Some Results of The Compact Graph

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ABSTRACT—Doubly stochastic matrix has many important applications, and the family of compact graphs has important research value which can be seen as the generalization of the famous Birkhoff theorem of doubly stochastic matrix in combinatorial matrix theory. Determining whether a graph is a compact graph is a difficult problem, and only few compact graphs are known at present. We have studied the compact graph and have obtained some results: the graph constructed by the disjoint union of any compact graph and some isolated points is a compact graph, the graph constructed by adding one pendant edge to each vertex of any compact graph is also a compact graph, and some compact graphs also are obtained using above results. In this paper, based on previous studies, the results of compact graphs are further given: the graph constructed by adding n pendant edges to each vertex of the complete graph is a compact graph, and the graph constructed by adding two pendant edges to each vertex of any compact graph is a compact graph. The disjoint union of any number of non-isomorphic complete graphs is a compact graph. Combined with these results, some results of compact graphs and super-compact graph are given.

Index Terms—Compact graph, doubly stochastic matrix, super compact graph, permanent

I Introduction

IN this paper, the simple and undirected graph $G=(V_n,E)$ with the vertex set $V_n=\{1,2,\cdots,n\}$ and the edge set E is considered. The adjacency matrix $A=A(G)$ of a graph G is a (0,1) matrix of order n, whose element is $a_{ij}=1$ or 0, if an edge $<i,j>\in E$ or not. Thus, a graph of order n corresponds to an adjacency matrix $A=\left(a_{ij}\right)_{n\times n}$. The graphs G and H are isomorphism if and only if their adjacency matrices are permutation similarity. That is, if A and B are the adjacency matrices of the graphs G and H respectively, then G and H are isomorphism if and only if there exists a permutation matrix P such that $AP=PB$, where P is called the self-isomorphic permutation matrix of A.

Let p_n be the set of all permutation matrices of order n, and $\mathrm{P}(A)$ be the set of all self-isomorphism of graph G, i.e., $\mathrm{P}(A)=\{X \mid X\in \mathrm{P}_n, AX=XA\}$.

Let $\overline{\mathrm{P}}(A)$ denote

$\overline{\mathrm{P}}(A)=\left\{\sum c_iP_i \mid \sum c_i=1, P_i\in \mathrm{P}(A), c_i>0\right\}$.

A non-negative square matrix X is called doubly stochastic matrix, if the X is the solution of the Linear programming equation $Xe=X^Te=e$, where e is the n-dimensional vector whose elements are all 1.

Let Ω_n be the set of all doubly stochastic matrices of order n, and

$$\Omega(A)=\{X \mid X\in\Omega_n, XA=AX\}.$$

Obviously, $\overline{\mathrm{P}}(A)\subseteq\Omega(A)$. If $\overline{\mathrm{P}}(A)=\Omega(A)$, graph G is called compact graph. Compact graph can be seen as the generalization of Birkhoff theorem[1] of doubly stochastic matrix in combinatorial matrix theory.

Theorem 1.1.(Birkhoff theorem) [1] Let A be a doubly stochastic matrix of order n, then A can be expressed as the convex linear combination of several permutation matrices of order n, i.e.

$$A=\sum_i^t c_iP_i,$$

where P_i is the permutation matrix of order n, $\sum_i^t c_i=1$, and $c_i(i=1,2,\cdots,t)$ is positive.

Let G be a complete graph of order n, then its adjacency matrix $A=J_n-I_n$, where J_n is a square matrix whose all elements are 1. It is easily to be seen that $\Omega(A)=\Omega_n$ and $\mathrm{P}(A)=\mathrm{P}_n$. So, they are equivalent that $\overline{\mathrm{P}}(A)=\Omega(A)$ and Birkhoff theorem. Therefore, the compact graph is indeed the extension of the Birkhoff theorem. Birkhoff theorem can be obtained by compact graph.

Theorem 1.2.(Tinhöfer theorem) [2] The disjoint union of the same compact graph is compact graph.

By the definition of compact graph, K_2 (Complete graph of order two) is compact graph. By theorem 1.2, if G is the disjoint union of n copies of K_2, then G is compact graph, and its adjacency matrix is

$$A=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}.$$

It is easily to be seen that

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$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

if and only if

$$X_1{=}X_4, X_2{=}X_3 .$$

Let $X = S + T$ be a doubly stochastic matrix of order n, where S, T are non-negative matrices, then

$$\begin{pmatrix} S & T \\ T & S \end{pmatrix} \in \Omega(A).$$

By the compactness of G, the following results hold

$$\begin{pmatrix} S & T \\ T & S \end{pmatrix} = \sum_{i=1}^{t} c_i \begin{pmatrix} P_i^{(1)} & P_i^{(2)} \\ P_i^{(2)} & P_i^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} P_i^{(1)} & P_i^{(2)} \\ P_i^{(2)} & P_i^{(1)} \end{pmatrix} \in P(A),$$

where $c_i (i = 1, 2, \cdots, t)$ is positive number and $\sum_{i}^{t} c_i = 1$.

If $P_i = P_i^{(1)} + P_i^{(2)} \quad (i = 1, 2, \cdots, t)$, then P_i is permutation matrix for all i and $X = \sum_{i}^{t} c_i P_i$. It is truly Birkhoff theorem.

Let A be the adjacency matrix of graph G. If there exists a non-negative square matrix X such that $XA = AX$, then the X is called the non-negative self-isomorphism of A. All non-negative self-isomorphisms of G are denoted by

$Cone(A)$

$= \{X \mid AX = XA, X \text{ is a non-negative matrix}\}$.

The self-isomorphic set $\mathrm{P}(A)$ of G generates

$$\hat{\mathrm{P}}(A) = \{\sum c_i P_i \mid P_i \in \mathrm{P}(A), c_i \geq 0\}.$$

It is easily to be seen that $\hat{\mathrm{P}}(A) \subseteq Cone(A)$. Then, for what kind of graph does the equation $\hat{\mathrm{P}}(A) = Cone(A)$ hold? A graph G is called super-compact graph if its adjacency matrix A satisfies $\hat{\mathrm{P}}(A) = Cone(A)$. Obviously $\overline{\mathrm{P}}(A) \subseteq \hat{\mathrm{P}}(\mathrm{A})$, $\Omega(A) \subseteq Cone(A)$, and it can be proved that if the adjacency matrix A of the graph G satisfies $\hat{\mathrm{P}}(A) = Cone(A)$, there must be $\overline{\mathrm{P}}(A) = \Omega(A)$. So a super-compact graph must be a compact graph. But not all the graphs are compact graphs and the compact graphs are not necessarily super-compact graphs. Sometimes there is little difference between the non-compact and the compact and the super-compact . We will illustrate this in the following.

In 1986, the concept of compact graph is proposed by G.Tinhöfer[2]. In 1988, R.A.Brualdi systematically introduced the compact graph in [3]. In 1990, Bai-lian Liu related some results on the compact graph and gave some new results in [4]. In 1997, C.D.Godsil discussed the compact graph on the view of algebraic combination in [5]. After that, Xiu-ping Zhang and Wei-cheng Lu gave some methods of constructing compact graph in and some results in [6], [7], [8], [9],[10]. But until now, only few families of compact graphs are known. We have studied the compactness of a graph in [12-14]. Particularly, the following two important results are given in [13] :

Theorem 1.3.[13] The disjoint union of any compact graph and some isolate vertices is compact graph.

Theorem 1.4.[13] The graph obtained by attaching one pendant edge to each vertex of compact graph is compact.

Based on the above results, we obtained some useful results such as any wheel graph is compact graph and any windmill graph is compact graph.

In this paper, based on the previous research, the following results will been given: the graph obtained by attaching n pendant edges to each vertex of a complete graph is compact graph, and the graph obtained by attaching two pendant edges to each vertex of any compact graph is compact graph, and the disjoint union of any number of non-isomorphic complete graphs is a compact graph. The relations between non-compact graph and compact graph and super-compact graph will be discussed.

II Definitions and preliminary lennas

Definition 2.1. [4] The maximum number of non-zero elements in different rows and different columns of a nonnegative matrix is called the term rank of the matrix.

Definition 2.2. [4] Let $A = \left(a_{ij}\right)_{m\times n}$ $(m \leq n)$ be a matrix, we call

$$PerA = \sum_{i_1, i_2, \cdots, i_m \in P_m^n} a_{1i_1} a_{2i_2} \cdots a_{mi_m}$$

the permanent of A, where P_m^n presents the set of all permutations of m elements in $\{1, 2, \cdots, n\}$.

Lemma 2.1. [4] The permanent of doubly stochastic matrix is positive.

Lemma 2.2.[4] If the graph G is compact, then the complementary graph G^c of G is compact.

Lemma 2.3.[4] The complete graph K_n, circle graph C_n, tree graph T, bipartite graph $K_{n,n}$ and graph $\overline{K}_{n,n}$ are all compact graphs, where $\overline{K}_{n,n}$ is the graph obtained by deleting 1 factor from $K_{n,n}$.

Lemma 2.4.[9] A graph G is a super compact graph if and only if G is a compact and connected regular graph.

Lemma 2.5. [10] Let G_1 and G_2 be connected k-regular compact graphs of order n and m respectively, $V(G_1) \cap V(G_2) = \phi$, u be a vertex of G_1, v be a vertex of G_2, then the graph $\overline{G}$ obtained by adding edge uv to the graph $G_1 \cup G_2$ is also compact where $n \neq m$.

Lemma 2.6.[11] Let $\delta(G)$ be the minimum degree of the vertex of graph G with order n. If $\delta(G) > \left[\frac{n}{2}\right] - 1$, then G is a connected graph.

Lemma 2.7. A non-negative matrix must be a square matrix, if its row sum equals to its column sum but not equals to zero.

Proof. Let $A=(a_{ij})_{n\times m}$, and the row sum and column sum be both $r(\neq 0)$, then $a_{i1}=r-\sum_{j=2}^{m}a_{ij}$. Hence

$$\sum_{i=1}^{n}a_{i1}=nr-\sum_{i=1}^{n}\sum_{j=2}^{m}a_{ij}$$

$$\Rightarrow \sum_{i=1}^{n}a_{i1}=nr-\sum_{j=2}^{m}\sum_{i=1}^{n}a_{ij}$$

$$\Rightarrow r=nr-(m-1)r\Rightarrow n=m.$$

Therefore, $A=(a_{ij})_{n\times m}$ is a square matrix.

Lemma 2.8. Let G be a compact graph of order n and its adjacency matrix be A, $X\in\Omega(A)$, $X=S+T$, where S,T are non-negative matrices, then there are permutation matrices $P_1,P_2,\cdots,P_t\in \mathrm{P}(A)$ such that

$$X=\sum_{i}^{t}c_iP_i,\ \begin{pmatrix}S & T\\ T & S\end{pmatrix}=\sum_{i=1}^{t}c_i\begin{pmatrix}P_i^{(1)} & P_i^{(2)}\\ P_i^{(2)} & P_i^{(1)}\end{pmatrix},$$

where $\begin{pmatrix}P_i^{(1)} & P_i^{(2)}\\ P_i^{(2)} & P_i^{(1)}\end{pmatrix}$ is a permutation matrix of order $2n$, $P_i=P_i^{(1)}+P_i^{(2)}$, $c_i(i=1,2,\cdots,t)$ are positive numbers and $\sum_{i}^{t}c_i=1$.

Proof. Let $Y=\begin{pmatrix}S & T\\ T & S\end{pmatrix}$, by lemma 2.1, the term rank $\rho_Y=2n$. We use mathematical induction on the numbers $\sigma(Y)$ of non-zero elements of Y.

(i) If $\sigma(Y)=2n$, then X,Y are permutation matrices. Lemma 2.8 holds.

(ii) If $\sigma(Y)>2n$. Since $X\in\Omega(A)$ and G is a compact graph, so there is a permutation matrix $P_1\in\mathrm{P}(A)$ such that the positive elements of P_1 correspond to the n positive independent vectors of $X=S+T$. Decompose P_1 into the sum of two matrices $P_1^{(1)}$, $P_1^{(2)}$ whose elements are 0 and 1 such that the positive elements of $P_1^{(1)}$ correspond to the positive elements of S, the positive elements of $P_1^{(2)}$ correspond to the positive elements of T.So $\begin{pmatrix}P_1^{(1)} & P_1^{(2)}\\ P_1^{(2)} & P_1^{(1)}\end{pmatrix}$ is a permutation matrix and its positive elements group correspond to the independent group of $2n$ positive elements of $\begin{pmatrix}S & T\\ T & S\end{pmatrix}$:

$$\left\{a_{1j_1},a_{2j_2},\cdots,a_{2n,j_{2n}}\right\}.$$

Denote $c_1=\min\left\{a_{1j_1},a_{2j_2},\cdots,a_{2n,j_{2n}}\right\}$, then $0<c_1<1$.

Let

$$X_2=\frac{1}{1-c_1}(X-c_1P_1),Y_2=\frac{1}{1-c_1}(Y-c_1\begin{pmatrix}P_1^{(1)} & P_1^{(2)}\\ P_1^{(2)} & P_1^{(1)}\end{pmatrix})$$

then $X_1\in\Omega(A)$ and Y_1 are all doubly stochastic matrices, and

$$X_2=\frac{1}{1-c_1}(X-c_1P_1)=\frac{1}{1-c_1}(S-c_1P_1^{(1)})+\frac{1}{1-c_1}(T-c_1P_1^{(2)})$$

$$\sigma(Y_1)\le\sigma(Y)-2$$

Let $S_2=\frac{1}{1-c_1}(S-c_1P_1^{(1)}),T_2=\frac{1}{1-c_1}(T-c_1P_1^{(2)})$, then $X_2=S_2+T_2$.

If $\sigma(Y_2)>2n$, then make $P_2\in\mathrm{P}(A)$ such that the positive elements of P_2 correspond to the independent group of n positive elements of $X_2=S_2+T_2$. Decompose P_2 into the sum of two matrices $P_2^{(1)}$, $P_2^{(2)}$ whose elements are 0 and 1 such that the positive elements of $P_2^{(1)}$ correspond to the positive elements of S_2 and the positive elements of $P_2^{(2)}$ correspond to the positive elements of T_2. So $\begin{pmatrix}P_1^{(1)} & P_1^{(2)}\\ P_1^{(2)} & P_1^{(1)}\end{pmatrix}$ is permutation matrix, and its positive elements group correspond to the independent group of $2n$ positive elements of $Y_2=\begin{pmatrix}S_2 & T_2\\ T_2 & S_2\end{pmatrix}$:

$$\left\{a^{(1)}_{1j_1^{(1)}}a^{(1)}_{2j_2^{(1)}},\cdots,a^{(1)}_{2n,j_{2n}^{(1)}}\right\}.$$

Denote $c_2=\min\left\{a^{(1)}_{1j_1^{(1)}},a^{(1)}_{2j_2^{(1)}},\cdots,a^{(1)}_{2n,j_{2n}^{(1)}}\right\}$, then $0<c_2<1$.

Let

$$X_3=\frac{1}{1-c_2}(X_1-c_2P_2),Y_3=\frac{1}{1-c_2}(Y_1-c_2\begin{pmatrix}P_2^{(1)} & P_2^{(2)}\\ P_2^{(2)} & P_2^{(1)}\end{pmatrix})$$

$$\sigma(Y_3)\le\sigma(Y_2)-2,$$

then $X_3\in\Omega(A)$ and Y_3 are all doubly stochastic matrices.

Repeating the above process, the following iterative formula can be got:

$$\begin{cases}X=X_1,X_1=S+T;\\ Y=Y_1,Y_1=\begin{pmatrix}S & T\\ T & S\end{pmatrix},\end{cases}$$

$$\begin{cases} X_i = \dfrac{X_{i-1} - c_{i-1}P_{i-1}}{1-c_{i-1}} = S_i + T_i, \\ S_i = \dfrac{S_{i-1} - c_{i-1}P_{i-1}^{(1)}}{1-c_{i-1}}, T_i = \dfrac{T_{i-1} - c_{i-1}P_{i-1}^{(2)}}{1-c_{i-1}} (i = 2,3,\cdots) \\ Y_i = \dfrac{Y_{i-1} - c_{i-1}\begin{pmatrix} P_{i-1}^{(1)} & P_{i-1}^{(2)} \\ P_{i-1}^{(2)} & P_{i-1}^{(1)} \end{pmatrix}}{1-c_{i-1}} = \begin{pmatrix} S_i & T_i \\ T_i & S_i \end{pmatrix} (i = 2,3,\cdots \end{cases}$$

$$\Rightarrow \begin{cases} X_{i-1} = c_{i-1}P_{i-1} + (1-c_{i-1})X_i (i = 2,3,\cdots); \\ Y_{i-1} = c_{i-1}\begin{pmatrix} P_{i-1}^{(1)} & P_{i-1}^{(2)} \\ P_{i-1}^{(2)} & P_{i-1}^{(1)} \end{pmatrix} + (1-c_{i-1})Y_i (i = 2,3,\cdots), \end{cases}$$

where $P_{i-1} = P_{i-1}^{(1)} + P_{i-1}^{(2)}; \sigma(Y_i) \le \sigma(Y) - 2(i-1)$.

Since Y_i is doubly stochastic matrix, there is a t such that $\sigma(Y_t) = 2n$, i.e., Y_t is permutation matrix. So by

$$X_i = S_i + T_i,$$

we know X_t is also permutation matrix.

Let $Y_t = \begin{pmatrix} P_t^{(1)} & P_t^{(2)} \\ P_t^{(2)} & P_t^{(1)} \end{pmatrix}$, then $X_t = P_t^{(1)} + P_t^{(2)}$, X_t is denoted by P_t. Iterating the above formula, there is $c_i > 0 (i = 1,2,3,\cdots,t)$, $\sum_{i=1}^{t} c_i = 1$ such that

$$\begin{cases} X = c_1P_1 + c_2P_2 + c_3P_3 + \cdots + c_tP_t; \\ \tilde{Y} = c_1\begin{pmatrix} P_1^{(1)} & P_1^{(2)} \\ P_1^{(2)} & P_1^{(1)} \end{pmatrix} + c_2\begin{pmatrix} P_2^{(1)} & P_2^{(2)} \\ P_2^{(2)} & P_2^{(1)} \end{pmatrix} \\ \quad + c_3\begin{pmatrix} P_3^{(1)} & P_3^{(2)} \\ P_3^{(2)} & P_3^{(1)} \end{pmatrix} + \cdots + c_t\begin{pmatrix} P_t^{(1)} & P_t^{(2)} \\ P_t^{(2)} & P_t^{(1)} \end{pmatrix}, \end{cases}$$

In summary, Lemma 2.8 holds.

III Main results and their proof

Let G be any graph and G^* be the graph obtained by attaching n pendant edges to each vertices of G. Let A be the adjacency matrix of graph G, then by adjusting the order of vertices, we can obtain the adjacent matrix of G^*

$$A^* = \begin{pmatrix} A & I & I & \cdots & I \\ I & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ I & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let X^* be the doubly stochastic matrix with the same order as A^*. Perform the same partitioned mode of X^* as A^* such that

$$X^* = \begin{pmatrix} X & X_{12} & X_{13} & \cdots & X_{1,n+1} \\ X_{21} & X_{22} & X_{23} & \cdots & X_{2,n+1} \\ X_{31} & X_{32} & X_{33} & \cdots & X_{3,n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ X_{n+1,1} & X_{n+1,2} & X_{n+1,3} & \cdots & X_{n+1,n+1} \end{pmatrix}$$

If $A^*X^* = X^*A^*$, then

$$\begin{cases} X_{12} = X_{21} = X_{13} = X_{31} = \cdots = X_{1,n+1} = X_{n+1,1} = Y; \\ AX = XA; \\ YA + X_{i2} + X_{i3} + \cdots + X_{i,n+1} = X (i = 2,3,\cdots,n+1); \\ AY + X_{2j} + X_{3j} + \cdots + X_{n+1,j} = X (j = 2,3,\cdots,n+1). \end{cases}$$

Since X^* is doubly stochastic matrix, the row sum of $Y + \sum_{k=2}^{n+1} X_{ik}$ equals to the row sum of $X + nY$. Hence the row sum of $\sum_{k=2}^{n+1} X_{ik}$ is equal to or larger than the row sum of X. Since $YA \ge 0$, and

$$YA + \sum_{k=2}^{n+1} X_{ik} = X (i = 2,3,\cdots,n+1),$$

we know $YA = 0$. Thus $\sum_{k=2}^{n+1} X_{ik} = X (i = 2,3,\cdots,n+1)$.

Also, the row sum of $Y + \sum_{k=2}^{n+1} X_{ik}$ and $X + nY$ are all equal to 1, so if $n \ge 2$, then $Y = 0$. Therefore,

$$X^* = \begin{pmatrix} X & 0 & 0 & \cdots & 0 \\ 0 & X_{22} & X_{23} & \cdots & X_{2,n+1} \\ 0 & X_{32} & X_{33} & \cdots & X_{3,n+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & X_{n+1,2} & X_{n+1,3} & \cdots & X_{n+1,n+1} \end{pmatrix},$$

$$\begin{cases} XA = AX; \\ \sum_{k=2}^{n+1} X_{ik} = X (i = 2,3,\cdots,n+1); \\ \sum_{k=2}^{n+1} X_{jk} = X (j = 2,3,\cdots,n+1). \end{cases} \quad (*)$$

Whereas, if X^* satisfies the condition (*), then

$$A^*X^* = X^*A^*.$$

If G is complete graph, then $\Omega(A) = \Omega_n$. Obviously, the condition $XA = AX$ in (*) can be satisfied. Hence for complete graph G and the null graph G_0 with same order (Constructed by some isolated vertices), we have $\Omega(A^*) = \Omega(A_0^*)$, where A^* and A_0^* are the adjacency matrices of G^* and G_0^* respectively. Since G_0^* is the disjoint union of the same star graph, and the star graph is compact[4], by theorem 1.2, G_0^* is a compact graph. So G^* is also a compact graph and the following theorem holds:

Theorem 3.1. The graph obtained by attaching n pendant edges to each vertices of complete graph is a compact graph.

Whether the result similar as theorem 3.1 holds for any compact graph? It is an unsolved problem. For the special case, we give the following theorem after theorem 1.4:

Theorem 3.2. The graph obtained by attaching two pendant edges to each vertices of any compact graph is a compact graph.

Proof. Let G be a compact graph of order n, A be the adjacency matrix of G, G^* be the graph obtained by attaching two pendant edges to each vertices of G, and A^* be the adjacency matrix of G^*. By properly adjusting the order of the vertices, we can make

$$A^* = \begin{pmatrix} A & I & I \\ I & 0 & 0 \\ I & 0 & 0 \end{pmatrix}$$

Let $X^* \in \Omega(A^*)$. Perform the same partitioned mode of X^* as A^* such that

$$X^* = \begin{pmatrix} X & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$$

Since $A^* X^* = X^* A^*$, combining with above discussion, we know

$$X^* = \begin{pmatrix} X & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{pmatrix},$$

$$\begin{cases} XA = AX; \\ X_{i2} + X_{i3} = X (i = 2,3); \\ X_{2j} + X_{3j} = X (j = 2,3). \end{cases}$$

Hence $X_{22} = X_{33}$, $X_{32} = X_{23}$. Let $X_{22} = S, X_{32} = T$, then

$$X^* = \begin{pmatrix} X & 0 & 0 \\ 0 & S & T \\ 0 & T & S \end{pmatrix},$$

where $X \in \Omega(A)$, $\begin{pmatrix} S & T \\ T & S \end{pmatrix}$ is doubly stochastic matrix of order $2n$, and $X = S + T$.

Since G is a compact graph, by lemma 2.8, we have

$$X = \sum_{i=1}^{t} c_i P_i,\ \sum_{i=1}^{t} c_i = 1,\ P_i \in \mathrm{P}(A^*).$$

So G^* is a compact graph.

From lemma 2.4 and theorem 3.2, the graph obtained by adding two pendent edges to each vertex of any super-compact graph is a compact graph, but not a super --compact graph.

Theorem 3.3 The disjoint union of any number of non-isomorphic complete graphs is a compact graph.

Proof. Let G be the disjoint union of the n distinct complete graphs $G_1, G_2, \cdots, G_n$ with the adjacency matrices $A_1, A_2, \cdots, A_n$ respectively, where A_i is the matrix of order n_i. Then the adjacency matrix of G is

$$A = diag(A_1, A_2, \cdots, A_n).$$

Let $X = (X_{ij})_{n \times n} \in \Omega(A)$, then $A_i X_{ij} = X_{ij} A_j$, $(i, j = 1, 2, \cdots, n)$. Since $A_i = J_i - I_i$, $A_j = J_j - I_j$, so $J_i X_{ij} = X_{ij} J_j$. Then the row sum and the column sum of X_{ij} are same. According to Lemma 2.7, when $i \neq j$, $X_{ij} = 0$. Hence

$$X = diag(X_{11}, X_{22}, \cdots, X_{nn}),$$
$$X_{ii} \in \Omega(A_i)(i = 1, 2, \cdots, n).$$

By Lemma 2.3, $G_i (i = 1, 2, \cdots, n)$ are compact graphs. So there exist $P_i \in P(A_i)(i = 1, 2, \cdots, n)$ such that the positive elements of P_i correspond to the independent group

$$\left\{ x^{(i)}_{1\sigma_i(1)}, x^{(i)}_{1\sigma_i(2)}, \ldots, x^{(i)}_{1\sigma_i(n_i)} \right\}$$

of the positive elements of X_{ii}.

Let

$$\varepsilon_i = \min\left\{ x^{(i)}_{1\sigma_i(1)}, x^{(i)}_{1\sigma_i(2)}, \cdots, x^{(i)}_{1\sigma_i(n_i)} \right\};$$

$$\varepsilon = \min\left\{ \varepsilon_i \mid i = 1, 2, \cdots, n \right\}; P = diag\left\{ P_1, P_2, \cdots, P_n \right\}.$$

(1) If $\varepsilon = 1$, then $X = P \in \overline{\mathrm{P}}(A)$.

(2) If $\varepsilon < 1$, let $Y = \frac{1}{1-\varepsilon}(X - \varepsilon P)$, then it is obviously that $Y \in \mathrm{P}(A)$ and Y has at least one zero elements more than X. Using mathematical induction on the number of non-zero elements, we obtain $X \in \overline{\mathrm{P}}(A)$.

In summary, $\Omega(A) \subseteq \overline{\mathrm{P}}(A)$. That is $\Omega(A) = \overline{\mathrm{P}}(A)$, and G is a compact graph.

Example 3.1. By theorem3.3, the disjoint union of the complete graphs K_5 and K_6 is a compact graph. And by theorem 3.2, the following graph (Fig 1 Compact graph) is compact.

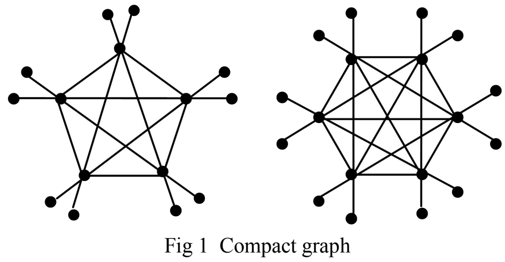

Fig 1 Compact graph

Theorem 3.4 The disjoint union of circle C_3 and circle $C_n (n > 3)$ is a non-compact graph.

Proof. Let A be the adjacency matrix of C_3, B be the adjacency matrix of $C_n(n>3)$, then $A=J_3-I_3$, $B=\left[b_{ij}\right]_n$, where b_{ij} satisfies that $b_{ij}=1$ if $j\equiv i+1(\mathrm{mod}\, n)$ or $j\equiv i-1(\mathrm{mod}\, n)$ and otherwise $b_{ij}=0$.

Let

$$\alpha=\begin{bmatrix}1 & 1\end{bmatrix},\ \beta=\begin{bmatrix}1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\end{bmatrix},$$

then

$$A=\begin{bmatrix}0 & \alpha\\ \alpha^T & A_1\end{bmatrix},\ B=\begin{bmatrix}0 & \beta\\ \beta^T & B_1\end{bmatrix},$$

It is easily to be seen that $\frac{1}{n+3}J_{n+3}\in\Omega(diag(A,B))$.

Therefore, if the disjoint union of circle C_3 and circle $C_n(n>3)$ is a compact graph, then

$$\frac{1}{n+3}J_{n+3}=\sum c_iP_i,\ \sum c_i=1,\ P_i\in \mathrm{P}(diag(A,B)).$$

Furthermore, there must be a permutation matrix P whose element $(1,4)$ is 1 in $\mathrm{P}(diag(A,B))$. Let

$$P=\begin{bmatrix}P_{3\times3} & P_{3\times n}\\ P_{n\times3} & P_{n\times n}\end{bmatrix}, P_{3\times n}=\begin{bmatrix}1 & 0\\ 0 & X\end{bmatrix},$$

then by $P\cdot diag(A,B)=diag(A,B)\cdot P$, we known

$$AP_{3\times n}=P_{3\times n}B$$

$$\Rightarrow\begin{bmatrix}0 & \alpha\\ \alpha^T & A_1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & X\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & X\end{bmatrix}\begin{bmatrix}0 & \beta\\ \beta^T & B_1\end{bmatrix}$$

$$\Rightarrow\begin{cases}\alpha X=\beta,\\ X\beta^T=\alpha^T,\Rightarrow \alpha A_1\alpha^T=\beta B_1\beta^T.\\ A_1X=XB_1.\end{cases}$$

However, $\alpha A_1\alpha^T=2$, $\beta B_1\beta^T=0$, $\alpha A_1\alpha^T\neq\beta B_1\beta^T$. Hence the disjoint union of circle C_3 and circle $C_n(n>3)$ is a non-compact graph. The proof of Theorem 3.4 is finished.

Since the complete graph $K_n(n\geq1)$ is a $n-1$ regular connected compact graph, there exists a n regular connected compact graph of order $n+1$ for any non-negative integer n. Therefore, the super-compact graphs of any order exist.

For non-compact graphs, the following conclusions can be drawn from Lemma 2.2 and Theorem 3.4:

Corollary 3.1 If $n\geq3$, the $n+1$ regular connected non-compact graph of order $n+4$ exist.

Sometimes, the difference between a non-compact graph and a compact graph is very small and maybe is only lost one edge, which can be seen from Lemma 2.5 and Theorem 3.4. For example:

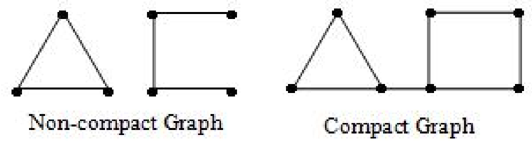

Fig 2 Difference between non-compact graph and compact gagraph

Similarly, the difference between the compact graph and the super-compact graph can be only lost one edge, which can be seen from Theorem 1.3, Lemma 2.2, Lemma 2.3 and Lemma 2.4. For example:

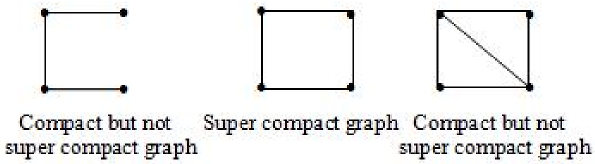

Fig 3 Difference between compact graph and super compact graph

The following two conclusions are also notable.

Corollary 3.2. For $n\geq5$, C_n and C_n^c both are simultaneously super-compact graphs .

Proof. When $n\geq5$, $\delta(C_n^c)=n-3>\left[\frac{n}{2}\right]-1$, so C_n^c is a connected graph. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, C_n and C_n^c both are simultaneously super-compact graphs .

Corollary 3.3. When $n\geq3$, $\bar{K}_{n,n}$ and $\bar{K}_{n,n}^c$ both are simultaneously super-compact graphs, where $\bar{K}_{n,n}$ is the graph obtained by deleting 1-factor from $K_{n,n}$.

Proof. Since $\delta(\bar{K}_{n,n}^c)=n+1>\left[\frac{2n}{2}\right]-1=n-1$, so $\bar{K}_{n,n}^c$ is a connected graph. When $n\geq3$, it is easy to know that $\bar{K}_{n,n}$ is a connected graph by mathematical induction. By Lemma 2.2, Lemma 2.3 and Lemma 2.4, $\bar{K}_{n,n}$ and $\bar{K}_{n,n}^c$ both are simultaneously super-compact graphs.

REFERENCES

[1] Birkhoff G, “Tres observaciones sobre el algebra lineal,” Univ.Nac. Tucumàn Rev. Sor. A, vol. 5, pp. 147-150, 1946.

[2] Tinhöfer G, “Gragh isomorphism and theorems of Birkhoff type,” Computing, vol. 36, pp. 285-300, 1986.

[3] Brualdi A R, “Some application of doubly stochastic matrices,” Linear Algebra Application, vol. 107, pp. 77-100, 1988.

[4] Liu Bolian,Combinatorial Matrix Theory. Beijing, Science Press, 2005.

[5] Godsil D C, “Compact graph and equitable partitions,” Linear Algebra Application, vol. 225, pp. 259-266, 1997.

[6] Zhang Xiuping, “Some results on compact graphs and supercompact graphs,” Journal of Beijing Normal University(Natural science), vol. 35, issue 1, pp. 16-21, 1999.

[7] Zhang Xiuping, “The compactness and supercompactness of Semi-Complement graph,” Journal of Beijing Normal University(Natural science), vol. 35, issue 3, pp. 316-319, 1999.

[8] Zhang Xiuping, “Some results on the compactness and supercompactness of (m,k) graph and their Semi-Complement graph,” Journal of Beijing Normal University(Natural science), vol. 36, issue 5, pp. 569-573, 2000.

[9] Lu Weicheng, Zhang Xuanhao, “The Theory of Compact Graph and Supper Compact Graph,” Science Technology And Engineering, vol. 11, issue 11, pp. 2399-2403, 2011.

[10] Zhang Xuanhao, Lu Weicheng, "A Method of Constructing the Compact Graph," Science Technology And Engineering, vol. 11, issue 26, pp. 6249-6252, 2011.

[11] Bondy J.A. and Murty U.S.R., Graph theory with applications. London: Macmillan Press, 1976.

[12] Sinqinbate, WANG Jing-yu. "Some Research Results about Compact Graph," Mathematics In Practice And Theory, vol. 45, issue 7, pp.296-303, 2015.

[13] Sinqinbate, WANG Jing-yu. "Two results of the compact graph and its applications," Operations Research Transactions, vol. 19, issue 4, pp.72-82, 2015.

[14] Siqinbate, WANG Jing-yu, "The compactness on complete graph," Mathematics In Practice And Theory, vol. 46, issue 18, pp. 229-234, 2016.