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} \times {}_{r+1}\phi_s\left(\begin{matrix} q^{-k}, a_R \\ b_S \end{matrix} \middle| q; x \right) \\ = & \sum_{k=0}^{n} (-1)^k q^{\frac{k(k+1)}{2}} \frac{(q^{-n};q)_k}{(q;q)_k} \times {}_{r+1}\phi_s\left(\begin{matrix} q^{-k}, a_R \\ b_S \end{matrix} \middle| q; x \right) \times G(n,k), \end{aligned} \tag{36}$$

where

$$\begin{aligned} & G(n,k) \\ = & \sum_{j=k}^{n}\sum_{m=0}^{j-k} C_j^m \theta^m \frac{(q^{k-n};q)_{j-k}(q^{j-m}a_R;q)_m}{(q^{j-m+1};q)_m (q;q)_{j-m-k}} \times \frac{q^{j-m-k}}{(q^{j-m}b_S;q)_m} \left((-1)^m q^{mj-m(m+1)/2}\right)^{s-r}. \end{aligned} \tag{37}$$

Exchanging the order of summation for (37) and replacing j by $l+m+k$, we obtain

$$\begin{aligned} & G(n,k) \\ = & \sum_{m=0}^{n-k}\sum_{l=0}^{n-k-m} C_{k+m+l}^m \theta^m \times \frac{(q^{k-n};q)_{m+l}(q^{k+l}a_R;q)_m}{(q^{k+l+1};q)_m (q;q)_l (q^{k+l}b_S;q)_m} \\ & \times q^l \left((-1)^m q^{m(k+m+l)-m(m+1)/2}\right)^{s-r}, \end{aligned} \tag{38}$$

which means that

$$\begin{aligned} & G(n,k) \\ = & \sum_{l=0}^{n-k} q^l \frac{(q^{k-n};q)_l}{(q;q)_l} \sum_{m=0}^{n-k-l} C_{k+m+l}^m \theta^m \times \frac{(q^{k+l-n};q)_m (q^{k+l}a_R;q)_m}{(q^{k+l+1};q)_m (q^{k+l}b_S;q)_m} \\ & \times \left((-1)^m q^{m(k+l)+m(m-1)/2}\right)^{s-r}. \end{aligned} \tag{39}$$

Combining (36), (39) with (26), we yield (29). Hence, the proof is complete. ■

Similarly, we can establish the multiplication and translation formulas for

$${}_{r+2}F_s\left(\begin{matrix} -n, MT(\theta,x), a_R \\ b_S \end{matrix} \middle| y \right),$$

$${}_{r+2}\phi_s\left(\begin{matrix} q^{-n}, MT(\theta,x), a_R \\ b_S \end{matrix} \middle| q; y \right),$$

where $MT(\theta,x) = \theta x$ or $\theta + x$.

Theorem 2.3 Let $n \in \mathbb{N}$ and $\theta, x, y \in \mathbb{R}\backslash\{0\}$. Then

$$\begin{aligned} & {}_{r+2}F_s\left(\begin{matrix} -n, \theta + x, a_R \\ b_S \end{matrix} \middle| y \right) \\ = & \sum_{k=0}^{n} C_n^k \; {}_{r+2}F_s\left(\begin{matrix} -k, x, a_R \\ b_S \end{matrix} \middle| y \right) \sum_{m=k}^{n} (-1)^{m-k} \times C_{n-k}^{m-k} \; {}_{r+2}F_s\left(\begin{matrix} m-n, \theta, a_R + mI_R \\ b_S + mI_S \end{matrix} \middle| y \right), \end{aligned} \tag{40}$$

and

$$\begin{aligned} & {}_{r+2}F_s\left(\begin{matrix} -n, \theta x, a_R \\ b_S \end{matrix} \middle| y \right) \\ = & \sum_{k=0}^{n} (-1)^k \; {}_{r+2}F_s\left(\begin{matrix} -k, x, a_R \\ b_S \end{matrix} \middle| y \right) \times \sum_{m=k}^{n} (-1)^m C_m^k \; {}^m_{r+1}F_s\left(\begin{matrix} -n, a_R \\ b_S \end{matrix} \middle| \theta; y \right), \end{aligned} \tag{41}$$

where

$$
\begin{aligned}
& {}^{m}_{r+1}F_s\left(\begin{array}{c}-n, a_R\\ b_S\end{array}\middle|\,\theta; y\right)\\
&= \sum_{l=m}^{n} C_n^l y^{l-m}\frac{(a_R+mI_R)_{l-m}}{(b_S+mI_R)_{l-m}}\\
&\quad\times\sum_{j=m}^{l} S_1(l,j)S_2(j,m)\theta^j.
\end{aligned}
\tag{42}
$$

Proof. 1) With the aid of the ChuVandermonde identity ([7], pp.509)

$$
(a+x)_n = \sum_{m=0}^{n} C_n^m (a)_{n-m}(x)_m, \tag{43}
$$

(10) becomes

$$
\begin{aligned}
& {}_{r+2}F_s\left(\begin{array}{c}-n, \theta+x, a_R\\ b_S\end{array}\middle|\, y\right)\\
&= \sum_{l=0}^{n}(-1)^l C_n^l\frac{(a_R)_l}{(b_S)_l}y^l\sum_{m=0}^{l} C_l^m(\theta)_{l-m}(x)_m\\
&= \sum_{m=0}^{n} C_n^m(-1)^m\frac{(a_R)_m}{(b_S)_m}(x)_m y^m\sum_{l=m}^{n}(-1)^{l-m}\\
&\quad\times C_{n-m}^{l-m}\frac{(a_R+mI_R)_{l-m}}{(b_S+mI_S)_{l-m}}(\theta)_{l-m}y^{l-m}.
\end{aligned}
\tag{44}
$$

It follows by variable substitution $u = l - m$ and (15) that

$$
\begin{aligned}
& {}_{r+2}F_s\left(\begin{array}{c}-n, \theta+x, a_R\\ b_S\end{array}\middle|\, y\right)\\
&= \sum_{m=0}^{n} C_n^m(-1)^m\frac{(a_R)_m}{(b_S)_m}(x)_m y^m\\
&\quad\times\sum_{u=0}^{n-m}(-1)^u C_{n-m}^u\frac{(a_R+mI_R)_u}{(b_S+mI_S)_u}(\theta)_u y^u\\
&= \sum_{m=0}^{n} C_n^m(-1)^m\sum_{k=0}^{m}\ {}_{r+2}F_s\left(\begin{array}{c}-k, x, a_R\\ b_S\end{array}\middle|\, y\right)\\
&\quad\times C_m^k(-1)^k\sum_{u=0}^{n-m}(-1)^u C_{n-m}^u\sum_{j=0}^{u} C_u^j(-1)^j\\
&\quad\times\ {}_{r+2}F_s\left(\begin{array}{c}-j, \theta, a_R+mI_R\\ b_S+mI_S\end{array}\middle|\, y\right)\\
&= \sum_{k=0}^{n} C_n^k\ {}_{r+2}F_s\left(\begin{array}{c}-k, x, a_R\\ b_S\end{array}\middle|\, y\right)\\
&\quad\times\sum_{m=k}^{n} C_{n-k}^{m-k}(-1)^{m-k}\sum_{j=0}^{n-m} C_{n-m}^j\\
&\quad\times\ {}_{r+2}F_s\left(\begin{array}{c}-j, \theta, a_R+mI_R\\ b_S+mI_S\end{array}\middle|\, y\right)\\
&\quad\times\sum_{u=j}^{n-m} C_{n-m-j}^{u-j}(-1)^{u-j}.
\end{aligned}
\tag{45}
$$

Next by applying the following identity

$$
\sum_{u=j}^{n-m} C_{n-m-j}^{u-j}(-1)^{u-j} = \left\{\begin{array}{ll} 1, & j = n-m,\\ 0, & j \neq n-m,\end{array}\right. \tag{46}
$$

(45) reduces to (40).

2) Making use of (17), we get

$$
\begin{aligned}
& (\theta x)_l\\
&= \sum_{j=0}^{l}(-1)^{l-j}S_1(l,j)\theta^j\\
&\quad\times\sum_{m=0}^{j}(-1)^{j-m}S_2(j,m)(x)_m\\
&= \sum_{m=0}^{l}(-1)^{l-m}(x)_m\sum_{j=m}^{l} S_1(l,j)S_2(j,m)\theta^j.
\end{aligned}
\tag{47}
$$

Combining (10), (47) and (15), we have

$$
\begin{aligned}
& {}_{r+2}F_s\left(\begin{array}{c}-n, \theta x, a_R\\ b_S\end{array}\middle|\, y\right)\\
&= \sum_{l=0}^{n}(-1)^l C_n^l\frac{(a_R)_l}{(b_S)_l}y^l\sum_{m=0}^{l}(-1)^{l-m}(x)_m\\
&\quad\times\sum_{j=m}^{l} S_1(l,j)S_2(j,m)\theta^j\\
&= \sum_{m=0}^{n}(-1)^m\frac{(a_R)_m}{(b_S)_m}y^m(x)_m\sum_{l=m}^{n} C_n^l y^{l-m}\\
&\quad\times\frac{(a_R+mI_R)_{l-m}}{(b_S+mI_S)_{l-m}}\sum_{j=m}^{l} S_1(l,j)S_2(j,m)\theta^j\\
&= \sum_{m=0}^{n}(-1)^m\sum_{k=0}^{m}\ {}_{r+2}F_s\left(\begin{array}{c}-k, x, a_R\\ b_S\end{array}\middle|\, y\right)\\
&\quad\times C_m^k(-1)^k\cdot {}^{m}_{r+1}F_s\left(\begin{array}{c}-n, a_R\\ b_S\end{array}\middle|\,\theta; y\right),
\end{aligned}
\tag{48}
$$

which implies that (41) holds. ■

Lemma 2.4 The translation formula of the q-shifted factorial can be expressed as follows,

$$
(a+x;q)_n = \sum_{k=0}^{n}(-1)^k q^{\frac{k(k+1)}{2}} M_{n,k}(a;q)(x;q)_k, \tag{49}
$$

where $n \in \mathbb{N}$, $a, x \in \mathbb{R}$, $0 < |q| < 1$ and

$$
\begin{aligned}
M_{n,k}(a;q) = & \sum_{m=k}^{n}\left[\begin{array}{c} n\\ m\end{array}\right]_q(-1)^m q^{\frac{m(m-1)}{2}-mk}\\
& \times\sum_{l=0}^{m-k} C_m^l\left[\begin{array}{c} m-l\\ k\end{array}\right]_q(aq^k)^l.
\end{aligned}
\tag{50}
$$

Proof. Taking into account (18) and (19), we acquire

$$
\begin{aligned}
& (a+x;q)_n\\
&= \sum_{m=0}^{n}\left[\begin{array}{c} n\\ m\end{array}\right]_q(-1)^m q^{m(m-1)/2}\\
&\quad\times\sum_{j=0}^{m} C_m^j a^{m-j}x^j\\
&= \sum_{m=0}^{n}\left[\begin{array}{c} n\\ m\end{array}\right]_q(-1)^m q^{m(m-1)/2}\sum_{j=0}^{m} C_m^j a^{m-j}\\
&\quad\times\sum_{k=0}^{j}\left[\begin{array}{c} j\\ k\end{array}\right]_q(-1)^k q^{k(k+1-2j)/2}(x;q)_k.
\end{aligned}
\tag{51}
$$

This means that

$$\begin{aligned}
&(a+x;q)_n \\
=&\sum_{k=0}^{n}(-1)^k q^{k(k+1)/2}(x;q)_k \\
&\times\sum_{m=k}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q (-1)^m q^{m(m-1)/2-mk} \\
&\times\sum_{j=k}^{m}\left[\begin{array}{c} j \\ k \end{array}\right]_q C_m^j (aq^k)^{(m-j)}.
\end{aligned}\tag{52}$$

Replacing j by $m-l$ in (52), we obtain (49). ■

Theorem 2.5 Let $n\in\mathbb{N}$, $\theta, x, y\in\mathbb{R}\backslash\{0\}$ and $0<|q|<1$. Then

$$\begin{aligned}
&{}_{r+2}\phi_s\left(\begin{array}{c} q^{-n},\theta x, a_R \\ b_S \end{array}\middle| q;y\right) \\
=&\sum_{k=0}^{n}(-1)^k q^{\frac{k(k+1)}{2}}\theta^k\frac{(q^{-n};q)_k}{(q;q)_k} \\
&\times{}_{r+2}\phi_s\left(\begin{array}{c} q^{-k},x, a_R \\ b_S \end{array}\middle| q;y\right)\sum_{m=0}^{n-k}(\theta q)^m\frac{(q^{k-n};q)_m}{(q;q)_m} \\
&\times{}_{r+2}\phi_s\left(\begin{array}{c} q^{m+k-n},\theta, q^{k+m}a_R \\ q^{k+m}b_S \end{array}\middle| q;q^{(k+m)(s-r-1)}y\right),
\end{aligned}\tag{53}$$

and

$$\begin{aligned}
&{}_{r+2}\phi_s\left(\begin{array}{c} q^{-n},\theta+x, a_R \\ b_S \end{array}\middle| q;y\right) \\
=&\sum_{k=0}^{n}(-1)^k q^{\frac{k(k-1)}{2}}{}_{r+2}\phi_s\left(\begin{array}{c} q^{-k},x, a_R \\ b_S \end{array}\middle| q;y\right) \\
&\times\sum_{t=k}^{n}(-1)^t q^{t(t+3)/2}\frac{(q^{-n};q)_t}{(q;q)_t}\left[\begin{array}{c} t \\ k \end{array}\right]_q \\
&\times{}_{r+2}^{\;t}\widetilde{\phi}_s\left(\begin{array}{c} q^{-n}, a_R \\ b_S \end{array}\middle| q;\theta;y\right),
\end{aligned}\tag{54}$$

where

$$\begin{aligned}
&{}_{r+2}^{\;t}\widetilde{\phi}_s\left(\begin{array}{c} q^{-n}, a_R \\ b_S \end{array}\middle| q;\theta;y\right) \\
=&\sum_{l=0}^{n-t}M_{l+t,t}(\theta;q)\left((-1)^l q^{\frac{l(l-1)}{2}}\right)^{s-r-1} \\
&\times(q^{t(s-r-1)}y)^l\frac{(q^{t-n};q)_l(q^t a_R;q)_l}{(q^{t+1};q)_l(q^t b_S;q)_l}.
\end{aligned}\tag{55}$$

Proof. 1) We recall the multiplication formula of the q-shifted factorial $(x;q)_n$

$$(ax;q)_n=\sum_{m=0}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q a^m(a;q)_{n-m}(x;q)_m,\tag{56}$$

which is Eq. (7) in [7]. Using (11) and (56), we find

$$\begin{aligned}
&{}_{r+2}\phi_s\left(\begin{array}{c} q^{-n},\theta x, a_R \\ b_S \end{array}\middle| q;y\right) \\
=&\sum_{j=0}^{n}\frac{(q^{-n};q)_j(a_R;q)_j}{(b_S;q)_j(q;q)_j}\left((-1)^j q^{j(j-1)/2}\right)^{s-r-1} \\
&\times y^j\sum_{t=0}^{j}\left[\begin{array}{c} j \\ t \end{array}\right]_q\theta^t(\theta;q)_{j-t}(x;q)_t \\
=&\sum_{t=0}^{n}(q\theta)^t\frac{(q^{-n};q)_t}{(q;q)_t}\cdot(q^{-1}y)^t \\
&\times\left((-1)^t q^{t(t-1)/2}\right)^{s-r-1}\frac{(a_R;q)_t}{(b_S;q)_t}(x;q)_t \\
&\times\sum_{j=t}^{n}\frac{(q^{t-n};q)_{j-t}(q^t a_R;q)_{j-t}}{(q^t b_S;q)_{j-t}(q;q)_{j-t}}y^{j-t}(\theta;q)_{j-t} \\
&\times\left((-1)^{j-t}q^{(j-t)(j-t-1)/2+(j-t)t}\right)^{s-r-1}.
\end{aligned}\tag{57}$$

By substituting $j=l+t$ and using (16), (11), we yield

$$\begin{aligned}
&{}_{r+2}\phi_s\left(\begin{array}{c} q^{-n},\theta x, a_R \\ b_S \end{array}\middle| q;y\right) \\
=&\sum_{t=0}^{n}(q\theta)^t\frac{(q^{-n};q)_t}{(q;q)_t}\cdot\sum_{k=0}^{t}\left[\begin{array}{c} t \\ k \end{array}\right]_q(-1)^k q^{\frac{k(k-1)}{2}} \\
&\times{}_{r+2}\phi_s\left(\begin{array}{c} q^{-k},x, a_R \\ b_S \end{array}\middle| q;y\right) \\
&\times\sum_{l=0}^{n-t}\frac{(q^{t-n};q)_l(q^t a_R;q)_l}{(q^t b_S;q)_l(q;q)_l} \\
&\times\left((-1)^l q^{l(l-1)/2+lt}\right)^{s-r-1}y^l(\theta;q)_l \\
=&\sum_{k=0}^{n}(-1)^k q^{\frac{k(k+1)}{2}}\theta^k\frac{(q^{-n};q)_k}{(q;q)_k} \\
&\times{}_{r+2}\phi_s\left(\begin{array}{c} q^{-k},x, a_R \\ b_S \end{array}\middle| q;y\right) \\
&\times\sum_{t=k}^{n}(q\theta)^{t-k}\frac{(q^{k-n};q)_{t-k}}{(q;q)_{t-k}} \\
&\times{}_{r+2}\phi_s\left(\begin{array}{c} q^{t-n},\theta, q^t a_R \\ q^t b_S \end{array}\middle| q;q^{t(s-r-1)}y\right).
\end{aligned}\tag{58}$$

It follows by variable substitution $m=t-k$ that (53) holds.

2) Using (11) and (49), one gets

$$\begin{aligned}
&{}_{r+2}\phi_s\left(\begin{array}{c} q^{-n},\theta+x, a_R \\ b_S \end{array}\middle| q;y\right) \\
=&\sum_{j=0}^{n}\frac{(q^{-n};q)_j(a_R;q)_j}{(b_S;q)_j(q;q)_j}\left((-1)^j q^{j(j-1)/2}\right)^{s-r-1} \\
&\times y^j\sum_{t=0}^{j}(-1)^t q^{t(t+1)/2}M_{j,t}(\theta;q)(x;q)_t \\
=&\sum_{t=0}^{n}(-1)^t q^{t(t+3)/2}\cdot\left((-1)^t q^{t(t-1)/2}\right)^{s-r-1} \\
&\times\frac{(q^{-n};q)_t}{(q;q)_t}(q^{-1}y)^t\frac{(a_R;q)_t}{(b_S;q)_t}(x;q)_t \\
&\times\sum_{j=t}^{n}\frac{(q^{t-n};q)_{j-t}(q^t a_R;q)_{j-t}}{(q^t b_S;q)_{j-t}(q^{t+1};q)_{j-t}}y^{j-t}M_{j,t}(\theta;q) \\
&\times\left((-1)^{j-t}q^{(j-t)(j-t-1)/2+(j-t)t}\right)^{s-r-1}.
\end{aligned}\tag{59}$$

It follows by variable substitution $l=j-t$ and (16) that

$$
\begin{aligned}
&{}_{r+2}\phi_s\left(\begin{array}{c} q^{-n},\theta+x,a_R \\ b_S \end{array}\middle| q;y\right) \\
=&\sum_{t=0}^{n}(-1)^t q^{t(t+3)/2}\frac{(q^{-n};q)_t}{(q;q)_t}\sum_{k=0}^{t}\begin{bmatrix} t \\ k \end{bmatrix}_q (-1)^k \\
&\times q^{\frac{k(k-1)}{2}}\cdot {}_{r+2}\phi_s\left(\begin{array}{c} q^{-k},x,a_R \\ b_S \end{array}\middle| q;y\right) \\
&\times\sum_{l=0}^{n-t}\frac{(q^{t-n};q)_l(q^t a_R;q)_l}{(q^t b_S;q)_l(q^{t+1};q)_l}y^l M_{l+t,t}(\theta;q) \\
&\times\left((-1)^l q^{l(l-1)/2+lt}\right)^{s-r-1} \\
=&\sum_{k=0}^{n}(-1)^k q^{\frac{k(k-1)}{2}}\,{}_{r+2}\phi_s\left(\begin{array}{c} q^{-k},x,a_R \\ b_S \end{array}\middle| q;y\right) \\
&\times\sum_{t=k}^{n}(-1)^t q^{t(t+3)/2}\frac{(q^{-n};q)_t}{(q;q)_t}\begin{bmatrix} t \\ k \end{bmatrix}_q \\
&\times {}^{t}_{r+2}\widetilde{\phi}_s\left(\begin{array}{c} q^{-n},a_R \\ b_S \end{array}\middle| q;\theta;y\right),
\end{aligned}
\tag{60}
$$

which means that (54) holds. ■

III. Applications

Many orthogonal polynomials can be represented by the hypergeometric functions or the basic hypergeometric series. So, if we can determine the nature of coefficients of multiplication and translation formulas , it is easily to construct the multiplication and translation formulas for some orthogonal polynomials based on the reslut in Section II.

Suppose that the polynomial $p_n(x)$ and its inversion formula are given by

$$p_n(x)=\sum_{k=0}^{n}\tau_k E_k(x),\quad E_n(x)=\sum_{k=0}^{n}\sigma_k p_k(x), \tag{61}$$

where $E_n(x)=x^n,(x)_n$ or $(x;q)_n$, then the polynomial $q_n(x)=A_n p_n(x)$ and its inversion formula can be expressed by

$$
\begin{aligned}
q_n(x)&=A_n\sum_{k=0}^{n}\tau_k E_k(x),\\
E_n(x)&=\sum_{k=0}^{n}\frac{\sigma_k}{A_k}A_k p_k(x)=\sum_{k=0}^{n}\frac{\sigma_k}{A_k}q_k(x).
\end{aligned}
\tag{62}
$$

By (1) and (2), the multiplication and translation formulas and inversion formula for $q_n(x)$,

$$q_n(ax)=A_n p_n(ax)=A_n\sum_{m=0}^{n}\frac{D_m(n,a)}{A_m}q_m(x), \tag{63}$$

and

$$q_n(a+x)=A_n p_n(a+x)=A_n\sum_{m=0}^{n}\frac{D_m(n,a)}{A_n}q_m(x). \tag{64}$$

Denote $s:=\alpha+\beta+1$. For $\alpha,\beta>-1$, the hypergeometric representations and the inverse formula of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ [7](pp.500) are given by

$$
\begin{aligned}
&P_n^{(\alpha,\beta)}(x)\\
=&\frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n,n+s \\ \alpha+1 \end{array}\middle| \frac{1-x}{2}\right),\\
&\left(\frac{1-x}{2}\right)^n\\
=&\sum_{l=0}^{n}\frac{(s+2l)(\alpha+l+1)_{n-l}(-n)_l}{(s+l)_{n+1}}P_l^{(\alpha,\beta)}(x).
\end{aligned}
\tag{65}
$$

Corollary 3.1 Let $\alpha,\beta>-1$, $\theta,x\in\mathbb{R}\backslash\{0\}$ and $n\in\mathbb{N}$. Then

$$
\begin{aligned}
&P_n^{(\alpha,\beta)}(\theta+x)\\
=&\sum_{j=0}^{n}\frac{(s+2j)(\alpha+j+1)_{n-j}}{(n-j)!}P_j^{(\alpha,\beta)}(x)\\
&\times\sum_{m=j}^{n}(-1)^{m-j}C_{n-j}^{n-m}\frac{(n+s)_m}{(s+j)_{m+1}}\\
&\times {}_2F_1\left(\begin{array}{c} m-n,n+s+m \\ \alpha+m+1 \end{array}\middle| -\frac{\theta}{2}\right),
\end{aligned}
\tag{66}
$$

and

$$
\begin{aligned}
&P_n^{(\alpha,\beta)}(\theta x)\\
=&\sum_{j=0}^{n}\frac{(s+2j)(\alpha+j+1)_{n-j}}{(n-j)!}P_j^{(\alpha,\beta)}(x)\\
&\times\sum_{m=j}^{n}(-1)^{m-j}C_{n-j}^{n-m}\frac{\theta^m(n+s)_m}{(s+j)_{m+1}}\\
&\times {}_2F_1\left(\begin{array}{c} m-n,n+s+m \\ \alpha+m+1 \end{array}\middle| 1-\frac{\theta}{2}\right).
\end{aligned}
\tag{67}
$$

Proof. 1) By using (10) and the Vandermonde summation formula [9](pp.7)

$${}_2F_1\left(\begin{array}{c} -n,b \\ c \end{array}\middle| 1\right)=\frac{(c-b)_n}{(c)_n}, \tag{68}$$

one has

$$
\begin{aligned}
&\sum_{l=j}^{k}(-1)^l C_{k-j}^{k-l}\frac{(n+s)_l}{(s+j)_{l+1}}\\
&=\frac{(-1)^j(n+s)_j}{(s+j)_{j+1}}\,{}_2F_1\left(\begin{array}{c} j-k,n+s+j \\ s+2j+1 \end{array}\middle| 1\right)\\
&=(-1)^j\frac{(n+s)_j(j+1-n)_{k-j}}{(s+j)_{k+1}}.
\end{aligned}
\tag{69}
$$

Combining (10) and (65), we encounter

$$
\begin{aligned}
&{}_2F_1\left(\begin{array}{c} -k,n+s \\ \alpha+1 \end{array}\middle| \frac{1-x}{2}\right)\\
=&\sum_{l=0}^{k}(-1)^l C_k^l\frac{(n+s)_l}{(\alpha+1)_l}\\
&\times\sum_{j=0}^{l}\frac{(s+2j)(\alpha+j+1)_{l-j}(-l)_j}{(s+j)_{l+1}}P_j^{(\alpha,\beta)}(x)\\
=&\sum_{j=0}^{k}(-1)^j\frac{(s+2j)k!}{(\alpha+1)_j(k-j)!}P_j^{(\alpha,\beta)}(x)\\
&\times\sum_{l=j}^{k}(-1)^l C_{k-j}^{k-l}\frac{(n+s)_l}{(s+j)_{l+1}}.
\end{aligned}
\tag{70}
$$

It follows by (69) that

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} -k, n+s \\ \alpha+1 \end{array} \middle| \frac{1-x}{2}\right) \\ = \; & k! \sum_{j=0}^{k} (-1)^{k-j} C_{n-1-j}^{k-j} \\ & \times \frac{(s+2j)(n+s)_j}{(\alpha+1)_j (s+j)_{k+1}} P_j^{(\alpha,\beta)}(x). \end{aligned} \tag{71}$$

Making use of (65), (21) and (71), we obtain

$$\begin{aligned} & P_n^{(\alpha,\beta)}(\theta+x) \\ = \; & \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} C_n^k (-1)^k \\ & \times {}_2F_1\left(\begin{array}{c} -k, n+s \\ \alpha+1 \end{array} \middle| \frac{1-x}{2}\right) \sum_{m=k}^{n} C_{n-k}^{n-m} \\ & \times (-1)^m \cdot {}_2F_1\left(\begin{array}{c} m-n, n+m+s \\ \alpha+m+1 \end{array} \middle| -\frac{\theta}{2}\right) \\ = \; & \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!} \sum_{j=0}^{k} C_{n-1-j}^{k-j} (-1)^{k-j} \\ & \times \frac{(s+2j)(\alpha+j+1)_{n-j}(n+s)_j}{(s+j)_{k+1}} P_j^{(\alpha,\beta)}(x) \\ & \times \sum_{m=k}^{n} C_{n-k}^{n-m} (-1)^m \\ & \times {}_2F_1\left(\begin{array}{c} m-n, n+m+s \\ \alpha+m+1 \end{array} \middle| -\frac{\theta}{2}\right) \\ = \; & \sum_{j=0}^{n} (-1)^j \frac{(s+2j)(\alpha+j+1)_{n-j}}{(n-j)!} P_j^{(\alpha,\beta)}(x) \\ & \times \sum_{m=j}^{n} (-1)^m C_{n-j}^{n-m} \cdot V(n,s,j,m) \\ & \times {}_2F_1\left(\begin{array}{c} m-n, n+m+s \\ \alpha+m+1 \end{array} \middle| -\frac{\theta}{2}\right), \end{aligned} \tag{72}$$

where

$$\begin{aligned} & V(n,s,j,m) \\ = \; & (n+s)_j \sum_{k=j}^{m} (-1)^{k-j} C_{m-j}^{m-k} \frac{(j+1-n)_{k-j}}{(s+j)_{k+1}}. \end{aligned} \tag{73}$$

Using (69), one has

$$\begin{aligned} & V(n,s,j,m) \\ = \; & \sum_{k=j}^{m} (-1)^k C_{m-j}^{m-k} \sum_{l=j}^{k} (-1)^l C_{k-j}^{k-l} \frac{(n+s)_l}{(s+j)_{l+1}} \\ = \; & \sum_{l=j}^{m} C_{m-j}^{l-j} \frac{(n+s)_l}{(s+j)_{l+1}} \sum_{k=l}^{m} (-1)^{k-l} C_{m-l}^{k-l} \\ = \; & \frac{(n+s)_m}{(s+j)_{m+1}}. \end{aligned} \tag{74}$$

Inserting (74) into (72), we arrive at the required result (66).

2) Using (20), (70) and (10), we come to

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} -m, n+s \\ \alpha+1 \end{array} \middle| \frac{\theta(1-x)}{2}\right) \\ = \; & \sum_{k=0}^{m} C_m^k \theta^k (1-\theta)^{m-k} \\ & \times {}_2F_1\left(\begin{array}{c} -k, n+s \\ \alpha+1 \end{array} \middle| \frac{1-x}{2}\right) \\ = \; & m! \sum_{k=0}^{m} \frac{\theta^k (1-\theta)^{m-k}}{(m-k)!} \sum_{j=0}^{k} (-1)^{k-j} C_{n-1-j}^{k-j} \\ & \times \frac{(s+2j)(n+s)_j}{(\alpha+1)_j (s+j)_{k+1}} P_j^{(\alpha,\beta)}(x) \\ = \; & m! \sum_{j=0}^{m} \frac{(s+2j)(n+s)_j \theta^j (1-\theta)^{m-j}}{(\alpha+1)_j} \\ & \times P_j^{(\alpha,\beta)}(x) \sum_{k=j}^{m} C_{n-1-j}^{k-j} \\ & \times \frac{1}{(m-k)!(s+j)_{k+1}} \left(\frac{\theta}{\theta-1}\right)^{k-j}. \end{aligned} \tag{75}$$

It follows by variable substitution $l = k - j$ and (10) that

$$\begin{aligned} & \frac{1}{m!} \cdot {}_2F_1\left(\begin{array}{c} -m, n+s \\ \alpha+1 \end{array} \middle| \frac{\theta(1-x)}{2}\right) \\ = \; & \sum_{j=0}^{m} \frac{(s+2j)(n+s)_j \theta^j (1-\theta)^{m-j}}{(m-j)!(\alpha+1)_j (s+j)_{j+1}} P_j^{(\alpha,\beta)}(x) \\ & \times \sum_{l=0}^{m-j} (-1)^l C_{m-j}^l \frac{(j+1-n)_l}{(s+2j+1)_l} \left(\frac{\theta}{\theta-1}\right)^l \\ = \; & \sum_{j=0}^{m} \frac{(s+2j)(n+s)_j \theta^j (1-\theta)^{m-j}}{(m-j)!(\alpha+1)_j (s+j)_{j+1}} P_j^{(\alpha,\beta)}(x) \\ & \times {}_2F_1\left(\begin{array}{c} j-m, j+1-n \\ s+2j+1 \end{array} \middle| \frac{\theta}{\theta-1}\right). \end{aligned} \tag{76}$$

With the aid of (65), (21) and (76), we deduce that

$$\begin{aligned} & P_n^{(\alpha,\beta)}(\theta x) \\ = \; & \frac{(\alpha+1)_n}{n!} \sum_{m=0}^{n} C_n^m (-1)^m \\ & \times {}_2F_1\left(\begin{array}{c} -m, n+s \\ \alpha+1 \end{array} \middle| \frac{\theta(1-x)}{2}\right) \sum_{l=m}^{n} C_{n-m}^{n-l} \\ & \times (-1)^l \cdot {}_2F_1\left(\begin{array}{c} l-n, n+l+s \\ \alpha+l+1 \end{array} \middle| \frac{1-\theta}{2}\right) \\ = \; & \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!} \sum_{j=0}^{m} \frac{(s+2j)(\alpha+j+1)_{n-j}}{(m-j)!(s+j)_{j+1}} \\ & \times (n+s)_j \theta^j (1-\theta)^{m-j} P_j^{(\alpha,\beta)}(x) \\ & \times {}_2F_1\left(\begin{array}{c} j-m, j+1-n \\ s+2j+1 \end{array} \middle| \frac{\theta}{\theta-1}\right) \\ & \times \sum_{l=m}^{n} C_{n-m}^{n-l} (-1)^l \\ & \times {}_2F_1\left(\begin{array}{c} l-n, n+l+s \\ \alpha+l+1 \end{array} \middle| \frac{1-\theta}{2}\right) \\ = \; & \sum_{j=0}^{n} \frac{(s+2j)(\alpha+j+1)_{n-j}}{(n-j)!} P_j^{(\alpha,\beta)}(x) \\ & \times \sum_{l=j}^{n} (-1)^{l-j} C_{n-j}^{n-l} \cdot U(n,s,j,l,\theta) \\ & \times {}_2F_1\left(\begin{array}{c} l-n, n+l+s \\ \alpha+l+1 \end{array} \middle| \frac{1-\theta}{2}\right), \end{aligned} \tag{77}$$

where

$$\begin{aligned} & U(n,s,j,l,\theta) \\ = & \frac{(n+s)_j\theta^j}{(s+j)_{j+1}}\sum_{m=j}^{l} C_{l-j}^{l-m}(-1)^m(1-\theta)^{m-j} \\ & \times {}_2F_1\left(\begin{array}{c} j-m, j+1-n \\ s+2j+1 \end{array}\middle| \frac{\theta}{\theta-1}\right). \end{aligned} \quad (78)$$

Using (10) and (74), we proceed to get

$$\begin{aligned} & U(n,s,j,l,\theta) \\ = & \frac{(n+s)_j\theta^j}{(s+j)_{j+1}}\sum_{t=0}^{l-j} C_{l-j}^{t}(-1)^{t+j}(1-\theta)^{t} \\ & \times\sum_{k=0}^{t}(-1)^k C_t^k \frac{(j+1-n)_k}{(s+2j+1)_k}\left(\frac{\theta}{\theta-1}\right)^k \\ = & (-1)^j\frac{(n+s)_j\theta^j}{(s+j)_{j+1}}\sum_{k=0}^{l-j}(-1)^k C_{l-j}^k \\ & \times\frac{(j+1-n)_k}{(s+2j+1)_k}\theta^k\sum_{t=k}^{l-j} C_{l-j-k}^{t-k}(\theta-1)^{t-k} \\ = & (-1)^j\frac{(n+s)_j\theta^l}{(s+j)_{j+1}} \\ & \times\sum_{k=0}^{l-j}(-1)^k C_{l-j}^k\frac{(j+1-n)_k}{(s+2j+1)_k} \\ = & \frac{\theta^l(n+s)_l}{(s+j)_{l+1}}. \end{aligned} \quad (79)$$

Combining (79) with (77), we deduce that (67) holds. ■

In the next part, we will discuss the multiplication and translation formulas for Little q-Jacobi polynomial [8] defined by

$$\begin{aligned} p_n(x;a,b;q) = & (-1)^n q^{\frac{n(n-1)}{2}}\frac{(aq;q)_n}{(abq^{n+1};q)_n} \\ & \times{}_2\phi_1\left(\begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array}\middle| q;qx\right), \end{aligned} \quad (80)$$

and it's inversion formula can be expressed as follows,

$$x^n = \sum_{m=0}^{n}\left[\begin{array}{c} n \\ m \end{array}\right]_q \frac{(aq^{m+1};q)_{n-m}}{(abq^{2(m+1)};q)_{n-m}} p_m(x;a,b;q). \quad (81)$$

Corollary 3.2 Let $\theta, x, a, b \in \mathbb{R}\backslash\{0\}$, $0<|q|<1$ and $n\in\mathbb{N}$, then the multiplication and translation formulae of Little q-Jacobi polynomial can be expressed as follows,

$$\begin{aligned} & p_n(\theta x;a,b;q) \\ = & (-1)^n q^{n(n-1)/2}\sum_{m=0}^{n}(\theta q)^m \\ & \times\frac{(aq^{m+1};q)_{n-m}(q^{-n};q)_m}{(abq^{n+m+1};q)_{n-m}(q;q)_m}\cdot p_m(x;a,b;q) \\ & \times{}_2\phi_1\left(\begin{array}{c} q^{m-n}, abq^{m+n+1} \\ abq^{2m+2} \end{array}\middle| q;\theta q\right), \end{aligned} \quad (82)$$

and

$$\begin{aligned} & p_n(\theta+x;a,b;q) \\ = & (-1)^n q^{n(n-1)/2}\sum_{u=0}^{n}\frac{(aq^{u+1};q)_{n-u}}{(abq^{n+u+1};q)_{n-u}(q;q)_u} \\ & \times p_u(x;a,b;q)\sum_{m=u}^{n}\frac{q^m(q^{-n};q)_m}{(q;q)_{m-u}} \\ & \times {}^{m}_{2}\phi_1\left(\begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array}\middle| q;\theta q\right) \\ & \times{}_2\phi_1\left(\begin{array}{c} q^{u-m}, q^{u+1-n} \\ abq^{2u+2} \end{array}\middle| q;abq^{n+m+1}\right), \end{aligned} \quad (83)$$

where ${}^{m}_{2}\phi_1$ can be given by (26).

Proof. 1) Combining (80) with (28), we yield

$$\begin{aligned} & p_n(\theta x;a,b;q) \\ = & (-1)^n q^{\frac{n(n-1)}{2}}\frac{(aq;q)_n}{(abq^{n+1};q)_n} \\ & \times{}_2\phi_1\left(\begin{array}{c} q^{-n}, abq^{n+1} \\ aq \end{array}\middle| q;q\theta x\right) \\ = & \frac{\theta^n(aq;q)_n}{(abq^{n+1};q)_n}\sum_{k=0}^{n} q^{kn}\frac{(q^{-n};q)_k(\theta^{-1};q)_{n-k}}{(q;q)_k} \\ & \times{}_2\phi_1\left(\begin{array}{c} q^{-k}, abq^{n+1} \\ aq \end{array}\middle| q;qx\right). \end{aligned} \quad (84)$$

Using (81) and (11), we have

$$\begin{aligned} & {}_2\phi_1\left(\begin{array}{c} q^{-k}, abq^{n+1} \\ aq \end{array}\middle| q;qx\right) \\ = & \sum_{l=0}^{k}\frac{(q^{-k};q)_l(abq^{n+1};q)_l}{(aq;q)_l(q;q)_l}(qx)^l \\ = & \sum_{l=0}^{k}\frac{(q^{-k};q)_l(abq^{n+1};q)_l}{(aq;q)_l(q;q)_l}q^l\sum_{m=0}^{l}\left[\begin{array}{c} l \\ m \end{array}\right]_q \\ & \times\frac{(aq^{m+1};q)_{l-m}}{(abq^{2(m+1)};q)_{l-m}}p_m(x;a,b;q) \\ = & \sum_{m=0}^{k}\frac{(q^{-k};q)_m(abq^{n+1};q)_m}{(aq;q)_m(q;q)_m}q^m\cdot p_m(x;a,b;q) \\ & \times{}_2\phi_1\left(\begin{array}{c} q^{m-k}, abq^{n+m+1} \\ abq^{2m+2} \end{array}\middle| q;q\right). \end{aligned} \quad (85)$$

Inserting (85) into (84), we obtain

$$\begin{aligned} & p_n(\theta x;a,b;q) \\ = & \frac{\theta^n(aq;q)_n}{(abq^{n+1};q)_n}\sum_{k=0}^{n} q^{kn}\frac{(q^{-n};q)_k(\theta^{-1};q)_{n-k}}{(q;q)_k} \\ & \times\sum_{m=0}^{k}\frac{(q^{-k};q)_m(abq^{n+1};q)_m}{(aq;q)_m(q;q)_m}p_m(x;a,b;q) \\ & \times q^m\cdot{}_2\phi_1\left(\begin{array}{c} q^{m-k}, abq^{n+m+1} \\ abq^{2m+2} \end{array}\middle| q;q\right) \\ = & \sum_{m=0}^{n}(\theta q)^m\frac{(aq^{m+1};q)_{n-m}(q^{-n};q)_m}{(abq^{n+1+m};q)_{n-m}(q;q)_m} \\ & \times p_m(x;a,b;q)\cdot Y(n,m,q,\theta,a,b), \end{aligned} \quad (86)$$

where

$$\begin{aligned} & Y(n,m,q,\theta,a,b) \\ = & \sum_{k=m}^{n}\theta^{n-m}q^{kn}\frac{(q^{-n+m};q)_{k-m}(\theta^{-1};q)_{n-k}}{(q;q)_k} \\ & \times(q^{-k};q)_m\cdot{}_2\phi_1\left(\begin{array}{c} q^{m-k}, abq^{n+m+1} \\ abq^{2m+2} \end{array}\middle| q;q\right). \end{aligned} \quad (87)$$

It follows by variable substitution $t = k - m$ that

$$\begin{aligned} & Y(n,m,q,\theta,a,b) \\ = & \sum_{t=0}^{n-m} \theta^{n-m} q^{(m+t)n} \frac{(q^{-n+m};q)_t(\theta^{-1};q)_{n-m-t}}{(q;q)_t} \\ & \times \frac{(q^{-m-t};q)_m}{(q^{t+1};q)_m} {}_2\phi_1\left(\begin{matrix} q^{-t}, abq^{n+m+1} \\ abq^{2m+2} \end{matrix} \,\middle|\, q;q\right). \end{aligned} \tag{88}$$

Using the definition of the q-shifted factorial (6), we have

$$\frac{(q^{-m-t};q)_m}{(q^{t+1};q)_m} = (-1)^m q^{-m(t+1)-\frac{m(m-1)}{2}}. \tag{89}$$

Inserting (89) into (88) and using (28), one gets

$$\begin{aligned} & Y(n,m,q,\theta,a,b) \\ = & (-1)^n q^{n(n-1)/2} \\ & \times {}_2\phi_1\left(\begin{matrix} q^{m-n}, abq^{m+n+1} \\ abq^{2m+2} \end{matrix} \,\middle|\, q;\theta q\right) \end{aligned} \tag{90}$$

Combining (90) with (86), we conclude that (82) holds.

2) Using (80) and (29), we derive

$$\begin{aligned} & p_n(\theta+x;a,b;q) \\ = & (-1)^n q^{\frac{n(n-1)}{2}} \frac{(aq;q)_n}{(abq^{n+1};q)_n} \\ & \times {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q;q\theta+qx\right) \\ = & (-1)^n q^{\frac{n(n-1)}{2}} \sum_{k=0}^{n} \frac{(aq;q)_n(q^{-n};q)_k}{(abq^{n+1};q)_n(q;q)_k} \\ & \times(-1)^k q^{\frac{k(k+1)}{2}} {}_2\phi_1\left(\begin{matrix} q^{-k}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q;qx\right) \\ & \times \sum_{t=0}^{n-k} \frac{q^t(q^{k-n};q)_t}{(q;q)_t} \\ & \times {}^{t+k}_{\ 2}\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q;q\theta\right). \end{aligned} \tag{91}$$

By substituting $m = k + t$ and using (85), we yield

$$\begin{aligned} & p_n(\theta+x;a,b;q) \\ = & (-1)^n q^{\frac{n(n-1)}{2}} \sum_{k=0}^{n} \frac{(aq;q)_n(q^{-n};q)_k}{(q;q)_k(abq^{n+1};q)_n}(-1)^k \\ & \times q^{\frac{k(k+1)}{2}} \sum_{u=0}^{k} \frac{(q^{-k};q)_u(abq^{n+1};q)_u}{(aq;q)_u(q;q)_u} q^u \\ & \times {}_2\phi_1\left(\begin{matrix} q^{u-k}, abq^{n+u+1} \\ abq^{2u+2} \end{matrix} \,\middle|\, q;q\right) \\ & \times p_u(x;a,b;q) \sum_{m=k}^{n} \frac{q^{m-k}(q^{k-n};q)_{m-k}}{(q;q)_{m-k}} \\ & \times {}^{m}_{2}\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q;q\theta\right). \end{aligned} \tag{92}$$

Exchanging the order of summation, (92) becomes,

$$\begin{aligned} & p_n(\theta+x;a,b;q) \\ = & (-1)^n q^{\frac{n(n-1)}{2}} \sum_{u=0}^{n} \frac{(aq^{u+1};q)_{n-u}}{(q;q)_u(abq^{n+1+u};q)_{n-u}} \\ & \times p_u(x;a,b;q) \cdot J(n,u,a,b,q,\theta), \end{aligned} \tag{93}$$

where

$$\begin{aligned} & J(n,u,a,b,q,\theta) \\ = & \sum_{k=u}^{n} (-1)^k q^{\frac{k(k+1)}{2}} \frac{(q^{-k};q)_u(q^{-n};q)_k}{(q;q)_k} q^u \\ & \times {}_2\phi_1\left(\begin{matrix} q^{u-k}, abq^{n+u+1} \\ abq^{2u+2} \end{matrix} \,\middle|\, q;q\right) \\ & \times \sum_{m=k}^{n} \frac{q^{m-k}(q^{k-n};q)_{m-k}}{(q;q)_{m-k}} \\ & \times {}^{m}_{2}\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q;q\theta\right) \\ = & \sum_{m=u}^{n} \frac{(q^{-n};q)_m}{(q;q)_{m-u}} q^m \cdot H(n,m,u,a,b,q) \\ & \times {}^{m}_{2}\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q;q\theta\right), \end{aligned} \tag{94}$$

where

$$\begin{aligned} & H(n,m,u,a,b,q) \\ = & \sum_{k=u}^{m} (-1)^k q^{\frac{k(k+1)}{2}} \frac{(q^{m-k+1};q)_{k-u}(q^{-k};q)_u}{(q;q)_k} \\ & \times q^{u-k} \cdot {}_2\phi_1\left(\begin{matrix} q^{u-k}, abq^{n+u+1} \\ abq^{2u+2} \end{matrix} \,\middle|\, q;q\right). \end{aligned} \tag{95}$$

Using the following identity [9]

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ a \end{matrix} \,\middle|\, q;q\right) = \frac{b^n(b^{-1}a;q)_n}{(a;q)_n}, \tag{96}$$

we have

$$\begin{aligned} & {}_2\phi_1\left(\begin{matrix} q^{u-k}, abq^{n+u+1} \\ abq^{2u+2} \end{matrix} \,\middle|\, q;q\right) \\ = & \frac{\left(abq^{n+u+1}\right)^{k-u}(q^{u+1-n};q)_{k-u}}{(abq^{2u+2};q)_{k-u}}. \end{aligned} \tag{97}$$

With the aid of (97), (32) and (11), (95) reduces to

$$\begin{aligned} & H(n,m,u,a,b,q) \\ = & \sum_{k=u}^{m} (-1)^k q^{\frac{k(k+1)}{2}} \frac{(q^{-k};q)_u(q^{m-k+1};q)_{k-u}}{(q^{k-u+1};q)_u(q^{u-m};q)_{k-u}} \\ & \times \left(q^{u-m-1}\right)^{k-u} \cdot \left(abq^{n+m+1}\right)^{k-u} \\ & \times \frac{(q^{u+1-n};q)_{k-u}(q^{u-m};q)_{k-u}}{(abq^{2u+2};q)_{k-u}(q;q)_{k-u}} \\ = & \sum_{k=u}^{m} \frac{(q^{u+1-n};q)_{k-u}(q^{u-m};q)_{k-u}}{(abq^{2u+2};q)_{k-u}(q;q)_{k-u}} \\ & \times \left(abq^{n+m+1}\right)^{k-u} \\ = & {}_2\phi_1\left(\begin{matrix} q^{u-m}, q^{u+1-n} \\ abq^{2u+2} \end{matrix} \,\middle|\, q;abq^{n+m+1}\right). \end{aligned} \tag{98}$$

Combining (93), (94) and (98), we arrive at the required result (83). ■

IV. Conclusion

As is well-konwn, many orthogonal polynomials [7], [9], [10] can be represented by the generalized hypergeometric functions ${}_rF_s$ or the basic hypergeometric series ${}_r\phi_s$, such as the classical continuous orthogonal polynomials: the Jacobi, Laguerre, Hermite and Bessel polynomials; the classical discrete orthogonal polynomials: the Hahn, Meixner, Krawtchouk and Charlier polynomials; the q-classical orthogonal polynomials: the little q-Jacobi, q-Hahn, q-Meixner,

q-Laguerre, q-Charlier and Stieltjes-Wigert polynomials and so on.

In this paper, we aim to establish the multiplication and translation formulas for the generalized hypergeometric polynomials ${}_rF_s$ and the basic hypergeometric polynomials ${}_r\phi_s$ with the aid of their inversion formula.

Based on these results and the nature of coefficients of multiplication and translation formulas, it is easily to construct the multiplication and translation formulas for the above metioned orthogonal polynomials directly.

Acknowledgment

The authors are grateful to the anonymous referees for constructive and helpful comments that improved this paper.

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