# Finite Volume Approximation of the Signorini Problem

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***Abstract*—In this work, we will use the finite volume method to approximate the problem of unilateral contact called Signorini problem. Furthermore, an error estimate with respect to the mesh is given.**

***Index Terms*—Signorini Problem, Unilateral Contact, Finite Volume Method, Error Estimate.**

## I. Introduction

THE study of problems in solid mechanics has become one of the most considered disciplinary subjects in physics and mathematics. Their models are governed by partial differential equations (PDEs), where their exact and/ or approximate solutions that are still of great importance, are unfortunately not very current. They are usually limited to simple situations.

In our work, we will try to approximate the problem of Signorini by a sheme based one finite volume method. To solve this problem, finite element shemes have been mostly used, see for example [2], [12], [13]. Also, in 2001, R. Herbin et al. [6] have approximated a class of variational inequalities by the finite volume method taking as conditions at the boundary Signorini conditions with the diffusion equation, taking in consideration recent works (see [3]–[5]).

However, the application of finite volume method to the Signorini problem is not an easy task due to the contact conditions on the boundary(see [7], [8]).

In 2015, T. Zhang et al. [9] have done a generalized finite difference analysis for the Signorini problem where some ambiguity are removed (see [10], [11]).
Then, in this context we will take the problem of Signorini analyzed by T. Zhang et al. [9] and we will approximate it with a finite volume sheme where an analysis for the numerical scheme is done as well as an error estimate is given.

Finally numerical test is given to improve the theoretical study.

## II. Contact Problem

The Signorini problem which we considered here can be stated as follows.

Let $\Omega$ be a polygonal domain of $\mathbb{R}^d$, $d = 2$ or 3, so that the boundary $\partial\Omega$ be composed of three non overlapping portions $\Gamma_1, \Gamma_2$ and $\Gamma_3$ which is the candidate for being in contact with a rigid frictionless obstacle.

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Now we consider the following Signorini problem

$$\left\{\begin{array}{l} -\operatorname{div}\left[a\left(x\right)\nabla u\right] = f \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_1 \\ a(x)\nabla u.\eta = 0 \text{ on } \Gamma_2 \\ \left.\begin{array}{c} u \geq \alpha \\ a(x)\nabla u.\eta \geq \beta \\ \left(u-\alpha\right)\left(a(x)\nabla u.\eta - \beta\right) = 0 \end{array}\right\} \text{ on } \Gamma_3 \end{array}\right. \tag{1}$$

under the following assumptions:

1) $\Omega$ is a bounded open polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3.
2) The boundary $\partial\Omega$ of $\Omega$ is composed of three non empty, non intersecting connected sets $\Gamma_1, \Gamma_2$ and $\Gamma_3$, such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3} = \partial\Omega$.
3) $f \in L^2\left(\Omega\right)$, $\eta$ is the unit outward normal to $\partial\Omega$.
4) $\alpha \in \mathbb{R}_-$ and $\beta \in \mathbb{R}$
5) Under the previous assumptions, the following hypothesis on $a$ are given:
   - $a$ is a piecewise $C^1$ function from $\overline{\Omega}$ to $\mathbb{R}^{d\times d}$ such that for all $x \in \overline{\Omega}$, $a\left(x\right)$ is a symmetric matrix.
   - There exists $a_0 \in \mathbb{R}_+^*$ such that $a\left(x\right)\xi.\xi \geq a_0$ for $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$.

## III. Preliminary

To establish the theoretical study, we need to state some definitions and lemmas.[4 ]-[6 ]

*Definition 1:* An admissible finite volume mesh of $\Omega$, denoted by $\mathcal{T}$ is given by a finite family of "control volumes", which are non intersecting open polygonal convex subsets of $\Omega$, a finite family of non intersecting subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, denoted by $\mathcal{E}$(these are the sides of the control volumes), with strictly positive $(d-1)$ dimensional measure, and a family of points of $\Omega$ denoted by $\mathcal{P}$ satisfying the following properties:

(i) The closure of the union of all the control volumes is $\overline{\Omega}$.
(ii) For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \overline{K} \setminus K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$ and $\cup_{K\in\mathcal{T}}\mathcal{E}_K = \mathcal{E}$.
(iii) For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$ either the $(d-1)$ dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is 0 or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, which will be denoted by $K \setminus L$.
(iv) The family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ is such that $x_K \in K$ (for all $K \in \mathcal{T}$) and if $K$ and $L$ are two neighbouring control volumes, it is assumed that $x_K \neq x_L$ and the straight line $\mathcal{D}_{K,L}$ going $x_K$ and $x_L$ is assumed to be orthogonal to $K \setminus L$.
(v) For any $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega$, there exists $i \in \{1, 2, 3\}$ such that $\sigma \subset \Gamma_i$.
(vi) For any $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega$, let $K$ be the control volume such that $\sigma \in \mathcal{E}_K$ and $\mathcal{D}_{K,\sigma}$ be the straight

line going through $x_K$ and orthogonal to $\sigma$; then $y_\sigma = \mathcal{D}_{K,\sigma} \cap \sigma$.

(vii) For any $K \in \mathcal{T}$, the restriction $a \mid_K$of the function $a$ to any given control volume $K$ belongs to $C^1\left(\overline{K}\right)$. Furthermore if $a$ is a piecewise $C^1$ function from $\overline{\Omega}$ to $\mathbb{R}^{d\times d}$, the orthogonality conditions $(iv)$ and $(vi)$ have to be modified into:

(iv)' For any $K \in \mathcal{T}$, let $a_K$ denote the mean value of $a$ on $K$, that is

$$a_K = \left| \frac{1}{m(K)} \int_K a(x)\, dx \right| \tag{2}$$

The set $\mathcal{T}$ is such that there exists a family of points $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ such that $x_K = \cap_{\sigma\in\mathcal{E}_K} \mathcal{D}_{K,\sigma,a} \in \overline{K}$.
Where $\mathcal{D}_{K,\sigma,a}$ is the straight line perpendicular to $\sigma$ with respect to the scalar product induced by $a_K^{-1}$ such that $\mathcal{D}_{K,\sigma,a} \cap \mathcal{D}_{L,\sigma,a} \neq \emptyset$ if $\sigma = K \setminus L$. And if $\sigma = K \setminus L$, let $y_\sigma = \mathcal{D}_{K,\sigma,a} \cap \sigma\, (= \mathcal{D}_{L,\sigma,a} \cap \sigma)$ and assume that $x_K \neq x_L$.

(vi) For any $\sigma \in \mathcal{E}_{ext}$, let $K$ be the control volume such that $\sigma \in \mathcal{E}_K$ and let $\mathcal{D}_{K,\sigma,a}$ be the straight line going through $x_K$ and orthogonal to $\sigma$ with respect to the scalar product induced by $a_K^{-1}$; then there exists $y_\sigma \in \sigma \cap \mathcal{D}_{K,\sigma,a}$; let $g_\sigma = g(y_\sigma)$.

*Notation 2:* Let size($\mathcal{T}$)=sup$\{diam(K), K \in \mathcal{T}\}$
For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $m(K)$ is the $d$-dimensional Lebesgue measure of $K$ and $m(\sigma)$ is the $(d-1)$-dimensional Lebesgue measure of $\sigma$.
The set of interior( resp. boundary) edges is denoted by $\mathcal{E}_{int}$ (resp. $\mathcal{E}_{ext}$), that is $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$.The set of neighbors of $K$ is denoted by $\mathcal{N}(K)$, that is $\mathcal{N}(K) = \left\{L \in \mathcal{T}; \exists\sigma \in \mathcal{E}_K : \sigma = \overline{K} \cap \overline{L}\right\}$.
If $\sigma = K \setminus L$, we denote by $d_\sigma$ or $d_{K\setminus L}$ the Euclidean distance between $x_K$ and $x_L$.If $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{ext}$, we denote by $d_\sigma$the Euclidiean distance between $x_K$ and $y_\sigma$. For any control volume $K$ and any edge $\sigma \in \mathcal{E}_K$, we shall denote by $d_{K,\sigma}$the distance between $x_K$ and $\sigma$.
For any $\sigma \in \mathcal{E}$; the transmissivity through $\sigma$ is defined by $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$.

*Definition 3:* Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1. We define $X(\mathcal{T})$ as the set of functions from $\Omega_\mathcal{T} = \cup_{K\in\mathcal{T}} K \cup \cup_{\sigma\subset\Gamma_3} \sigma$ to $\mathbb{R}$ which are constant over each control volume of the mech and over each edge of $\mathcal{E}_{ext}$ which is included in $\Gamma_3$.

*Definition 4:* Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1. We define the operator $\gamma$ from $X(\mathcal{T})$ to $L^2(\partial\Omega)$ such that: let $u \in X(\mathcal{T})$, let $u_K$ be the value of $u$ in the control volume $K$ and $u_\sigma$ be the value of $u$ in the edge $\sigma$, for $\sigma \subset \Gamma_3$; let us define

$$\left\{ \begin{array}{c} \gamma(u) = u_\sigma \text{ on } \sigma, \text{ if } \sigma \subset \Gamma_3 \\ \gamma(u) = u_K \text{ on } \sigma, \text{ if } \sigma \subset \Gamma_2 \text{ and } \sigma \in \mathcal{E}_K \\ \gamma(u) = 0 \text{ on } \sigma, \text{ if } \sigma \subset \Gamma_1 \end{array} \right. \tag{3}$$

*Definition 5:* Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1. For $u \in X(\mathcal{T})$ the discrete $H_0^1$ norm is defined by

$$\|u\|_{1,\mathcal{T}} = \left( \sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma u)^2 \right)^{\frac{1}{2}}, \tag{4}$$

with

$$\left\{ \begin{array}{c} |D_\sigma u| = |u_K - u_L| \text{ if } \sigma \in \mathcal{E}_{int}, \sigma = K \setminus L \\ D_\sigma u = -u_K \text{ if } \sigma \subset \Gamma_1, \sigma \in \mathcal{E}_K \\ D_\sigma u = 0 \text{ if } \sigma \subset \Gamma_2 \\ D_\sigma u = u_\sigma - u_K \text{ if } \sigma \subset \Gamma_3, \sigma \in \mathcal{E}_K \end{array} \right. \tag{5}$$

*Lemma 6:* Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1, and $u \in X(\mathcal{T})$, then

$$\|u\|_{L^2(\Omega)} \leq diam(\Omega) \|u\|_{1,\mathcal{T}}. \tag{6}$$

*Lemma 7:* Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1 and $u \in X(\mathcal{T})$. Let us denote by $u_K$ the value of $u$ in the control volume $K$ and $u_\sigma$ the value of $u$ in the edge $\sigma$, for $\sigma \subset \Gamma_3$. Let $\gamma(u) \in L^2(\partial\Omega)$ (see Definition 4), then there exists $C$ depending only on $\Omega$, such that

$$\|\gamma(u)\|_{L^2(\partial\Omega)} \leq C \left( \|u\|_{1,\mathcal{T}} + \|u\|_{L^2(\Omega)} \right). \tag{7}$$

## IV. Finite volume scheme

In order to obtain the finite volume scheme, we first integrating the first equation of problem (1) over each control volume $K$, then we approximate the normal derivative on each cell boundary by finite differences and by taking into account the boundary conditions of problem (1), we obtain the following scheme

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = m(K) f_K, \tag{8}$$

where

$$f_K = \frac{1}{m(K)} \int_K f(x)\, dx, \tag{9}$$

such that on internal edges

$$F_{K,\sigma} = -\tau_\sigma (u_L - u_K) \text{ if } \sigma \in \mathcal{E}_{int}, \sigma = K \setminus L \tag{10}$$

with

$$\tau_\sigma = \left\{ \begin{array}{l} m(\sigma) \frac{a_{K,\sigma} a_{L,\sigma}}{a_{K,\sigma} d_{L,\sigma} + a_{L,\sigma} d_{K,\sigma}} \\ \\ \text{if } y_\sigma \neq x_K \text{ and } y_\sigma \neq x_L, \\ m(\sigma) \frac{a_{K,\sigma}}{d_{K,\sigma}} \\ \text{if } y_\sigma \neq x_K \text{ and } y_\sigma = x_L, \end{array} \right. \tag{11}$$

and on boundary edges:

$$F_{K,\sigma} = \left\{ \begin{array}{ll} -\tau_\sigma (u_\sigma - u_K) & \text{if } \sigma \in \mathcal{E}_{ext}, \sigma \in \mathcal{E}_K \\ \tau_\sigma u_K, & \forall\sigma \subset \Gamma_1, \sigma \in \mathcal{E}_K \\ 0, & \forall\sigma \subset \Gamma_2, \sigma \in \mathcal{E}_K \\ -\tau_\sigma (u_\sigma - u_K), & \forall\sigma \subset \Gamma_3, \sigma \in \mathcal{E}_K \end{array} \right. \tag{12}$$

with

$$\tau_\sigma = m(\sigma) \frac{a_{K,\sigma}}{d_{K,\sigma}} \tag{13}$$

$$u_\sigma \geq \alpha, \forall\sigma \subset \Gamma_3 \tag{14}$$

$$-F_{K,\sigma} \geq m(\sigma)\beta, \forall\sigma \subset \Gamma_3 \tag{15}$$

$$(u_\sigma - \alpha) \left( \frac{F_{K,\sigma}}{m(\sigma)} + \beta \right) = 0, \forall\sigma \subset \Gamma_3 \tag{16}$$

## V. Existence and uniqueness of discrete solution

In order to show the existence and uniqueness of $U = \left((u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}\right)$ where $u_K$ and $u_\sigma$ satisfy equations (8)-(16), we derive an equivalent variational formulation.

*Lemma 8:* Under assumptions 1-5, let $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1; and let $u_\mathcal{T} \in X(\mathcal{T})$ defined by $u_\mathcal{T} = u_K$ for $x \in K$, for all $K \in \mathcal{T}$ and by $u_\mathcal{T}(x) = u_\sigma$ for $x \in \sigma$, for all $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$, then $(u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}$ is solution to problem $(8)-(14)$ if and only if $u_\mathcal{T}$ is solution to the following problem:

$$\left\{\begin{array}{c} u_\mathcal{T} \in \mathcal{K}_\mathcal{T} = \{v \in X(\mathcal{T}), s.t. v_\sigma \geq \alpha, \forall\sigma \subset \Gamma_3\}, \\ \text{such that} \\ A(u_\mathcal{T}, v - u_\mathcal{T}) \geq L(v - u_\mathcal{T}), \forall v \in \mathcal{K}_\mathcal{T}, \end{array}\right. \tag{17}$$

where

$$\begin{aligned} A(u,v) = & \sum_{\sigma=K\backslash L\in\mathcal{E}_{int}} \tau_\sigma (u_K - u_L)(v_K - v_L) \\ & + \sum_{\sigma\in\mathcal{E}_{ext}} \tau_\sigma (D_\sigma u)(D_\sigma v) \;\; \forall u, v \in \mathcal{K}_\mathcal{T}, \end{aligned} \tag{18}$$

and

$$L(u) = \sum_{\sigma\subset\Gamma_3} \beta u_\sigma m(\sigma) + \sum_{K\in\mathcal{T}} m(K) f_K u_K, \forall u \in \mathcal{K}_\mathcal{T}. \tag{19}$$

where $\tau_\sigma$ is defined in (11).

*Proof:* Let $u_\mathcal{T} \in X(\mathcal{T})$ (see Definition 3) defined by $u_\mathcal{T} = u_K$ for $x \in K$ and for all $K \in \mathcal{T}$ and by $u_\mathcal{T}(x) = u_\sigma$ for $x \in \sigma$, for all $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$.

Let us assume that $(u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}$ satisfy $(8)-(14)$ and assume that $f_K = 0$, let us show that $u_\mathcal{T}$ satisfies problem (15), $u_\mathcal{T}$ is clearly in $\mathcal{K}_\mathcal{T}$. Let $v \in \mathcal{K}_\mathcal{T}$, multiplying (8) by $(v_K - u_K)$ and summing over $K$ leads to

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(v_K - u_K) = 0. \tag{20}$$

Which gives

$$A(u_\mathcal{T}, v - u_\mathcal{T}) = \sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \tau_\sigma (D_\sigma u_\mathcal{T})(v_\sigma - u_\sigma). \tag{21}$$

Let us show that

$$\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \tau_\sigma (D_\sigma u_\mathcal{T})(v_\sigma - u_\sigma) \geq L(v - u_\mathcal{T}) \tag{22}$$

Let $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$ and $K \in \mathcal{T}$ such that $\sigma \in \mathcal{E}_K$, then

$$\begin{aligned} -F_{K,\sigma}(v_\sigma - u_\sigma) = & (-F_{K,\sigma} - m(\sigma)\beta)(v_\sigma - \alpha) \\ & + (-F_{K,\sigma} - m(\sigma)\beta)(\alpha - u_\sigma) \quad (23) \\ & + m(\sigma)\beta(v_\sigma - u_\sigma). \quad (24) \end{aligned}$$

Using equation (16), we obtain

$$(-F_{K,\sigma} - m(\sigma)\beta)(\alpha - u_\sigma) = 0 \tag{25}$$

and

$$(-F_{K,\sigma} - m(\sigma)\beta)(v_\sigma - \alpha) \geq 0. \tag{26}$$

Thus, $\forall\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$, we have

$$\tau_\sigma (D_\sigma u_\mathcal{T})(v_\sigma - u_\sigma) \geq m(\sigma)\beta(v_\sigma - u_\sigma). \tag{27}$$

It follows that $u_\mathcal{T}$ satisfy problem (17).

Now assume that $u_\mathcal{T} \in X(\mathcal{T})$ satisfies problem (17), then $(u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}$ is a solution to problem (8)-(16).

Indeed, let $K_0 \in \mathcal{T}$ and let us prove that

$$\sum_{\sigma\in\mathcal{E}_{K_0}} F_{K_0,\sigma} = 0,$$

where $F_{K_0,\sigma}$ is defined by (11)-(12).

For that, let $v = u_\mathcal{T} \pm w, v \in \mathcal{K}_\mathcal{T}$ with $w \in X(\mathcal{T})$, such that

$$\left\{\begin{array}{l} w_{K_0} = 1 \\ w_K = 0 \end{array}\right. \forall K \in \mathcal{T}, K \neq K_0$$

and $w_\sigma = 0, \forall\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$.
One may take $v = u_\mathcal{T} + w$ (respectively $v = u_\mathcal{T} - w$) in (17), to get $A(u_\mathcal{T}, w) \geq 0$ (respectively $A(u_\mathcal{T}, w) \leq 0$).
Thus, $A(u_\mathcal{T}, w) = 0$
In other terms we obtain

$$A(u_\mathcal{T}, w) = \sum_{\sigma\in\mathcal{E}_{K_0}} F_{K_0,\sigma} = 0.$$

Also, since $u_\mathcal{T} \in \mathcal{K}_\mathcal{T}$ it is clear that $u_\mathcal{T}$ satisfies equations (8)-(14).

Now, let us check that $u_\mathcal{T}$ satisfies equation (15).
Let $\sigma_0 \in \mathcal{E}_{ext} \cap \mathcal{E}_K, \sigma_0 \subset \Gamma_3$ and let $v = u_\mathcal{T} + w, v \in \mathcal{K}_\mathcal{T}$ with $w \in X(\mathcal{T})$ such that $w_K = 0, \forall K \in \mathcal{T}$ and

$$\left\{\begin{array}{l} w_{\sigma_0} = 1 \\ w_\sigma = 0 \end{array}\right., \forall\sigma_0 \subset \Gamma_3, \sigma \neq \sigma_0.$$

Substituting $v$ in (17) yields that

$$\tau_{\sigma_0}(u_{\sigma_0} - u_K) \geq \beta m(\sigma_0).$$

Let $\Gamma_\alpha = \{\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3 \; s.t \, u_\alpha \geq \alpha\}$, $\sigma_0 \in \Gamma_\alpha$ and let $v = u_\tau - \mu w$, $v \in \mathcal{K}_\mathcal{T}$ with $w \in X(\mathcal{T})$, such that $w_K = 0, \forall K \in \mathcal{T}$ and

$$\left\{\begin{array}{l} w_{\sigma_0} = 1 \\ w_\sigma = 0 \end{array}\right., \forall\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3, \sigma, \neq \sigma_0$$

and $\mu = u_{\tau_0} - \alpha$.
By taking $v$ in (17), we obtain

$$\tau_{\sigma_0}(u_{\sigma_0} - u_K) \geq \beta m(\sigma_0).$$

Which proves that

$$\tau_{\sigma_0}(u_{\sigma_0} - u_K) = \beta m(\sigma_0).$$

■

We will use Lemma 3 in the following result.

*Proposition 9 (Existence, Uniqueness and stability):* Under assumptions 1-5, let $\mathcal{T}$ be an admissible finite volume mesh in the sense of Definition 1; there exists a unique solution $(u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}$ to problem (8)-(16). We may then define $u_\mathcal{T} \in X(\mathcal{T})$ by $u_\mathcal{T} = u_K$ for $x \in K$, for all $K \in \mathcal{T}$ and by $u_\mathcal{T}(x) = u_\sigma$ for $x \in \sigma$, for all $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$. There exists $c > 0$ only depending on $\Omega$, $f$ and $\beta$, such that

$$\|u_\mathcal{T}\|_{1,\mathcal{T}} \leq c \text{ and } \|u_\mathcal{T}\|_{L^2(\Omega)} \leq c \tag{28}$$

where $\|.\|_{1,\mathcal{T}}$ is defined in Definition 5.

*Proof: Step 1 (existence and uniqueness)*
The bilinear form $A$ defined by (18) and the linear form $L$ defined by (19) are continuous on the Hilbert space $X(\mathcal{T})$. Furthermore $A(u_\mathcal{T}, u_\mathcal{T}) = \|u_\mathcal{T}\|_{1,\mathcal{T}}^2, \forall u_\mathcal{T} \in X(\mathcal{T})$.
By Stampacchia's theorem there exists a unique solution $u_\mathcal{T}$ to problem (17).

Hence by lemma 3, there exists a unique solution $(u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}$ to problem (8)-(16).

*Step 2 (stability)*
Now, we will prove (28). Let $u_\mathcal{T}$ be a solution to (17). By taking $v = 0$ we get

$$\|u_\mathcal{T}\|_{1,\mathcal{T}}^2 \le L(u_\mathcal{T}). \tag{29}$$

Also, from (19) and using the triangular inequality, we obtain

$$|L(u_\mathcal{T})| \le \left|\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \beta u_\sigma m(\sigma)\right| + \left|\sum_{K\in\mathcal{T}} m(K) f_K u_K\right|. \tag{30}$$

Using the Cauchy-Schwartz inequality, it yields that

$$\left|\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \beta u_\sigma m(\sigma)\right| \le |\beta| \left(\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} u_\sigma^2 m(\sigma)\right)^{\frac{1}{2}} \times \left(\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} m(\sigma)\right)^{\frac{1}{2}} \tag{31}$$

and

$$\left|\sum_{K\in\mathcal{T}} m(K) f_K u_K\right| \le \left(\sum_{K\in\mathcal{T}} m(K) u_K^2\right)^{\frac{1}{2}} \times \left(\sum_{K\in\mathcal{T}} m(K) f_K^2\right)^{\frac{1}{2}}. \tag{32}$$

Then, we have

$$\left|\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \beta u_\sigma m(\sigma)\right| \le |\beta| \sqrt{m(\Gamma_3)}\, \|\gamma(u_\mathcal{T})\|_{L^2(\Gamma_3)} \tag{33}$$

and

$$\left|\sum_{K\in\mathcal{T}} m(K) f_K u_K\right| \le \|u\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}. \tag{34}$$

Using the trace and Poincaré Inequalities [1], [6], this yields we obtain

$$|L(u_\mathcal{T})| \le c\, \|u_\mathcal{T}\|_{1,\mathcal{T}}. \tag{35}$$

By the discrete Poincaré Inequality the second inequality follows immediately, see Lemma 1. ■

## VI. Error estimate

To establish the error estimate we refer to [5], [6], [14], [15].

*Theorem 10:* Under assumptions 1-5, let $\mathcal{T}$ be an admissible mesh as defined in Definition 1, let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3.
Let $\lambda = \min_{x\in K}\min_{\sigma\in\mathcal{E}_K} \frac{d_{K,\sigma}}{diam(K)}$ and $u_\mathcal{T} \in X(\mathcal{T})$ be defined by $u_\mathcal{T}(x) = u_K$ for a.e $x \in K$, for all $K \in \mathcal{T}$ and by $u_\mathcal{T}(x) = u_\sigma$ for a.e $x \in \sigma$, for all $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$ where $\left((u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}\right)$ is the solution to (8)-(16).
Assume that the unique variational solution of problem (1) satisfies $u \in H^2(\Omega)$. For each $K \in \mathcal{T}$, let $e_K = u(x_K) - u_K$ and for each $\sigma \in \mathcal{E}_{ext}$ such that $\sigma \subset \Gamma_3$ let $e_\sigma = u(y_\sigma) - u_\sigma$ and let $e_\mathcal{T} \in X(\mathcal{T})$ be defined by $e_\mathcal{T}(x) = e_K$ for $x \in K$, for all $K \in \mathcal{T}$ and by $e_\mathcal{T}(x) = e_\sigma$ for $x \in \sigma$, for all $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$.
Then, there exists $C \in \mathbb{R}^+$ depending only on $u, \beta, \lambda, \lambda_1, \lambda_2, f$ and $\Omega$ such that

$$\|e_\mathcal{T}\|_{1,\mathcal{T}} \le C\ size(\mathcal{T}) \tag{36}$$

and

$$\|e_\mathcal{T}\|_{L^2(\Omega)} \le C\ size(\mathcal{T}). \tag{37}$$

where $\|.\|_{1,\mathcal{T}}$ is the discrete $H^2$ norm defined in preliminary E, $size(\mathcal{T}) = \sup_{K\in\mathcal{T}} diam(K)$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$ satisfy

$$\begin{aligned} \lambda_1 (size(\mathcal{T}))^2 &\le m(K) \le \lambda_2 (size(\mathcal{T}))^2 \\ \lambda_1 size(\mathcal{T}) &\le m(\sigma) \le \lambda_2 size(\mathcal{T}) \\ \lambda_1 size(\mathcal{T}) &\le d_\sigma \le \lambda_2 size(\mathcal{T}). \end{aligned}$$

*Proof:* Let $u_\mathcal{T} \in X(\mathcal{T})$ be defined in $\Omega$ by $u_\mathcal{T}(x) = u_K$ for $x \in K$, for all $K \in \mathcal{T}$ and by $u_\mathcal{T}(x) = u_\sigma$ for $x \in \sigma$ for all $\sigma \in \mathcal{E}_{ext}, \sigma \subset \Gamma_3$, where $\left((u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}\right)$ is the solution to (8)-(16).
So for any $K \in \mathcal{T}$, we define the exact diffusion flux by $\overline{F}_{K,\sigma}$, where $\overline{F}_{K,\sigma} = -\int_\sigma a(x) \nabla u(x) \eta_{K,\sigma} d\gamma(x)$, and suppose that for all $K \in \mathcal{T}$, $\sum_{K\in\mathcal{T}} m(K) f_K = 0$, then we have

$$\sum_{\sigma\in\mathcal{E}_K} \overline{F}_{K,\sigma} = 0. \tag{38}$$

Let $F^*_{K,\sigma}$ be defined by

$$F^*_{K,\sigma} = \begin{cases} -\tau_\sigma (u(x_L) - u(x_K)) & \text{if } \sigma \in \mathcal{E}_{int}, \sigma = K \setminus L \\ \tau_\sigma u(x_K), & \forall \sigma \subset \Gamma_1, \sigma \in \mathcal{E}_K \\ 0, & \forall \sigma \subset \Gamma_2, \sigma \in \mathcal{E}_K \\ -\tau_\sigma (u(y_\sigma) - u(x_K)) & \forall \sigma \subset \Gamma_3, \sigma \in \mathcal{E}_K \end{cases}$$

Therefore, the consistency error on the diffusion flux may be defined as

$$R_{K,\sigma} = \frac{1}{m(\sigma)} \left(\overline{F}_{K,\sigma} - F^*_{K,\sigma}\right). \tag{39}$$

Assuming that the unique variational solution $u$ to problem (1) belongs to $H^2(\Omega)$ and thanks to the regularity of $u$, there exists $C_1 \in \mathbb{R}^+$, depending only on $\left\|D^2 u\right\|_{L^2(\Omega)}, \lambda$ and $\lambda_2$, such that

$$m(\sigma) d_\sigma |R_{K,\sigma}|^2 \le C_1 (size(\mathcal{T}))^2, \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K. \tag{40}$$

Subtracting (8) from the equation (38), using (39) and the regularity of $u$, it yields that

$$\sum_{\sigma\in\mathcal{E}_K} \left(F^*_{K,\sigma} - F_{K,\sigma}\right) = -\sum_{\sigma\in\mathcal{E}_K} m(\sigma) R_{K,\sigma}. \tag{41}$$

Let $H_{K,\sigma} = F^*_{K,\sigma} - F_{K,\sigma}$ such that

$$H_{K,\sigma} = \begin{cases} -\tau_\sigma (e_L - e_K) & \text{if } \sigma \in \mathcal{E}_{int}, \sigma = K \setminus L \\ \tau_\sigma e_K & \forall \sigma \subset \Gamma_1, \sigma \in \mathcal{E}_K \\ = 0 & \forall \sigma \subset \Gamma_2, \sigma \in \mathcal{E}_K \\ -\tau_\sigma (e_\sigma - e_K) & \forall \sigma \subset \Gamma_3, \sigma \in \mathcal{E}_K. \end{cases}$$

Multiplying (41) by $e_K$, summing for $K \in \mathcal{T}$, and thanks to the conservativity of $H_{K,\sigma}$ i.e. $H_{K,\sigma} = -H_{L,\sigma}$ for any $\sigma \in \mathcal{E}_{int}$ such that $\sigma = K \setminus L$ , then

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = -\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} H_{K,\sigma}e_\sigma - \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} m(\sigma) R_{K,\sigma}e_\sigma, \tag{42}$$

where $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}$ is defined by (4)-(5).
Reordering the terms in (42) and using the property of conservativity of $H_{K,\sigma}$, one has for any $\sigma \in \mathcal{E}_{int}$ such that $\sigma = K \setminus L$

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \le \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \left(-H_{K,\sigma} - m(\sigma) R_{K,\sigma}\right) e_\sigma + \sum_{\sigma\in\mathcal{E}} m(\sigma) |R_{K,\sigma}| |D_\sigma e_{\mathcal{T}}| .$$

Now, using the fact that

$$\left(-H_{K,\sigma} - m(\sigma) R_{K,\sigma}\right) e_\sigma = \left(F_{K,\sigma} - \overline{F}_{K,\sigma}\right) e_\sigma$$

and using the last equation in (1) and (16), we obtain

$$\begin{aligned}
&(-H_{K,\sigma} - m(\sigma) R_{K,\sigma})e_\sigma &(43)\\
&\quad = \left(F_{K,\sigma} + m(\sigma)\beta\right)\left(u(y_\sigma) - \alpha\right)\\
&\qquad + \left(F_{K,\sigma} + m(\sigma)\beta\right)\left(\alpha - u_\sigma\right)\\
&\qquad + \int_\sigma \left(a(x)\nabla u(x)\eta_{K,\sigma}d\gamma(x) - \beta\right)\left(u(y_\sigma) - \alpha\right)\\
&\qquad + \int_\sigma \left(a(x)\nabla u(x)\eta_{K,\sigma}d\gamma(x) - \beta\right)\left(\alpha - u_\sigma\right) &(44)
\end{aligned}$$

Introducing the Signorini conditions stated in (1),(14)-(16), the previous equation becomes

$$\begin{aligned}
&\left(-H_{K,\sigma} - m(\sigma) R_{K,\sigma}\right) e_\sigma\\
&\le \int_\sigma \left(a(x)\nabla u(x)\eta_{K,\sigma}d\gamma(x) - \beta\right)\left(u(y_\sigma) - \alpha\right).
\end{aligned}$$

If $u \neq \alpha$ on $\sigma$, for a given edge $\sigma$, then $a(x)\nabla u(x)\eta_{K,\sigma}d\gamma(x) = 0$ on $\sigma$, so

$$\left(-H_{K,\sigma} - m(\sigma) R_{K,\sigma}\right) e_\sigma \le 0.$$

And if $u = \alpha$ on a subset of $\sigma$ whose measure is non zero, then by the regularity of $u$, there exists $C_2 > 0$ only depending on $D^2u$, $u$ and $\beta$ such that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{ext},\sigma\subset\Gamma_3} \left(-H_{K,\sigma} - m(\sigma) R_{K,\sigma}\right) e_\sigma\\
&\le C_2 m(\Gamma_3)\left(size(\mathcal{T})\right)^2. &(45)
\end{aligned}$$

Using Cauchy-Schwartz inequality and last inequality we get the following estimate

$$\begin{aligned}
\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \le &\left(\sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma |R_{K,\sigma}|^2\right)^{\frac{1}{2}}\\
&\times \left(\sum_{\sigma\in\mathcal{E}} \frac{m(\sigma)}{d_\sigma} |D_\sigma e_{\mathcal{T}}|^2\right)^{\frac{1}{2}} + C_3\left(size(\mathcal{T})\right)^2
\end{aligned} \tag{46}$$

Using (40), there exists $C_4 > 0$ only depending on $u, \lambda, \lambda_1, \lambda_2$ and $\Omega$, such that

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \le C_4 size(\mathcal{T}) \|e_{\mathcal{T}}\|_{1,\mathcal{T}} + C_3\left(size(\mathcal{T})\right)^2.$$

By Young's inequality, there exists $C > 0$ depending only on $u, \beta, \lambda, \lambda_1, \lambda_2$ and $\Omega$ such that

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \le C size(\mathcal{T}). \tag{47}$$

The second inequality follows immediately from Lemma 1. ■

## VII. Numerical test

In order to satisfy theoretical study, we consider the following problem:

$$\left\{\begin{array}{l}
-\operatorname{div}\left[a(x)\nabla u\right] = f \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \Gamma_1,\\
a(x)\nabla u.\eta = 0 \quad \text{on } \Gamma_2,\\
\left.\begin{array}{l}
u \ge \alpha\\
a(x)\nabla u.\eta \ge \beta\\
(u-\alpha)\left(a(x)\nabla u.\eta - \beta\right) = 0
\end{array}\right\} \quad \text{on } \Gamma_3.
\end{array}\right. \tag{48}$$

with $\Omega = ]0, x_m[ \times ]0, y_m[$ , the boundary $\partial\Omega$ of $\Omega$ is composed of three non empty, non intersecting connected sets $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, such that [4]:

$$\begin{aligned}
\Gamma_1 &= \left\{(x,y) \in \overline{\Omega} \text{ s.t. } x = 0\right\},\\
\Gamma_2 &= \left\{(x,y) \in \overline{\Omega} \text{ s.t. } x = x_m\right\} \cup \left\{(x,y) \in \overline{\Omega} \text{ s.t. } y = y_m\right\},\\
\Gamma_3 &= \left\{(x,y) \in \overline{\Omega} \text{ s.t. } y = 0\right\},
\end{aligned}$$

$\alpha, \beta \in \mathbb{R}$, $f \in L^2(\Omega)$ , $\eta$ is the unit normal vector to $\partial\Omega$ outward to $\Omega$.

Hence for some $\alpha, \beta < 0$, $u$ satisfies problem (48), with $f \in C\left(\overline{\Omega}\right)$.

The function $f$ and the matrix $a(x)$ are chosen such that problem (48) has a solution $u$ which satisfies:

1) $u \in C^2\left(\overline{\Omega}\right)$,
2) $u(x,0) \ge \alpha$, $a(x)\nabla u(x,0).\eta = \beta$, $\forall x \in \left[0, \frac{x_m}{2}\right]$,
3) $u(x,0) = \alpha$, $a(x)\nabla u(x,0).\eta \ge \beta$, $\forall x \in \left[\frac{x_m}{2}, x_m\right]$,
4) $a(x)\nabla u(x,y).\eta = 0$, $\forall x \in \Gamma_2$.

Let $\mathcal{T}$ be an admissible mesh, in the sense of Definition1; we choose $x_K$ at the center of the control volume $K \in \mathcal{T}$ and assume that each cell $K$ is a rectangle.

The finite volume scheme of problem (48) is given as follow:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = m(K) f_K , \quad \forall K \in \mathcal{T},$$

where $F_{K,\sigma}$ and $f_K$ are defined by $(9) - (16)$.

The continous problem which we wish to solve is non linear because of the signorini conditions. Thanks to the monotony algorithm proposed in [3]-[6],[16], we calculate the solution $U = \left((u_K)_{K\in\mathcal{T}}, (u_\sigma)_{\sigma\subset\Gamma_3}\right)$ of problem $(48), (9) - (16)$.
We give below graphical representation of $u$, of the trace of $u$ on $\Gamma_3$.

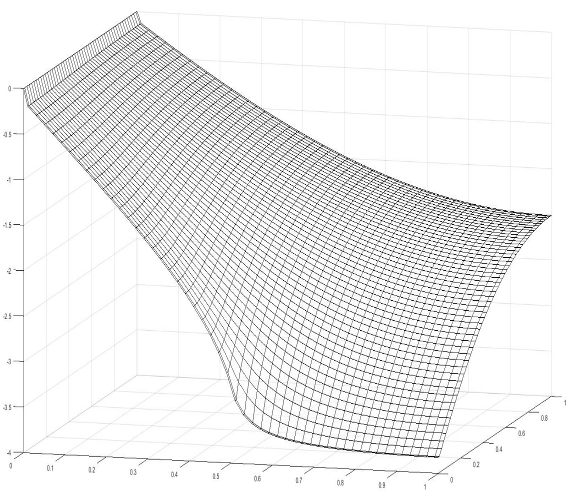

Figure 1: $z = u(x, y) ; \forall (x, y) \in \Omega$.

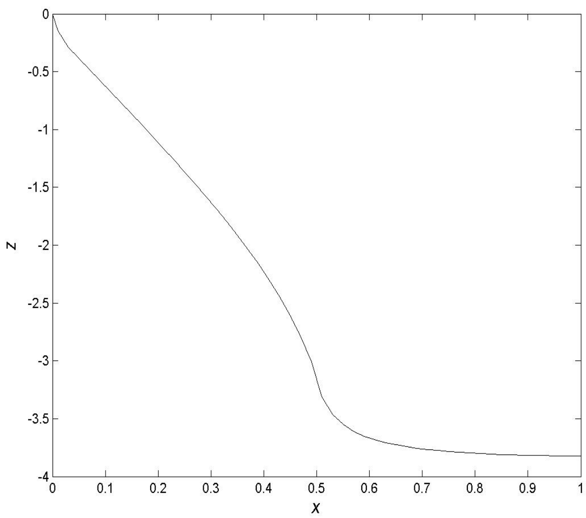


Figure 2: $z = u(x, 0) ; \forall x \in [0, x_m]$.

Figure1 represent a configuration of the deformed solid, and Figure2 shows the deplacement along the x-axis (deformation in the contact zone).

## VIII. Conclusion

By starting with the model for a deformed elastic solid with a unilateral contact of a rigid body, an approximate numerical scheme by the finite volume method has been proposed.

Then, the analysis of the approximate scheme have been proved as well as an estimate of order 1 for the discrete $H_0^1$ norm (resp. $L^2$ norm) of the error on the solution is done.

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