Stabilisation by Stochastic Feedback Control Based on Discrete-time Observations

Chao Wei

***Abstract*—This paper is concerned with the stabilization of linear and nonlinear stochastic systems by linear stochastic feedback control from discrete-time observations. By using Itô formula, Borel-Cantelli lemma, Burkholder-Davis-Gundy inequality, Hölder inequality and Gronwall inequality, the almost sure exponential stabilization of the stochastic systems are studied and the sufficient conditions are provided. The results are extended to stochastic feedback control with Lévy noises as well.**

***Index Terms*—Nonlinear stochastic system, almost sure exponential stabilization, discrete-time state observation.**

I. Introduction

It is well known that noise can be used to stabilize a given unstable system or to make a system even more stable when it is already stable [2], [7], [9], [18]–[20]. For example, the nonlinear scalar differential equation $\frac{dx(t)}{dt} = f(x(t))$ is unstable but it can be stabilized by a Brownian motion $Ax(t)dB(t)$, namely, the nonlinear stochastic differential equation

$$dx(t) = f(x(t))dt + Ax(t)dB(t)$$

is stable. From the point of control theory, it is the stochastic feedback control $Ax(t)dB(t)$ that stabilizes the unstable system $\frac{dx(t)}{dt} = f(x(t))$. During the past few decades, some authors have studied the stabilization of the system. Mao [15] developed a general theory on stabilization and destabilization by a linear stochastic feedback control. Appleby and Mao [3] stabilized a class of functional differential equations by noise. Mao et al. [16], [17] used Lyapunov method to solve the stabilization problems. Wu et al. [21] investigated the stabilization issue of stochastic coupled systems with Markovian switching by using feedback control.

We observe that a common feature of the stochastic feedback controls is that the controls depend on the current state $x(t)$ continuously. However, the state of the given system is in fact observed only at discrete times such as $0, \tau, 2\tau, ...,$, where $\tau > 0$ is the duration between two consecutive observations. It also costs less if τ is larger. In the past few decades, the stabilization of system by discrete-time stochastic feedback control have been discussed in some literatures. For example, Hagiwara and Araki [6] designed the stable state feedback controller based on the multirate sampling of the plant output. Allwright et al. [1] studied the asymptotic stabilization of linear systems by periodic, piecewise constant, output feedback. Ebihara et al. [5] discussed the periodically time-varying controller synthesis for multiobjective H_2/H_∞ control of discrete-time systems. Some other work in this area was made by Li et al. [8], Dong [4], Xie and Jiang [22]. However, the almost surely stochastic stabilization problem for nonlinear stochastic system has not been studied so far. In this paper, we studied the stabilization of nonlinear stochastic system by linear stochastic feedback control from discrete-time observations. The almost sure exponential stabilization of the nonlinear stochastic system is discussed and the sufficient conditions are provided.

This paper is constructed in the following way. In Section 2, the nonlinear stochastic system is introduced, some mathematical preliminaries and basic assumptions are given. Section 3 discusses the almost sure exponential stabilization of the nonlinear stochastic system and provides the sufficient conditions. In Section 4, The results are extended to stochastic feedback control with Lévy noises. The conclusion is given in Section 5.

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Chao Wei is with the School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China.(Email: chaowei0806@aliyun.com..

II. Problem Formulation and Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions that it is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let $B(t) = (B_1(t), \cdots, B_n(t))^T$ be an n-dimensional Brownian motion defined on the probability space.

Let us consider an unstable ODE system:

$$\frac{dy(t)}{dt} = f(y(t)) \tag{1}$$

where $f : R^n \to R^n$. Now we design a state feedback stochastic control $Ax(\delta_t)dB(t)$ based on the discrete-time observations and a Brownian motion to stabilize the system:

$$dx(t) = f(x(t))dt + Ax(\delta_t)dB(t) \quad t \geq 0, \tag{2}$$

where $A \in R^{n\times n}$, $x(0) = x_0$ and

$$\delta_t = [\frac{t}{\tau}]\tau, \tag{3}$$

where τ is the discrete time gap between two adjacent observations, $[\frac{t}{\tau}]$ is the integer part of $\frac{t}{\tau}$. To discuss the stochastic stabilisation theory, we also impose the following assumptions.

Assumption 1: Assume that the drift functions $f(x)$ is globally Lipschitz continuous

$$|f(x) - f(y)| \leq K_1|x - y| \tag{4}$$

for all $x, y \in R^n$, where K_1 is a positive constant.

Due to the discussion of stability, we also assume the origin is an equilibrium point $f(0) = 0$. It is easy to see that Assumption 1 implies the following linear growth condition

$$|f(x)| \leq K_1|x| \tag{5}$$

for all $x \in R^n$. Besides, we also require some conditions on the diffusion coefficient, which is our controller function $Ax(t)$.

Assumption 2: Assume that the diffusion coefficient fulfill the following

$$|Ax|^2 \leq K_2|x|^2 \qquad \text{and} \qquad |x^T Ay|^2 \geq K_3|x|^2|y|^2 \tag{6}$$

for all $x, y \in R^n$, where both K_2 and K_3 are positive constants.

Because the controller function is human designed, thus the assumption we required is reasonable. One may known from [10], [11] that it is not difficult to find the examples of the square matrix A that fulfils Assumption 2.

Definition 1: The solution of system (31) is said to be almost sure exponential stability if it satisfies

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(|x(t)|) < 0 \qquad a.s.$$

We also need some lemmas such as Burkholder-Davis-Gundy inequality, Borel-Cantelli lemma and Hölder inequality as follows.

Lemma 1: (Burkholder-Davis-Gundy inequality) [12] For $t \geq 0$, let $x(t) = \int_0^t g(s)dB(s)$ $A(t) = \int_0^t |g(s)|^2 ds$. Then, for any $p > 0$, there exist positive constants c_p and C_p satisfying

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}(\sup_{0\leq s\leq t} |x(s)|^p) \leq C_p \mathbb{E}|A(t)|^{\frac{p}{2}},$$

where

$$\begin{cases} c_p = (\frac{p}{2})^p, \quad C_p = (\frac{32}{p})^{\frac{p}{2}}, \quad 0 < p < 2; \\ c_p = 1, \quad C_p = 4, \quad p = 2; \\ c_p = (2p)^{-\frac{p}{2}}, \quad C_p = (\frac{p^{p+1}}{2(p-1)^{p-1}}))^{\frac{p}{2}}, \quad p > 2. \end{cases}$$

Lemma 2: (Borel-Cantelli lemma) [15] For the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$,

(1) if $\{\mathcal{A}_k\} \subset \mathscr{F}$ and $\sum_{k=1}^{\infty} \mathrm{P}(\mathcal{A}_k) < \infty$, then

$$\mathrm{P}(\limsup_{k\to\infty} \mathcal{A}_k) = 0.$$

Namely, there exist a positive constant k_0 and set Ω_0, where $\Omega_0 \in \mathscr{F}$ and satisfying $\mathrm{P}(\Omega_0) = 1$, for any $\omega \in \Omega_0$, it follows that

$$\omega \notin \mathcal{A}_k \quad k \geq k_0.$$

(2) If $\{\mathcal{A}_k\} \subset \mathscr{F}$ are independent and $\sum_{k=1}^{\infty} \mathrm{P}(\mathcal{A}_k) = \infty$, then

$$\mathrm{P}(\limsup_{k\to\infty} \mathcal{A}_k) = 1.$$

Namely, there exist a set Ω_1 satisfying $\mathrm{P}(\Omega_1) = 1$, and $\mathcal{A}_{k_i}$, for any $\omega \in \Omega_1$, it follows that

$$\omega \in \mathcal{A}_{k_i} \quad \forall i \in \mathcal{I}_+.$$

Lemma 3: (Hölder inequality) [14] If $p > 1, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$,$X \in \mathrm{L}^p$, $Y \in \mathrm{L}^q$, $X, Y \in (\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathbb{E}|XY| \leq (\mathbb{E}|X|^P)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{\frac{1}{q}}.$$

III. Main results

To illustrate our main theorem, let us present an useful lemma first.

Lemma 4: Let Assumptions 1 and 2 hold. Let us define

$$K(\tau) = (8\tau^2 K_1^2 + 4\tau K_2)e^{4\tau^2 K_1^2}, \tag{7}$$

If $\tau > 0$ is sufficiently small such that for $K(\tau) < 1$, then the solution of (31) satisfies

$$\mathbb{E}|x(s) - x(\delta(s))|^2 ds \leq \frac{K(\tau)}{1 - K(\tau)} \mathbb{E}|x(s)|^2 \tag{8}$$

for any $t \geq 0$.

Proof: Fix any $t \geq 0$, there must exist an integer $k \geq 0$ such that $t \in [k\tau, (k+1)\tau)$, thus we have $\delta(t) = k\tau$. It is easy to see from (31) that

$$\begin{aligned} & x(t) - x(\delta(t)) \\ & = x(t) - x(k\tau) \\ & = \int_{k\tau}^t f(x(s))ds + \int_{k\tau}^t Ax(k\tau)dB(s) \end{aligned}$$

By the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, Hölder's inequality and Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} & \mathbb{E}|x(t) - x(\delta(t))|^2 \\ & \leq 2\mathbb{E}|\int_{k\tau}^t f(x(s))ds|^2 + 2\mathbb{E}|\int_{k\tau}^t Ax(k\tau)dB(s)|^2 \\ & \leq 2\tau \int_{k\tau}^t \mathbb{E}|f(x(s))|^2 ds + 2\int_{k\tau}^t \mathbb{E}|Ax(k\tau)|^2 ds \\ & \leq 2\tau K_1^2 \int_{k\tau}^t \mathbb{E}|x(s)|^2 ds + 2\tau K_2 \mathbb{E}|x(k\tau)|^2 \\ & \leq 4\tau K_1^2 \int_{k\tau}^t \mathbb{E}|x(s) - x(\delta(t))|^2 ds \\ & + (4\tau^2 K_1^2 + 2\tau K_2)\mathbb{E}|x(k\tau)|^2 \end{aligned}$$

By the Gronwall inequality, it follows that

$$\begin{aligned} & \mathbb{E}|x(t) - x(\delta(t))|^2 \\ & \leq (4\tau^2 K_1^2 + 2\tau K_2)e^{4\tau^2 K_1^2} \mathbb{E}|x(k\tau)|^2 \\ & \leq (8\tau^2 K_1^2 + 4\tau K_2)e^{4\tau^2 K_1^2} (\mathbb{E}|x(t) - x(k\tau)|^2 \\ & + \mathbb{E}|x(k\tau)|^2) \\ & \leq \frac{K(\tau)}{1 - K(\tau)} \mathbb{E}|x(t)|^2 \end{aligned}$$

■

Theorem 1: Let Assumptions 1 and 2 hold, the controlled system (31) is almost sure exponential stable.

Proof: Since

$$\begin{aligned} & d|x(t)|^2 = [2x(t)^T f(x(t)) + |Ax(\delta(t))|^2]dt \\ & + 2x(t)^T Ax(\delta(t))dB(t). \end{aligned}$$

let $p \in (0,1)$, then we have

$$\begin{aligned}
&d|x(t)|^p \\
&= (\frac{p}{2}(|x(t)|^2)^{\frac{p}{2}-1}[2x(t)^T f(x(t)) + |Ax(\delta(t))|^2] \\
&+\frac{p}{4}(\frac{p}{2}-1)(|x(t)|^2)^{\frac{p}{2}-2}|2x(t)^T Ax(\delta(t))|^2)dt \\
&+\frac{p}{2}(|x(t)|^2)^{\frac{p}{2}-1}2x(t)^T Ax(\delta(t))dB(t) \\
&= (\frac{p}{2}|x(t)|^{p-2}[2x(t)^T f(x(t)) + |Ax(\delta(t))|^2] \\
&+p(\frac{p}{2}-1)|x(t)|^{p-4}|x(t)^T Ax(\delta(t))|^2)dt \\
&+p|x(t)|^{p-2}x(t)^T Ax(\delta(t))dB(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}(e^{\theta(k+1)\tau}|x((k+1)\tau)|^p) \\
&\le e^{\theta k\tau}\mathbb{E}|x(k\tau)|^p + \mathbb{E}\int_{k\tau}^{(k+1)\tau}(\theta e^{\theta s}|x(s)|^p \\
&+e^{\theta s}(\frac{p}{2}|x(s)|^{p-2}[2x(s)^T f(x(s)) + |Ax(\delta(s))|^2] \\
&+p(\frac{p}{2}-1)|x(s)|^{p-4}|x(s)^T Ax(\delta(s))|^2))ds.
\end{aligned}$$

Because of $\frac{p}{2}-1<0$ and according to all assumptions, it can be checked that

$$\begin{aligned}
&\mathbb{E}(e^{\theta(k+1)\tau}|x((k+1)\tau)|^p) \\
&\le e^{\theta k\tau}\mathbb{E}|x(k\tau)|^p + \mathbb{E}\int_{k\tau}^{(k+1)\tau}(\theta e^{\theta s}|x(s)|^p \\
&+e^{\theta s}(\frac{p}{2}|x(s)|^{p-2}[2K_1|x(s)|^2 + 2K_2|x(s)|^2 \\
&+2K_2|x(s)-x(\delta(s))|^2] \\
&+p(\frac{p}{2}-1)|x(s)|^{p-2}K_3|x(\delta(s))|^2))ds.
\end{aligned} \tag{9}$$

By the elementary inequality $a^2 \ge 1/2b^2 - |a-b|^2$ and notice that $\frac{p}{2}-1<0$ again, we have

$$\begin{aligned}
&p(\frac{p}{2}-1)|x(s)|^{p-2}K_3|x(\delta(s))|^2 \\
&\le p(\frac{p}{2}-1)|x(s)|^{p-2}K_3(\frac{1}{2}|x(s)|^2 \\
&-|x(s)-x(\delta(s))|^2).
\end{aligned} \tag{10}$$

Substituting (10) into (9), it follows that

$$\begin{aligned}
&\mathbb{E}(e^{\theta(k+1)\tau}|x((k+1)\tau)|^p) \\
&\le e^{\theta k\tau}\mathbb{E}|x(k\tau)|^p + \int_{k\tau}^{(k+1)\tau}[\theta + p(K_1+K_2 \\
&-\frac{K_3}{2}(1-\frac{p}{2}))]e^{\theta s}\mathbb{E}|x(s)|^p ds \\
&+\int_{k\tau}^{(k+1)\tau} p[K_2+(1-\frac{p}{2})K_3]e^{\theta s}\mathbb{E}(|x(s)|^{p-2} \\
&|x(s)-x(\delta(s))|^2)ds.
\end{aligned}$$

By Lemma 4, I wish we can somehow get

$$\begin{aligned}
&\int_{k\tau}^{(k+1)\tau} p[K_2+(1-\frac{p}{2})K_3]e^{\theta s}\mathbb{E}(|x(s)|^{p-2} \\
&|x(s)-x(\delta(s))|^2)ds \\
&\le \int_{k\tau}^{(k+1)\tau} p[K_2+(1-\frac{p}{2})K_3] \\
&\frac{K(\tau)}{1-K(\tau)}e^{\theta s}\mathbb{E}(|x(s)|^p)ds.
\end{aligned}$$

Then let p sufficient small so that

$$K_1+K_2 < \frac{K_3}{2}(1-\frac{p}{2}).$$

When τ is sufficient small, it follows that

$$[K_2+(1-\frac{p}{2})K_3]\frac{K(\tau)}{1-K(\tau)} < \frac{K_3}{2}(1-\frac{p}{2}) - (K_1+K_2).$$

Then, let

$$\begin{aligned}
\theta = &-p(K_1+K_2-\frac{K_3}{2}(1-\frac{p}{2}) \\
&+[K_2+(1-\frac{p}{2})K_3]\frac{K(\tau)}{1-K(\tau)}) > 0.
\end{aligned}$$

We can obtain

$$\mathbb{E}(e^{\theta(k+1)\tau}|x((k+1)\tau)|^p) \le e^{\theta k\tau}\mathbb{E}|x(k\tau)|^p, \tag{11}$$

which means

$$\mathbb{E}|x((k+1)\tau)|^p \le e^{\theta\tau}\mathbb{E}|x(k\tau)|^p. \tag{12}$$

By [11] we know that there exist a constant H such that

$$\mathbb{E}(\sup_{k\tau\le t\le(k+1)\tau}|x(t)|^p) \le H\mathbb{E}|x(k\tau)|^p \le H|x_0|^p e^{-k\theta\tau}, \tag{13}$$

for all $k \ge 1$.

So we have

$$P(\sup_{k\tau\le t\le(k+1)\tau}|x(t)|^p \ge e^{-\frac{1}{2}k\theta\tau}) \le H|x_0|^p e^{-\frac{1}{2}k\theta\tau}, \tag{14}$$

for all $k \ge 1$.

Then by the well-know Borel-Cantelli lemma, we have

$$\sup_{k\tau\le t\le(k+1)\tau}|x(t)|^p < e^{-\frac{1}{2}k\theta\tau}, \tag{15}$$

holds for all but finitely many k.

Thus, for almost all $\omega \in \Omega$, there exist an integer $k_0 = k_0(\omega)$ such that

$$\sup_{k\tau\le t\le(k+1)\tau}|x(t,\omega)|^p < e^{-\frac{1}{2}k\theta\tau}, \tag{16}$$

for all $k \ge k_0(\omega)$.

Therefore, for $k\tau \le t \le (k+1)\tau$ and $k \ge k_0$, we have

$$\frac{1}{t}\log(|x(t,\omega)|) < -\frac{\frac{1}{2}k\theta\tau}{p(k+1)\tau}. \tag{17}$$

Letting $t \to \infty$, we get

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t,\omega)|) < -\frac{\theta}{2p} < 0, \tag{18}$$

for almost all $\omega \in \Omega$. Which means that the system is almost sure exponential stable.

The proof is complete. ■

Remark 1: Let us consider an unstable linear ODE system:

$$\frac{dx(t)}{dt} = \alpha x(t). \tag{19}$$

Now we design a state feedback stochastic control $\beta x([t/\tau]\tau)dB(t)$ based on the a Brownian motion to stabilize the system:

$$dx(t) = \alpha x(t)dt + \beta x([t/\tau]\tau)dB(t) \quad t \ge 0, \tag{20}$$

where $x(0) = x_0 \in \mathbb{R}$, τ is a positive constant. Let us form this equation as a stochastic differential delay equation.

In the following theorem, the almost sure exponential stable of the stochastic system is proved.

Theorem 2: For any initial value $x_0 \in \mathbb{R}$, the controlled system (20) is almost sure exponential stable.

Proof: Let $t_k = k\tau$ for $k = 0, 1, 2, ...$ and set $x_k = x(t_k)$. For $t \in [t_k, t_{k+1}]$, $x(t)$ can be regarded as the solution to the following equation

$$dx(t) = \alpha x(t)dt + \beta x_k dB(t), \tag{21}$$

with initial value $x_k = x(t_k)$ at time t_k. Then, it can be checked that

$$x(t) = e^{\alpha(t-t_k)}x_k + \beta \int_{t_k}^{t} e^{\alpha(t-s)} x_k dB(s). \tag{22}$$

In particular,

$$\begin{aligned} & x_{k+1} \\ = \ & x_k e^{\alpha\tau} + \beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} x_k dB(s) \\ = \ & x_k (e^{\alpha\tau} + \beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} dB(s). \end{aligned}$$

Hence, for $p \in (0, 1)$, we obtain that

$$\mathbb{E}|x_{k+1}|^p = \mathbb{E}|x_k|^p \mathbb{E}|e^{\alpha\tau} + \beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} dB(s).$$

By the elementary inequality $|a+b+c|^p \leq 3^p(|a|^p + |b|^p + |c|^p)$ for any real numbers a, b and c and Burkholder-Davis-Gundy inequality, we derive

$$\begin{aligned} & \mathbb{E}|e^{\alpha\tau} + \beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} dB(s)|^p \\ \leq \ & 3^p \mathbb{E}(e^{\alpha\tau p} + |\beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} dB(s)|^p \\ \leq \ & 3^p e^{\alpha\tau p} + 3^p |\beta|^p c_p (\int_{t_k}^{t_{k+1}} e^{2\alpha(t_{k+1}-s)} ds)^{\frac{p}{2}} \\ \leq \ & e^{-\varepsilon\tau}, \end{aligned}$$

where c_p s a positive number dependent on p only and $\varepsilon > 0$.

Then, we obtain that

$$\mathbb{E}|x_{k+1}|^p = \mathbb{E}|x_k|^p e^{-\varepsilon\tau}, \forall k \geq 0. \tag{23}$$

Thus, we get

$$\mathbb{E}|x_{k+1}|^p = |x_0|^p e^{-\varepsilon(k+1)\tau}, \forall k \geq 0. \tag{24}$$

Note that

$$\begin{aligned} & \mathbb{E}(\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^p) = \mathbb{E}|x_k|^p \mathbb{E}(\sup_{t_k \leq t \leq t_{k+1}} |e^{\alpha\tau} \\ & + \beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} dB(s). \end{aligned}$$

By the same methods above, it follows that

$$\begin{aligned} & \mathbb{E}(\sup_{t_k \leq t \leq t_{k+1}} |e^{\alpha\tau} + \beta \int_{t_k}^{t} e^{\alpha(t_{k+1}-s)} dB(s) \\ \leq \ & 3^p e^{\alpha\tau p} + 3^p |\beta|^p c_p \tau^{\frac{p}{2}} e^{\alpha\tau p} \\ = \ & 3^p e^{\alpha\tau p}(1 + |\beta|^p c_p \tau^{\frac{p}{2}}). \end{aligned}$$

Let $3^p e^{\alpha\tau p}(1 + |\beta|^p c_p \tau^{\frac{p}{2}}) = C$, we obtain that

$$\mathbb{E}(\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^p) \leq C|x_0|^p e^{-\varepsilon k\tau}, \forall k \geq 0. \tag{25}$$

Since

$$\begin{aligned} & \mathbb{P}(\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^p \geq e^{-0.5\varepsilon k\tau}) \\ \leq \ & \frac{\mathbb{E}(\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^p)}{e^{-0.5\varepsilon k\tau}} \\ \leq \ & C|x_0|^p e^{-0.5\varepsilon k\tau}. \end{aligned}$$

By the Borel-Cantelli lemma, it can be checked that

$$\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^p < e^{-0.5\varepsilon k\tau}, \tag{26}$$

holds for all but finitely many k. That is, for almost all $\omega \in \Omega$, there is an integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_k \leq t \leq t_{k+1}} |x(t, \omega)|^p < e^{-0.5\varepsilon k\tau}, \forall k \geq k_0(\omega). \tag{27}$$

Therefore, for $t_k \leq t \leq t_{k+1}$ and $k \geq k_0$, we obtain that

$$\frac{1}{t} \log(|x(t, \omega)|) < -\frac{0.5\varepsilon k\tau}{p(k+1)\tau}. \tag{28}$$

Let $t \to \infty$, for almost all $\omega \in \Omega$, we get

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t, \omega)|) \leq -\frac{\varepsilon}{2p}. \tag{29}$$

■

IV. Linear Equations with Lévy Noises

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions that it is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let $B(t)$ be a scalar Brownian motion defined on the probability space. $N(t, y)$ is an l-dimensional $\mathscr{F}_t$-adapted Poisson random measure on $[0, +\infty) \times \mathbb{R}^l$ with compensator $\widetilde{N}(t, y)$ which satisfies $\widetilde{N}(dt, dy) = N(dt, dy) - \lambda\phi(dy)dt$, where λ is the probability density of Poisson process and ϕ is the probability distribution of y. $B(t)$ and $N(t, y)$ are independent.

Let us consider an unstable linear ODE system:

$$\frac{dx(t)}{dt} = \alpha x(t). \tag{30}$$

Now we design a state feedback stochastic control $\beta x([t/\tau]\tau)dB(t) + \int_Y x([t/\tau]\tau)N(dt, dy)$ based on the a Brownian motion and a Lévy noise to stabilize the system:

$$\begin{aligned} dx(t) &= \alpha x(t)dt + \beta x([t/\tau]\tau)dB(t) \\ &+ \int_Y x([t/\tau]\tau)N(dt, dy) \quad t \geq 0, \end{aligned} \tag{31}$$

where $x(0) = x_0 \in \mathbb{R}$, τ is a positive constant. Let us form this equation as a stochastic differential delay equation.

In the following theorem, the almost sure exponential stable of the stochastic system is proved.

Theorem 3: For any initial value $x_0 \in \mathbb{R}$, the controlled system (31) is almost sure exponential stable.

Proof: Let $t_k = k\tau$ for $k = 0, 1, 2, ...$ and set $x_k = x(t_k)$. For $t \in [t_k, t_{k+1}]$, $x(t)$ can be regarded as the solution to the following equation

$$dx(t) = \alpha x(t)dt + \beta x_k dB(t) + \int_Y x_k N(dt, dy), \tag{32}$$

with initial value $x_k = x(t_k)$ at time t_k. Then, it can be checked that

$$x(t) = e^{\alpha(t-t_k)}x_k + \beta\int_{t_k}^{t} e^{\alpha(t-s)}x_k dB(s) \quad (33)$$
$$+\int_{t_k}^{t}\int_Y e^{\alpha(t-s)}x_k N(ds,dy).$$

In particular,

$$\begin{aligned} x_{k+1} &= x_k e^{\alpha\tau} + \beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}x_k dB(s) \\ &\quad +\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}x_k N(ds,dy) \\ &= x_k(e^{\alpha\tau} + \beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}dB(s) \\ &\quad +\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}N(ds,dy)). \end{aligned}$$

Hence, for $p\in(0,1)$, we obtain that

$$\mathbb{E}|x_{k+1}|^p = \mathbb{E}|x_k|^p\mathbb{E}|e^{\alpha\tau} + \beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}dB(s)$$
$$+\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}N(ds,dy)|^p.$$

By the elementary inequality $|a+b+c|^p \le 3^p(|a|^p+|b|^p+|c|^p)$ for any real numbers a, b and c and Burkholder-Davis-Gundy inequality, we derive

$$\begin{aligned} &\mathbb{E}|e^{\alpha\tau} + \beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}dB(s) \\ &\quad +\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}N(ds,dy)|^p \\ \le\ & 3^p\mathbb{E}(e^{\alpha\tau p} + |\beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}dB(s)|^p \\ &\quad +|\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}N(ds,dy)|^p \\ \le\ & 3^p e^{\alpha\tau p} + 3^p|\beta|^p c_p(\int_{t_k}^{t_{k+1}} e^{2\alpha(t_{k+1}-s)}ds)^{\frac{p}{2}} \\ &\quad +3^p\lambda^{\frac{p}{2}}(\int_{t_k}^{t_{k+1}} e^{2\alpha(t_{k+1}-s)}ds)^{\frac{p}{2}} \\ \le\ & e^{-\varepsilon\tau}, \end{aligned}$$

where c_p s a positive number dependent on p only and $\varepsilon > 0$.

Then, we obtain that

$$\mathbb{E}|x_{k+1}|^p = \mathbb{E}|x_k|^p e^{-\varepsilon\tau}, \forall k\ge 0. \quad (34)$$

Thus, we get

$$\mathbb{E}|x_{k+1}|^p = |x_0|^p e^{-\varepsilon(k+1)\tau}, \forall k\ge 0. \quad (35)$$

Note from (33) that

$$\mathbb{E}(\sup_{t_k\le t\le t_{k+1}}|x(t)|^p) = \mathbb{E}|x_k|^p\mathbb{E}(\sup_{t_k\le t\le t_{k+1}}|e^{\alpha\tau}$$
$$+\beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}dB(s)$$
$$+\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}N(ds,dy)|^p).$$

By the same methods above, it follows that

$$\begin{aligned} &\mathbb{E}(\sup_{t_k\le t\le t_{k+1}}|e^{\alpha\tau} + \beta\int_{t_k}^{t} e^{\alpha(t_{k+1}-s)}dB(s) \\ &\quad +\int_{t_k}^{t}\int_Y e^{\alpha(t_{k+1}-s)}N(ds,dy)|^p) \\ \le\ & 3^p e^{\alpha\tau p} + 3^p|\beta|^p c_p\tau^{\frac{p}{2}}e^{\alpha\tau p} + 3^p\lambda^{\frac{p}{2}}e^{\alpha\tau p} \\ =\ & 3^p e^{\alpha\tau p}(1+|\beta|^p c_p\tau^{\frac{p}{2}} + \lambda^{\frac{p}{2}}). \end{aligned}$$

Let $3^p e^{\alpha\tau p}(1+|\beta|^p c_p\tau^{\frac{p}{2}} + \lambda^{\frac{p}{2}}) = C$, we obtain that

$$\mathbb{E}(\sup_{t_k\le t\le t_{k+1}}|x(t)|^p) \le C|x_0|^p e^{-\varepsilon k\tau}, \forall k\ge 0. \quad (36)$$

Since

$$\begin{aligned} &\mathbb{P}(\sup_{t_k\le t\le t_{k+1}}|x(t)|^p \ge e^{-0.5\varepsilon k\tau}) \\ \le\ & \frac{\mathbb{E}(\sup_{t_k\le t\le t_{k+1}}|x(t)|^p)}{e^{-0.5\varepsilon k\tau}} \\ \le\ & C|x_0|^p e^{-0.5\varepsilon k\tau}. \end{aligned}$$

By the Borel-Cantelli lemma, it can be checked that

$$\sup_{t_k\le t\le t_{k+1}}|x(t)|^p < e^{-0.5\varepsilon k\tau}, \quad (37)$$

holds for all but finitely many k. That is, for almost all $\omega\in\Omega$, there is an integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_k\le t\le t_{k+1}}|x(t,\omega)|^p < e^{-0.5\varepsilon k\tau}, \forall k\ge k_0(\omega). \quad (38)$$

Therefore, for $t_k\le t\le t_{k+1}$ and $k\ge k_0$, we obtain that

$$\frac{1}{t}\log(|x(t,\omega)|) < -\frac{0.5\varepsilon k\tau}{p(k+1)\tau}. \quad (39)$$

Let $t\to\infty$, for almost all $\omega\in\Omega$, we get

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t,\omega)|) \le -\frac{\varepsilon}{2p}. \quad (40)$$

The proof is complete. ■

V. Nonlinear equations with Lévy noises

In this section, we consider the stochastic feedback control with Lévy noises as follows:

$$\begin{aligned} dx(t) &= f(x(t))dt + Hx(\delta_t)dB(t) \\ &\quad +\int_Y x(\delta(t))N(dt,dy) \qquad t\ge 0, \end{aligned} \quad (41)$$

where $x(0) = x_0$, $H\in R^{n\times n}$, $N(t,y)$ is an l-dimensional $\mathscr{F}_t$-adapted Poisson random measure on $[0,+\infty)\times\mathbb{R}^l$ with compensator $\widetilde{N}(t,y)$ which satisfies $\widetilde{N}(t,y) = N(dt,dy) - \nu(dy)dt$, $\nu(dy)$ is a Lévy measure.

Then, the assumptions in Section 3 should be changed to

Assumption 3: Assume that the drift functions $f(x)$ is globally Lipschitz continuous

$$|f(x)-f(\xi)| + \int_Y |x-\xi|\nu(dy) \le K_1|x-\xi| \quad (42)$$

for all $x,\xi\in R^n$, where K_1 is a positive constant.

Assumption 4: Assume that the diffusion coefficient fulfill the following conditions

$$|Hx|^2 \le K_2|x|^2 \qquad \text{and} \qquad |x^T H\xi|^2 \ge K_3|x|^2|\xi|^2 \quad (43)$$

for all $x, \xi \in R^n$, where both K_2 and K_3 are positive constants.

Therefore, by using the technique of generalized Itô formula for Lévy stochastic integral, Borel-Cantelli lemma, Burkholder-Davis-Gundy inequality, Hölder inequality and Gronwall inequality, it fololows that

Lemma 5: Let Assumptions 3 and 4 hold. Let us define

$$K(\tau) = (12\tau^2 K_1^2 + 6\tau(K_2+1))e^{6\tau^2 K_1^2}, \tag{44}$$

If $\tau > 0$ is sufficiently small such that for $K(\tau) < 1$, then the solution of (19) satisfies

$$\mathbb{E}|x(s) - x(\delta(s))|^2 ds \le \frac{K(\tau)}{1-K(\tau)}\mathbb{E}|x(s)|^2 \tag{45}$$

for any $t \ge 0$.

Theorem 4: Let Assumptions 3 and 4 hold, the controlled system (19) is almost sure exponential stable.

Remark 2: The proof of Lemma 5 and Theorem 2 have some different places with Lemma 4 and Theorem 1 because of the Lévy noises. For example,

$$\begin{aligned}
&x(t) - x(\delta(t)) \\
&= x(t) - x(k\tau) \\
&= \int_{k\tau}^t f(x(s))ds + \int_{k\tau}^t Hx(k\tau)dB(s) \\
&+ \int_{k\tau}^t \int_Y x(k\tau)N(ds, dy).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}|x(t) - x(\delta(t))|^2 \\
&\le 3\mathbb{E}|\int_{k\tau}^t f(x(s))ds|^2 + 3\mathbb{E}|\int_{k\tau}^t Hx(k\tau)dB(s)|^2 \\
&+3\mathbb{E}|\int_{k\tau}^t \int_Y x(k\tau)N(ds, dy)|^2 \\
&\le 3\tau \int_{k\tau}^t \mathbb{E}|f(x(s))|^2 ds + 3\int_{k\tau}^t \mathbb{E}|Hx(k\tau)|^2 ds \\
&+3\mathbb{E}\int_{k\tau}^t \int_Y |x(k\tau)|\nu(dy)ds \\
&\le 3\tau K_1^2 \int_{k\tau}^t \mathbb{E}|x(s)|^2 ds + 3\tau K_2 \mathbb{E}|x(k\tau)|^2 \\
&+3\tau\mathbb{E}|x(k\tau)|^2 \\
&= 3\tau K_1^2 \int_{k\tau}^t \mathbb{E}|x(s)|^2 ds + 3\tau(K_2+1)\mathbb{E}|x(k\tau)|^2 \\
&\le 6\tau K_1^2 \int_{k\tau}^t \mathbb{E}|x(s) - x(\delta(t))|^2 ds \\
&+(6\tau^2 K_1^2 + 3\tau(K_2+1))\mathbb{E}|x(k\tau)|^2.
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}|x(t) - x(\delta(t))|^2 \\
&\le (6\tau^2 K_1^2 + 3\tau(K_2+1))e^{6\tau^2 K_1^2}\mathbb{E}|x(k\tau)|^2 \\
&\le (12\tau^2 K_1^2 + 6\tau(K_2+1))e^{6\tau^2 K_1^2}(\mathbb{E}|x(t) \\
&-x(k\tau)|^2 + \mathbb{E}|x(k\tau)|^2) \\
&\le \frac{K(\tau)}{1-K(\tau)}\mathbb{E}|x(t)|^2.
\end{aligned}$$

$$\begin{aligned}
&d|x(t)|^2 \\
&= [2x(t)^T f(x(t)) + |Hx(\delta(t))|^2]dt \\
&+2x(t)^T Hx(\delta(t))dB(t) + \int_Y x^2(\delta(t))\nu dy \\
&+2x(t)^T \int_Y x(\delta(t))N(dt, dy).
\end{aligned}$$

$$\begin{aligned}
&d|x(t)|^p \\
&= (\frac{p}{2}(|x(t)|^2)^{\frac{p}{2}-1}[2x(t)^T f(x(t)) + |Hx(\delta(t))|^2] \\
&+\frac{p}{4}(\frac{p}{2}-1)(|x(t)|^2)^{\frac{p}{2}-2}|2x(t)^T Hx(\delta(t))|^2)dt \\
&+\frac{p}{2}(|x(t)|^2)^{\frac{p}{2}-1}2x(t)^T Hx(\delta(t))dB(t) \\
&+\frac{p}{2}(|x(t)|^2)^{\frac{p}{2}-1}2x(t)^T \int_Y x(\delta(t))N(dt, dy) \\
&+\frac{p}{4}(\frac{p}{2}-1)(|x(t)|^2)^{\frac{p}{2}-2} \\
&(4(x(t)^T)^2 + 1)\int_Y x^2(\delta(t))\nu dy \\
&= (\frac{p}{2}|x(t)|^{p-2}[2x(t)^T f(x(t)) + |Hx(\delta(t))|^2] \\
&+p(\frac{p}{2}-1)|x(t)|^{p-4}|x(t)^T Hx(\delta(t))|^2)dt \\
&+p|x(t)|^{p-2}x(t)^T Hx(\delta(t))dB(t) \\
&+p|x(t)|^{p-2}x(t)^T \int_Y x(\delta(t))N(dt, dy) \\
&+\frac{p}{4}(\frac{p}{2}-1)|x(t)|^{p-2}(4(x(t)^T)^2 + 1)\int_Y x^2(\delta(t))\nu dy.
\end{aligned}$$

Remark 3: If the state feedback stochastic control is driven by α-stable noises as follows:

$$\begin{aligned}
&dx(t) = \alpha x(t)dt + \beta x([t/\tau]\tau)dB(t) \\
&+x([t/\tau]\tau)dZ(t) \quad t \ge 0,
\end{aligned}$$

where $Z = \{Z_t, t \ge 0\}$ is a strictly symmetric α-stable Lévy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$ and location parameter $\mu \in (-\infty, \infty)$ if it has the following characteristic function:

$$\phi_\eta(u) = \begin{cases} \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta sgn(u)\tan\frac{\alpha\pi}{2}) + i\mu u\} \\ \hfill if \alpha \ne 1, \\ \exp\{-\sigma|u|(1 + i\beta\frac{2}{\pi}sgn(u)\log|u|) + i\mu u\} \\ \hfill if \alpha = 1. \end{cases}$$

Then, the methods to prove the almost sure exponential stable is different from Lévy noises and it is more difficult. For example:

$$\begin{aligned}
&x(t) - x(\delta(t)) \\
&= x(t) - x(k\tau) \\
&= \int_{k\tau}^t f(x(s))ds + \int_{k\tau}^t Hx(k\tau)dB(s) \\
&+ \int_{k\tau}^t x(k\tau)dZ(s).
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta(t))|^2 \\
&\le 3\mathbb{E}|\int_{k\tau}^{t} f(x(s))ds|^2 + 3\mathbb{E}|\int_{k\tau}^{t} Hx(k\tau)dB(s)|^2 \\
&+3\mathbb{E}|\int_{k\tau}^{t} x(k\tau)dZ(s)|^2 \\
&\le 3\tau\int_{k\tau}^{t}\mathbb{E}|f(x(s))|^2 ds + 3\int_{k\tau}^{t}\mathbb{E}|Hx(k\tau)|^2 ds \\
&+3\mathbb{E}(\int_{k\tau}^{t}|x(k\tau)|^{2\alpha}ds)^{\frac{1}{\alpha}} \\
&\le 3\tau K_1^2\int_{k\tau}^{t}\mathbb{E}|x(s)|^2 ds + 3\tau K_2\mathbb{E}|x(k\tau)|^2 \\
&+3\tau^{\frac{1}{\alpha}}\mathbb{E}|x(k\tau)|^{2\alpha} \\
&\le 6\tau K_1^2\int_{k\tau}^{t}\mathbb{E}|x(s)-x(\delta(t))|^2 ds \\
&+(6\tau^2K_1^2+3\tau K_2+3\tau^{\frac{1}{\alpha}})\mathbb{E}|x(k\tau)|^2.
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta(t))|^2 \\
&\le (6\tau^2K_1^2+3\tau K_2+3\tau^{\frac{1}{\alpha}})e^{6\tau^2K_1^2}\mathbb{E}|x(k\tau)|^2 \\
&\le (12\tau^2K_1^2+6\tau K_2+6\tau^{\frac{1}{\alpha}})e^{6\tau^2K_1^2}(\mathbb{E}|x(t) \\
&-x(k\tau)|^2+\mathbb{E}|x(k\tau)|^2) \\
&\le \frac{K(\tau)}{1-K(\tau)}\mathbb{E}|x(t)|^2.
\end{aligned}$$

VI. Conclusions

In this paper, the almost sure exponential stable of nonlinear stochastic system by linear stochastic feedback control from discrete-time observations has been studied. By using Itô formula, Borel-Cantelli lemma, Burkholder-Davis-Gundy inequality, Hölder inequality and Gronwall inequality, the almost sure exponential stabilization of the nonlinear stochastic system has been discussed and the sufficient conditions have been provided. Moreover, the results have been extended to stochastic feedback control with Lévy noises. Further research topics will include the stabilisation problem of hybrid stochastic systems.

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