Inferences for Burr-X Model Based on Unified Hybrid Censored Data

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Abstract—**In this paper, Burr-X distribution with unified hybrid censored data is considered. The maximum likelihood method, the Bayesian and the E-Bayesian (the expectation of the Bayesian estimate) approaches are studied for the distribution parameter and the associated reliability function. The Bayesian and the E-Bayesian estimates are derived under LINEX and squared error loss (SEL) functions. We apply Markov chain Monte Carlo (MCMC) techniques to derive the Bayesian and the E-Bayesian estimates. Also, confidence intervals (CIs) of the maximum likelihood estimates, as well as credible intervals (CRIs) of the Bayesian and the E-Bayesian estimates, are computed. On the other hand, an example of a real data set is provided for the purpose of illustration. Finally, a numerical comparison among the proposed methods is performed.**

Index Terms—**Bayesian estimation, Burr-X distribution, E-Bayesian estimation, Unified hybrid censoring scheme, Maximum likelihood estimation, MCMC method.**

I. Introduction

GENERALIZED Type-I and Type-II hybrid censoring schemes (HCSs) are proposed by [1] to overcome the drawbacks of Type-I and Type-II HCSs, for more details about HCSs, one can refer to [2]. The generalized HSCs are an extension of Type-I and Type-II HCSs, therefore, we can notice two types of censoring schemes are described as follows. Generalized Type-I HCS: fix integers $r, k \in \{1, 2, \cdots, n\}$ such that $k < r < n$, and time $T \in (0, \infty)$. The experiment is terminated at $\min\{X_{r:n}, T\}$, if the k-th failure occurs before time T. If the k-th failure occurs after time T, the experiment is terminated at $X_{k:n}$.

Generalized Type-II HCS: fix integer $r \in \{1, 2, \cdots, n\}$ and time points $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$. The experiment is terminated at T_1, if the r-th failure occurs before time T_1. If the r-th failure occurs between T_1 and T_2, the experiment is terminated at $X_{r:n}$. Finally, if the r-th failure occurs after time T_2, the experiment is terminated at T_2.

Unified HCS (UHCS) is defined by [3] as a mixture of generalized Type-I HCS and generalized Type-II HCS which can be described as follows. Suppose n identical units are put on a test and the lifetime of each unit is independent and identically distributed (i.i.d) random variables. Fix integers $r, k \in \{1, 2, \cdots, n\}$ such that $k < r < n$, and time $T_1 < T_2 \in (0, \infty)$. The experiment is terminated at $\min\{\max\{X_{r:n}, T_1\}, T_2\}$, if the k-th failure occurs before time T_1. If the k-th failure occurs between T_1 and T_2, the experiment is terminated at $\min\{X_{r:n}, T_2\}$. Finally, if the k-th failure occurs after time T_2, the experiment is terminated at $X_{k:n}$. By using this censoring scheme, we guarantee that the maximum time for the experiment is T_2 and at least k failures are obtained. This scheme has been discussed earlier in many literatures, see for example, [4], [5], [6] and [7].

A. Burr-X as a lifetime model

Burr-X distribution is suggested by [8] as one of Burr distributions family. This model has a great importance in statistics and operations research. It also has applications in many fields such as health, agriculture and biology. The probability density function (PDF) of Burr-X distribution is given by,

$$f(x;\alpha) = 2\alpha x \exp(-x^2)(1-\exp(-x^2))^{\alpha-1}, \;\; x > 0, \;\; \alpha > 0, \tag{1}$$

hence, the cumulative distribution function (CDF) takes the form

$$F(x;\alpha) = (1 - \exp(-x^2))^{\alpha}, \;\; x > 0, \;\; \alpha > 0, \tag{2}$$

where, α is the shape parameter. The reliability function $R(t)$ of Burr-X distribution is given by,

$$R(t) = 1 - (1 - \exp(-t^2))^{\alpha}, \;\; t > 0. \tag{3}$$

Many literatures have been written on Burr-X distribution, see for example, [9], [10] and [11].

The rest of this paper is organized as follows. The Maximum likelihood function is presented in Section II. The Bayesian analysis under the SEL and the LINEX loss functions is described in Section III. In Section IV, The E-Bayesian estimation under the SEL and the LINEX loss functions is discussed. The MCMC method is presented in Section V. In Section VI, we present an illustrative example of real-life data. Concluding remarks are stated in Section VII.

II. Maximum Likelihood Function

Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ be a random sample of size n, drawn from Burr-X distribution based on UHCS, in which we observe one of the following six cases of the censored data

- Case (1): $\{0 < X_{k:n} < X_{r:n} < T_1 < T_2\}$, the experiment is terminated at T_1,
- Case (2) : $\{0 < X_{k:n} < T_1 < X_{r:n} < T_2\}$, the experiment is terminated at $X_{r:n}$,
- Case (3): $\{0 < X_{k:n} < T_1 < T_2 < X_{r:n}\}$, the experiment is terminated at T_2,

Manuscript received 6 April, 2019; revised 23 May, 2019. This work was substantially supported by the National Natural Sciences Foundations of China (No. 11771452, No. 11571372) and Natural Sciences Foundations of Hunan (No. 2017JJ2328)

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- Case (4): $\{0 < T_1 < X_{k:n} < X_{r:n} < T_2\}$, the experiment is terminated at $X_{r:n}$,
- Case (5): $\{0 < T_1 < X_{k:n} < T_2 < X_{r:n}\}$, the experiment is terminated at T_2,
- Case (6): $\{0 < T_1 < T_2 < X_{k:n} < X_{r:n}\}$, the experiment is terminated at $X_{k:n}$.

Thus, the likelihood function for the previous cases of UHCS, can be combined and written as follows.

$$\begin{aligned} L(\alpha; x) &= \frac{n!}{(n-m)!}\Big\{1-F(s)\Big\}^{n-m}\prod_{i=1}^{m} f(x_{i:n}), \\ &= \frac{n!(2\alpha)^m}{(n-m)!}\big\{1-\big(W_s\big)^{\alpha}\big\}^{n-m} \\ &\times \prod_{i=1}^{m} x_{i:n}\exp(-x_{i:n}^2)\big(W_{x_{i:n}}\big)^{\alpha-1}, \end{aligned} \tag{4}$$

where, $W_s = 1-\exp(-s^2)$ and $W_{x_{i:n}} = 1-\exp(-x_{i:n}^2)$,

$$m = \begin{cases} D_1, & \text{for Case (1),} \\ r, & \text{for Cases (2),(4),} \\ D_2, & \text{for Cases (3),(5),} \\ k, & \text{for Case (6),} \end{cases}$$

where, D_1 and D_2 number of failures occur at times T_1 and T_2, respectively,

$$s = \begin{cases} T_1, & \text{for Case (1),} \\ x_{r:n}, & \text{for Cases (2),(4),} \\ T_2, & \text{for Cases (3),(5),} \\ x_{k:n}, & \text{for Case (6),} \end{cases}$$

The log-likelihood function of Eq. (4), can be written as follows.

$$\begin{aligned} \ell &= \ln L(\alpha; x) \\ &= (n-m)\ln\big\{1-\big(W_s\big)^{\alpha}\big\} + m\ln(2\alpha) + \sum_{i=1}^{m}\ln(x_{i:n}) \\ &- \sum_{i=1}^{m} x_{i:n}^2 + (\alpha-1)\sum_{i=1}^{m}\ln\big(W_{x_{i:n}}\big). \end{aligned} \tag{5}$$

The maximum likelihood estimate (MLE) of the unknown parameter α is obtained by setting the first partial derivative of Eq. (5) with respect to α to zero and solving numerically the following equation.

$$\begin{aligned} 0 &= \frac{\partial \ell}{\partial \alpha} \\ &= \frac{m}{\alpha} - \frac{(n-m)\,W_s^{\alpha}\ln W_s}{1-W_s^{\alpha}} + \sum_{i=1}^{m}\ln\big(1-\exp(-x_{i:n}^2)\big), \end{aligned} \tag{6}$$

Substituting by $\hat{\alpha}$ the MLE of α in Eq.(3), we obtain $\hat{R(t)}$ the MLE of $R(t)$ as follows.

$$\hat{R(t)} = 1-(1-\exp(-t^2))^{\hat{\alpha}}. \tag{7}$$

III. Bayesian Analysis

In this section, we use the Bayesian method for estimating the parameter α and the reliability function $R(t)$ of Burr-X model based on UHCS. The Bayesian estimates are obtained under the SEL and the LINEX loss functions. We assume that the prior PDF of the parameter α is Gamma(a,b) given as follows.

$$\pi(\alpha|a,b) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}\exp(-b\alpha), \quad a,b>0. \tag{8}$$

From (4) and (8), the posterior PDF of α, can be written as follows.

$$\begin{aligned} \pi(\alpha|x) &= \frac{L(\alpha;x)\pi(\alpha|a,b)}{\displaystyle\int_{\alpha} L(\alpha;x)\pi(\alpha|a,b)d\alpha} \\ &= K^{-1}\alpha^{m+a-1}\exp(-b\alpha)\{1-W_s^{\alpha}\}^{n-m} \\ &\times \prod_{i=1}^{m} x_{i:n}\exp(-x_{i:n}^2)W_{x_{i:n}}^{\alpha-1}, \end{aligned} \tag{9}$$

where, $W_s = 1-\exp(-s^2)$, $W_{x_{i:n}} = 1-\exp(-x_{i:n}^2)$ and K is a normalizing constant given by,

$$\begin{aligned} K &= \int_{\alpha} L(\alpha;x)\pi(\alpha|a,b)d\alpha. \\ &= \int_0^{\infty}\alpha^{m+a-1}e^{-b\alpha}\Big\{1-W_s^{\alpha}\Big\}^{n-m} \\ &\times \prod_{i=1}^{m} x_{i:n}e^{-x_{i:n}^2}W_{x_{i:n}}^{\alpha-1}d\alpha. \end{aligned} \tag{10}$$

Under the SEL function, the Bayesian estimates of α and $R(t)$ are, respectively, given by,

$$\begin{aligned} \hat{\alpha}_{BS} &= E[\alpha|x] \\ &= \int_0^{\infty}\alpha\pi(\alpha|x)d\alpha \\ &= K^{-1}\int_0^{\infty}\alpha^{m+a}e^{-b\alpha}\{1-W_s^{\alpha}\}^{n-m} \\ &\times \prod_{i=1}^{m} x_{i:n}e^{-x_{i:n}^2}W_{x_{i:n}}^{\alpha-1}d\alpha, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \hat{R(t)}_{BS} &= E[R(t)|x] \\ &= \int_0^{\infty} R(t)\pi(\alpha|x)d\alpha \\ &= K^{-1}\int_0^{\infty}\{1-(1-\exp(-t^2))^{\alpha}\}\alpha^{m+a-1}e^{-b\alpha} \\ &\times \{1-W_s^{\alpha}\}^{n-m}\prod_{i=1}^{m} x_{i:n}e^{-x_{i:n}^2}W_{x_{i:n}}^{\alpha-1}d\alpha. \end{aligned} \tag{12}$$

Based on the LINEX loss function, the Bayesian estimates of α and $R(t)$ are, respectively, given by,

$$\begin{aligned} \hat{\alpha}_{BL} &= \frac{-1}{h}\ln E[\exp(-h\alpha)|x] \\ &= \frac{-1}{h}\ln\int_0^{\infty}\exp(-h\alpha)\pi(\alpha|x)d\alpha \\ &= \frac{-1}{Kh}\ln\int_0^{\infty}\alpha^{m+a-1}e^{-\alpha(b+h)}\{1-W_s^{\alpha}\}^{n-m} \\ &\times \prod_{i=1}^{m} x_{i:n}e^{-x_{i:n}^2}W_{x_{i:n}}^{\alpha-1}d\alpha, \quad h\neq 0, \end{aligned} \tag{13}$$

where, h is the shape parameter of the loss function and

$$\begin{aligned}\hat{R(t)}_{BL} &= \frac{-1}{h}\ln E[\exp(-hR(t))|x] \\ &= \frac{-1}{h}\ln\int_0^\infty \exp(-hR(t))\pi(\alpha|x)d\alpha \\ &= \frac{-1}{Kh}\ln\int_0^\infty \alpha^{m+a-1}e^{(h[(1-\exp(-t^2))^\alpha-1]-b\alpha)} \\ &\times\left\{1-W_s^\alpha\right\}^{n-m}\prod_{i=1}^m x_{i:n}e^{-x_{i:n}^2}W_{x_{i:n}}^{\alpha-1}d\alpha, \quad h\neq 0.\end{aligned} \tag{14}$$

IV. E-Bayesian Estimation

Based on [12], the hyper parameters a and b of the prior PDF should be selected, so that $\pi(\alpha)$ is a decreasing function of α. The derivative of $\pi(\alpha)$ with respect to α is given by,

$$\frac{d\pi(\alpha)}{d\alpha} = \frac{b^a}{\Gamma(a)}\alpha^{a-2}\exp(-b\alpha)\left\{(a-1)-b\alpha\right\},$$

when $0 < a < 1$, $b > 0$ and $\frac{d\pi(\alpha)}{d\alpha} < 0$, thus $\pi(\alpha)$ is a decreasing function of α. We assume that the hyper parameters a and b are independent with bi-variate PDF as follows.

$$\pi(a,b) \quad = \quad \pi_1(a)\pi_2(b),$$

then, the E-Bayesian estimate of the parameter α and the reliability function $R(t)$ (the expectation of the Bayesian estimate of α and $R(t)$) are, respectively, written as follows.

$$\hat{\alpha}_{EB} = E[\hat{\alpha}_B|x] = \iint_Q \hat{\alpha}_B\pi(a,b)dadb, \tag{15}$$

and

$$\hat{R(t)}_{EB} = E[\hat{R(t)}_B|x] = \iint_Q \hat{R(t)}_B\pi(a,b)dadb, \tag{16}$$

where, Q is the set of all possible values of a and b, $\hat{\alpha}_B$ and $\hat{R(t)}_B$ are the Bayesian estimate of the parameter α and the reliability function $R(t)$, respectively, under the SEL and the LINEX loss functions. For more details, one can refer to [13], [14], [15], [16] and [17].

We consider the following prior PDFs of a and b to clarify the impact of these prior PDFs on the E-Bayesian estimates.

$$\left.\begin{aligned}\pi_1(a,b) &= \frac{2a}{c}, & 0<a<1, 0<b<c,\\ \pi_2(a,b) &= \frac{2b}{c^2}, & 0<a<1, 0<b<c,\\ \pi_3(a,b) &= \frac{3b^2}{c^3}, & 0<a<1, 0<b<c,\end{aligned}\right\} \tag{17}$$

where, $c > 0$ is a given upper bound, to be determined. The E-Bayesian estimate of α under the SEL function can be obtained from (11), (15) and (17) as follows.

$$\hat{\alpha}_{EBS_j} = \iint_Q \hat{\alpha}_{BS}\pi_j(a,b)dadb, \qquad j=1,2,3, \tag{18}$$

also, the E-Bayesian estimate of α under the LINEX loss function can be obtained from (13), (15) and (17) as follows.

$$\hat{\alpha}_{EBL_j} = \iint_Q \hat{\alpha}_{BL}\pi_j(a,b)dadb, \qquad j=1,2,3. \tag{19}$$

the E-Bayesian estimate of the reliability function $R(t)$ under the SEL function can be obtained from (12), (16) and (17) as follows.

$$\hat{R(t)}_{EBS_j} = \iint_Q \hat{R(t)}_{BS}\pi_j(a,b)dadb, \qquad j=1,2,3, \tag{20}$$

the E-Bayesian estimate of the reliability function $R(t)$ under the LINEX loss function can be obtained from (14),(16) and (17) as follows.

$$\hat{R(t)}_{EBL_j} = \iint_Q \hat{R(t)}_{BL}\pi_j(a,b)dadb, \qquad j=1,2,3. \tag{21}$$

It is noted that the Bayesian and E-Bayesian estimates cannot be obtained analytically, so we use the MCMC method to derive the these estimates of the parameter α and the reliability function $R(t)$.

V. MCMC Algorithm for Bayesian Estimation

In this section, we use the MCMC algorithm to derive the Bayesian and the E-Bayesian estimates of the parameter α and the reliability function $R(t)$. The full conditional posterior PDF of the parameter α can be obtained from (9) as follows.

$$\pi^*(\alpha|x) = \alpha^{m+a-1}\exp(-b\alpha)\left\{1-W_s^\alpha\right\}^{n-m}\prod_{i=1}^m W_{x_{i:n}}^\alpha. \tag{22}$$

It is noted from Eq. (22), that the full conditional posterior PDF of the parameter α cannot be reduced to a well-known distribution, so we consider the Metropolis-Hastings algorithm, one of the MCMC methods, to generate posterior samples of the parameter α from the full conditional posterior PDF then compute Bayesian estimates of the unknown parameter α and the reliability function $R(t)$. For more details, one can refer to [18] and [19].

The following steps indicate the Metropolis-Hastings algorithm for simulating the posterior samples, then derive the Bayesian estimates as follows.

1) Choose initial guess of the parameter α say $\alpha^{(0)}(=\hat{\alpha}_{MLE})$.
2) At iteration j, generate $\alpha^{(j)}$ from $\pi^*(\alpha^{(j-1)}|x)$ and generate $\alpha^{(*)}$ from a normal distribution as a proposal distribution.
3) Generate a sample u from $U(0,1)$.
4) Calculate the acceptance probability
$$r(\alpha^{j-1}|\alpha^{(*)}) = min\Big[1, \frac{\pi^*(\alpha^{(*)}|x)}{\pi^*(\alpha^{(j-1)}|x)}\Big] \tag{23}$$
5) If $u < r$ accept $\alpha^{(*)}$ as $\alpha^{(j)}$, otherwise, $\alpha^{(j)} = \alpha^{(j-1)}$
6) Compute $R(t)$ as follows.
$$R^{(j)}(t) = 1-(1-\exp(-t^2))^{\alpha^{(j)}}. \tag{24}$$
7) Repeat $(3-6)$ N times to obtain $\alpha^{(j)}$ and $R^{(j)}(t)$, $j = M+1, \cdots, N$.
8) The Bayesian estimates of the parameter α and the reliability function $R(t)$ under the SEL function are, respectively, given by,
$$\hat{\alpha}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^N \alpha^{(j)}. \tag{25}$$

$$\hat{R(t)}_{BS} = \frac{1}{N-M}\sum_{j=M+1}^{N} R^{(j)}(t), \quad (26)$$

where, M is an optional burn-in period.

9) The Bayesian estimates of the parameter α and the reliability function $R(t)$ under the LINEX loss function are, respectively, given by,

$$\hat{\alpha}_{BL} = \frac{-1}{h}\ln\left[\frac{1}{N-M}\sum_{j=M+1}^{N} e^{-h\alpha^{(j)}}\right], \quad (27)$$

$$\hat{R(t)}_{BL} = \frac{-1}{h}\ln\left[\frac{1}{N-M}\sum_{j=M+1}^{N} e^{-hR^{(j)}(t)}\right]. \quad (28)$$

10) Approximate $100(1-\gamma)\%$ confidence intervals (CIs) for the MLEs of α and $R(t)$ can be constructed as follows.

$$\hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\alpha})} \;\&\; \hat{R(t)} \pm z_{\frac{\gamma}{2}}\sqrt{Var(\hat{R(t)})}, \quad (29)$$

where, $z_{\frac{\gamma}{2}}$ is the $100(1-\frac{\gamma}{2})\%$ upper percentile of standard normal variate.

11) A $100(1-\gamma)\%$ credible intervals (CRIs) of the Bayesian and the E-Bayesian estimates of α and $R(t)$ can be constructed from the $(\frac{\gamma}{2})$ and $(1-\frac{\gamma}{2})$ sample quantiles of the empirical posterior PDF of MCMC draws, given by,

$$\left[\alpha_{N[\frac{\gamma}{2}]}, \alpha_{N[1-\frac{\gamma}{2}]}\right] \;\&\; \left[R_{N[\frac{\gamma}{2}]}, R_{N[1-\frac{\gamma}{2}]}\right], \quad (30)$$

where, N represents the number of draws.

A. Unified hybrid censored data

In this subsection, we present some simulation results for different values of $n = 20, 40, 60$; $r = 15, 30, 45$ and $T_1 = 0.7, 0.9$; $T_2 = 0.9, 1.5$. For a given value of c, values of a and b were generated from (17) and those of α was generated from Gamma (a, b). The resulting value of α is used as a true value to generate a unified hybrid censored sample from Burr-X distribution by using the inverse function method as follows.

$$X = \{-\ln(1-U^{\frac{1}{\alpha}})\}^{\frac{1}{2}}, \quad (31)$$

where, U is a random number generated from U(0,1). The $MLEs$ of the unknown parameter α and the reliability function $R(t)$ are obtained from (6) and (7), respectively. By using Metropolis-Hastings sampler, we generate a Markov chain with 11000 observations of the parameter α, discarding the first 1000 observations as a "burn-in" period. By using these values of the parameter α, we compute Bayesian estimates $\hat{\alpha}_{BS}$ and $\hat{R(t)}_{BS}$ under SEL function from (25) and (26), respectively, and Bayesian estimates $\hat{\alpha}_{BL}$ and $\hat{R(t)}_{BL}$ under LINEX loss function are obtained from (27) and (28), respectively. Also, E-Bayesian estimates $\hat{\alpha}_{EBS}$ and $\hat{R(t)}_{EBS}$ under SEL function are derived from (18) and (20), respectively, and E-Bayesian estimates $\hat{\alpha}_{EBL}$ and $\hat{R(t)}_{EBL}$ under LINEX loss function are given from (19) and (21), respectively. The mean squared error (*MSE*) is used to compare the estimators of the parameter α and the reliability function $R(t)$ and, respectively, given as follows.

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\alpha}_j - \alpha)^2$$

$$MSE(\hat{R(t)}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{R(t)}_j - R(t))^2,$$

where, $\hat{\alpha}$ and $\hat{R(t)}$ are the estimates of the parameter α and the reliability function $R(t)$, respectively. The 95% CIs of the MLEs of the parameter α and the reliability function $R(t)$ are computed from (29). Also, the 95% CRIs of the Bayesian and the E-Bayesian estimates of the parameter α and the reliability function $R(t)$ are computed from (30). All numerical results are listed in Tables (I-II). Where, in Table I, $\hat{\alpha}_{EBS1}$, $\hat{\alpha}_{EBS2}$ and $\hat{\alpha}_{EBS3}$ stand for the E-Bayesian estimates of α relative to SEL based on $\pi_1(a,b)$, $\pi_2(a,b)$ and $\pi_3(a,b)$, respectively, $\hat{\alpha}_{EBL1}$, $\hat{\alpha}_{EBL2}$ and $\hat{\alpha}_{EBL3}$ denote the E-Bayesian estimates of α relative to LINEX based on $\pi_1(a,b)$, $\pi_2(a,b)$ and $\pi_3(a,b)$, respectively. And $\hat{\alpha}_{BS}$ and $\hat{\alpha}_{BL}$ are the Bayesian estimates of α under SEL and LINEX loss, respectively. While in Table II, $\hat{R}_{EBS1}$, $\hat{R}_{EBS3}$ and $\hat{R}_{EBS2}$ stand for the E-Bayesian estimates of $R(t)$ relative to SEL based on $\pi_1(a,b)$, $\pi_2(a,b)$ and $\pi_3(a,b)$, respectively, $\hat{R}_{EBL1}$, $\hat{R}_{EBL2}$ and $\hat{R}_{EBL3}$ are the E-Bayesian estimates of $R(t)$ relative to LINEX based on $\pi_1(a,b)$, $\pi_2(a,b)$ and $\pi_3(a,b)$, respectively. And $\hat{R}_{BS}$ and $\hat{R}_{BL}$ denote the Bayesian estimates of $R(t)$ under SEL and LINEX loss, respectively.

VI. Application of Burr-X Model

In this section, we present an illustrative example of real-life data to clarify the performance of the proposed methods in the application. These data were obtained in the Semiconductor Electronics Division of the National Institute of Standards and Technology Electronics and Electrical Engineering Laboratory and taken from [20]. This data set represents minority electron mobility for $Ga_{1-x}Al_xAs$ semi-conductor. One data set at the mole fraction of 0.25 is considered here. This data set was used by [21], who proved that Burr-X distribution gives a good fit for this data set. The data set contains 21 observations as listed below.

Data Set (belongs to mole fraction 0.25): 3.051, 2.779, 2.6044, 2.371, 2.214,2.045, 1.715, 1.525, 1.296, 1.154, 1.0164, 0.7948, 0.7007, 0.6292, 0.6175, 0.6449, 0.8881,1.115, 1.397,1.506, and 1.528.

We assume that values of data set are failure lifetime observations following Burr-X distribution. By applying UHCS on this uncensored data considering $n = 21$, $r = 18$, $k = 15$, $\alpha = 1.6526$ and $R(t = 1.2)$, we observe the following cases:

- When $T_1 = 1.5$ and $T_2 = 2$, we observe that k number of failures occur after T_1, hence the experiment is terminated at random time $T^* = min\{X_{r:n}, T_2\} = min\{2.371, 2\} = 2$.
- When $T_1 = 2$ and $T_2 = 2.5$, we observe that k number of failures occur before T_1, hence the experiment is terminated at random time $T^* = min\{max\{X_{r:n}, T_1\}, T_2\} = min\{max\{2.371, 2\}, 2.5\} = min\{2.371, 2.5\} = 2.371$.

- When $T_1 = 2.5$ and $T_2 = 3$, we observe that k number of failures occur before T_1, hence the experiment is terminated at random time $T^* = min\{max\{X_{r:n}, T_1\}, T_2\} = min\{max\{2.371, 2.5\}, 3\} = min\{2.5, 3\} = 2.5$.

With respect to the real data set, all estimates are derived based on SEL and LINEX loss functions by the same previous methods and listed in Tables (III-IV).

TABLE I: Average estimates, MSE and 95% CIs and CRIs of MLEs, Bayesian and E-Bayesian estimates for α under LINEX and SEL functions when $\alpha = 1.15, a = 0.8, b = 0.7, h = 1.5, c = 1$.

(n, r, k)	(T_1, T_2)	Criteria	$\hat{\alpha}_{MLE}$	Squared Error Loss				LINEX Loss			
				$\hat{\alpha}_{BS}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{EBL1}$	$\hat{\alpha}_{EBL2}$	$\hat{\alpha}_{EBL3}$
$(20, 15, 10)$	$(0.7, 0.9)$	Mean	1.42127	1.40206	1.25625	1.09922	1.15418	1.35505	1.21812	1.0697	1.12176
		MSE	0.2888	0.11275	0.0508	0.03283	0.03337	0.07775	0.03425	0.02993	0.02637
		Lower	0.404682	0.9277	0.871	0.697	0.3315	0.4675	0.6195	0.2582	0.3908
		Upper	2.43786	1.8764	1.8391	1.8155	1.8669	1.8408	1.8168	1.8812	1.8527
		Length	2.03318	0.9488	0.9681	1.1185	1.5354	1.3733	1.1973	1.623	1.4619
	$(0.9, 1.5)$	Mean	0.78429	1.34879	1.20852	1.05745	1.11032	1.32846	1.19213	1.04485	1.09645
		MSE	0.15808	0.04588	0.00853	0.01248	0.00589	0.03751	0.00638	0.01463	0.00678
		Lower	0.442441	1.0217	0.9989	0.7811	0.3992	0.5397	0.7434	0.3651	0.5034
		Upper	1.12614	1.6759	1.658	1.6359	1.7157	1.6809	1.6408	1.7246	1.6895
		Length	0.683698	0.6542	0.6591	0.8548	1.3164	1.1412	0.8974	1.3594	1.1861
$(40, 30, 20)$	$(0.7, 0.9)$	Mean	0.99085	1.40502	1.2589	1.10153	1.15661	1.37389	1.23377	1.08219	1.13532
		MSE	0.15602	0.07055	0.01629	0.00574	0.00378	0.05481	0.01084	0.00758	0.00348
		Lower	0.198635	0.9971	0.9614	0.7603	0.3801	0.5213	0.7054	0.3292	0.4675
		Upper	1.78307	1.813	1.7864	1.7575	1.823	1.792	1.7621	1.8352	1.8031
		Length	1.58443	0.8159	0.8251	0.9971	1.4429	1.2707	1.0567	1.5059	1.3356
	$(0.9, 1.5)$	Mean	0.88756	1.33505	1.19621	1.04668	1.09902	1.31043	1.17632	1.03137	1.08217
		MSE	0.0902	0.04139	0.00787	0.01507	0.00744	0.03176	0.00561	0.01791	0.00881
		Lower	0.567599	0.9737	0.9458	0.7435	0.3754	0.5115	0.6993	0.3347	0.4685
		Upper	1.20752	1.6964	1.675	1.6489	1.718	1.6865	1.6533	1.728	1.6959
		Length	0.639921	0.7227	0.7292	0.9054	1.3426	1.175	0.954	1.3933	1.2274
$(60, 45, 30)$	$(0.7, 0.9)$	Mean	0.96224	1.33124	1.19279	1.04369	1.09587	1.31346	1.17846	1.03268	1.08375
		MSE	0.1472	0.06276	0.02585	0.02969	0.0232	0.05369	0.0227	0.03072	0.0230
		Lower	0.22904	1.034	1.0141	0.7893	0.4055	0.546	0.7559	0.3755	0.5140
		Upper	1.69543	1.6284	1.6128	1.5963	1.6819	1.6458	1.601	1.6899	1.6535
		Length	1.46638	0.5944	0.5987	0.8070	1.2765	1.0998	0.8451	1.3144	1.1395
	$(0.9, 1.5)$	Mean	1.18969	1.31624	1.1524	1.00835	1.05876	1.29909	1.16553	0.99818	1.04757
		MSE	0.06007	0.05703	0.00647	0.02501	0.01378	0.04873	0.02175	0.02755	0.01543
		Lower	0.659684	1.0255	0.9781	0.7607	0.3906	0.5413	0.7503	0.3629	0.4966
		Upper	1.7197	1.6069	1.5613	1.5441	1.6261	1.6257	1.5807	1.6335	1.5986
		Length	1.06001	0.5814	0.5832	0.7835	1.2356	1.0844	0.8304	1.2706	1.102

TABLE II: Average estimates, MSE and 95%CIs and CRIs of MLEs, Bayesian and E-Bayesian estimates for $R(t=1.2)$ under LINEX and SEL functions when $\alpha=1.15, a=0.8, b=0.7, h=1.5,\ c=1$.

(n, r, k)	(T_1, T_2)	Criteria	$\hat{R}_{MLE}$	Squared Error Loss				LINEX Loss			
				$\hat{R}_{BS}$	$\hat{R}_{EBS1}$	$\hat{R}_{EBS2}$	$\hat{R}_{EBS3}$	$\hat{R}_{BL}$	$\hat{R}_{EBL1}$	$\hat{R}_{EBL2}$	$\hat{R}_{EBL3}$
(20, 15, 10)	$(0.7, 0.9)$	Mean	0.313721	0.31267	0.28015	0.24513	0.25739	0.31112	0.2789	0.24418	0.25634
		MSE	0.009544	0.00365	0.00144	0.00147	0.00118	0.00344	0.00136	0.00147	0.00115
		Lower	0.125406	0.2288	0.2272	0.1738	0.0872	0.1192	0.1712	0.0848	0.1165
		Upper	0.50203	0.3965	0.3951	0.3865	0.4030	0.3956	0.3867	0.4036	0.3961
		Length	0.37662	0.1677	0.1679	0.2127	0.3158	0.2765	0.2155	0.3188	0.2796
	$(0.9, 1.5)$	Mean	0.19039	0.30474	0.27305	0.23891	0.25086	0.30402	0.27247	0.23847	0.25037
		MSE	0.007003	0.00162	0.00021	0.00094	0.00042	0.00156	0.0002	0.00096	0.00043
		Lower	0.11791	0.2440	0.2433	0.1861	0.0963	0.1290	0.1848	0.0951	0.1277
		Upper	0.26286	0.3655	0.3648	0.3599	0.3815	0.3727	0.3602	0.3819	0.3730
		Length	0.14495	0.1215	0.1215	0.1738	0.2852	0.2437	0.1754	0.2868	0.2453
(40, 30, 20)	$(0.7, 0.9)$	Mean	0.23142	0.31487	0.28212	0.24686	0.2592	0.31379	0.28125	0.24619	0.25847
		MSE	0.00667	0.00245	0.00037	0.00053	0.00019	0.00234	0.00034	0.00055	0.0002
		Lower	0.07053	0.2404	0.2393	0.1838	0.0942	0.1271	0.1818	0.0923	0.1252
		Upper	0.39232	0.3893	0.3883	0.3804	0.3996	0.3913	0.3807	0.4000	0.3918
		Length	0.32179	0.1489	0.1490	0.1966	0.3054	0.2642	0.1989	0.3077	0.2666
	$(0.9, 1.5)$	Mean	0.21275	0.30197	0.27057	0.23674	0.24858	0.30109	0.26986	0.2362	0.24798
		MSE	0.00395	0.00145	0.00021	0.00108	0.00052	0.00138	0.0002	0.00111	0.00053
		Lower	0.14412	0.2349	0.2340	0.1795	0.0923	0.1243	0.1779	0.0909	0.1227
		Upper	0.28138	0.3690	0.3681	0.3616	0.3812	0.3729	0.3619	0.3816	0.3733
		Length	0.13725	0.1341	0.1341	0.1821	0.2888	0.2487	0.1840	0.2907	0.2506
(60, 45, 30)	$(0.7, 0.9)$	Mean	0.22599	0.30093	0.26964	0.23593	0.24773	0.30032	0.26914	0.23555	0.24731
		MSE	0.00634	0.00217	0.00083	0.00162	0.00108	0.00211	0.00082	0.00163	0.00109
		Lower	0.07668	0.2459	0.2453	0.1871	0.0970	0.1297	0.1859	0.0960	0.1285
		Upper	0.37529	0.3559	0.3553	0.3522	0.3748	0.3658	0.3524	0.3751	0.3661
		Length	0.29861	0.1100	0.1100	0.1651	0.2778	0.2361	0.1665	0.2792	0.2375
	$(0.9, 1.5)$	Mean	0.27347	0.29813	0.26249	0.22968	0.24116	0.29753	0.26664	0.22931	0.24076
		MSE	0.00252	0.00197	0.00026	0.0016	0.00088	0.00192	0.00081	0.00162	0.0009
		Lower	0.16424	0.2441	0.2377	0.1814	0.0940	0.1287	0.1844	0.0930	0.1246
		Upper	0.38271	0.3521	0.3470	0.3436	0.3654	0.3622	0.3488	0.3656	0.3569
		Length	0.218472	0.1080	0.1093	0.1622	0.2713	0.2335	0.1644	0.2727	0.2323

TABLE III: For real data set : Average estimates, MSE and 95%CIs and CRIs of MLEs, Bayesian and E-Bayesian estimates for α under LINEX and SEL functions when $\alpha=1.66, a=0.8, b=1.4, h=1.5,\ c=2$

(n, r, k)	(T_1, T_2)	Criteria	$\hat{\alpha}_{MLE}$	Squared Error Loss				LINEX Loss			
				$\hat{\alpha}_{BS}$	$\hat{\alpha}_{EBS1}$	$\hat{\alpha}_{EBS2}$	$\hat{\alpha}_{EBS3}$	$\hat{\alpha}_{BL}$	$\hat{\alpha}_{EBL1}$	$\hat{\alpha}_{EBL2}$	$\hat{\alpha}_{EBL3}$
(21, 18, 15)	(2, 2.5)	Mean	2.21753	2.45016	2.19534	1.92092	2.01697	2.26428	2.04457	1.80418	1.88877
		MSE	0.87139	0.62438	0.28662	0.0681	0.12745	0.36518	0.14792	0.0208	0.05235
		Lower	0.576875	1.421	1.1725	1.0514	0.4597	0.6827	0.7441	0.1725	0.3821
		Upper	3.85818	3.4793	3.356	3.3393	3.3822	3.3512	3.345	3.4358	3.3954
		Length	3.28131	2.0584	2.1835	2.288	2.9225	2.6685	2.6009	3.2633	3.0133
	(2.5, 3)	Mean	2.15877	2.44508	2.19079	1.91694	2.01279	2.26161	2.04193	1.80164	1.88619
		MSE	0.7657	0.61638	0.28177	0.06604	0.12449	0.362	0.14592	0.0201	0.0512
		Lower	0.58324	1.422	1.1772	1.0528	0.4615	0.6844	0.7493	0.1777	0.3873
		Upper	3.7343	3.4681	3.346	3.3288	3.3724	3.3412	3.3346	3.4256	3.385
		Length	3.15106	2.0461	2.1688	2.276	2.9108	2.6567	2.5853	3.2478	2.9977

TABLE IV: For real data set : Average estimates, MSE and 95%CIs and CRIs of MLEs, Bayesian and E-Bayesian estimates for $R(t=1.2)$ under LINEX and SEL functions, $\alpha=1.66, a=0.8, b=1.4, h=1.5,\ c=2$.

(n, r, k)	(T_1, T_2)	Criteria	$\hat{R}_{MLE}$	Squared Error Loss				LINEX Loss			
				$\hat{R}_{BS}$	$\hat{R}_{EBS1}$	$\hat{R}_{EBS2}$	$\hat{R}_{EBS3}$	$\hat{R}_{BL}$	$\hat{R}_{EBL1}$	$\hat{R}_{EBL2}$	$\hat{R}_{EBL3}$
(21, 18, 15)	(2, 2.5)	Mean	0.472972	0.51653	0.46281	0.40496	0.42521	0.51237	0.45947	0.4024	0.42239
		MSE	0.025538	0.01504	0.00475	0.00012	0.00098	0.01404	0.0043	0.00007	0.00081
		Lower	0.168667	0.3706	0.3662	0.2829	0.1421	0.1943	0.2756	0.1353	0.1871
		Upper	0.77727	0.6625	0.6585	0.6428	0.6679	0.6562	0.6433	0.6695	0.6576
		Length	0.60861	0.2919	0.2923	0.3599	0.5258	0.4619	0.3677	0.5342	0.4705
	(2.5, 3)	Mean	0.46478	0.51586	0.46221	0.40443	0.42465	0.51173	0.4589	0.4019	0.42186
		MSE	0.02316	0.01488	0.00467	0.00011	0.00095	0.01389	0.00423	0.00006	0.00078
		Lower	0.169633	0.3706	0.3663	0.2829	0.1421	0.1943	0.2757	0.1355	0.1872
		Upper	0.75993	0.6611	0.6572	0.6415	0.6667	0.655	0.6421	0.6683	0.6565
		Length	0.59030	0.2905	0.291	0.3587	0.5246	0.4607	0.3664	0.5329	0.4692

VII. Concluding remarks

In this paper, we used the maximum likelihood method, as well as the Bayesian and the E-Bayesian approaches, for estimating the unknown parameter and the reliability function of Burr-X distribution based on UHCS. The Bayesian and E-Bayesian estimates are derived under the SEL and the LINEX loss functions. The MCMC algorithm is applied to obtain the Bayesian and the E-Bayesian estimates as they cannot be obtained analytically. Also, we constructed the CIs of MLEs and the CRIs of the Bayesian and the E-Bayesian estimates. Furthermore, an illustrative example is presented to investigate the performance of the proposed methods in the application. Based on the results obtained in Tables (I-II), we observe the following.

- The Bayesian estimator is better than MLE in terms of MSE.
- The E-Bayesian estimator is the best when compared with the Bayesian and the MLEs in the sense of having a smaller MSE.
- The MSE of all estimates decreases when the sample size n and time points T_1, T_2 increase.
- Also, the length of CIs and CRIs decreases when n, r, T_1 and T_2 increase.
- The E-Bayesian estimator is more efficient than the Bayesian and MLEs, as it has a smaller MSE.
- Based on the results obtained in Tables (III-IV), we observe that all the previous results are realized for the Bayesian, the E-Bayesian estimates and MLEs, that is the proposed methods behave well for a practical real data set.
- Moreover, the E-Bayesian estimator is the most efficient compared with the Bayesian estimator and MLE in the application.
- On the other hand, the length of CIs and CRIs decreases when n, r, T_1 and T_2 get larger.
- From Tables (I-IV), we can state that the E-Bayesian method is easy to apply and more efficient than both Bayesian and maximum likelihood methods.
- Also, a large sample size n gives better estimates with a smaller MSE.
- Furthermore, the E-Bayesian method can be applied to any censoring scheme as performed in [11], [16] and [17] with the same efficient behavior.

Acknowledgment

The authors would like to thank the associate editor, the editor and the referees for their constructive comments and suggestions that improved the content and the style of this paper.

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