Extended Generalized Adams-Type Second Derivative Boundary Value Methods

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Abstract—**In [14], we derived a family of second derivative generalized Adams-type methods (SDGAMs) with order $p = 2k + 2$ for all values of the step-length $k \geq 1$. These methods which are implemented as boundary value methods (BVMs) are all $0_{v,k-v}$-stable and $A_{v,k-v}$-stable requiring $(v, k-v)$-boundary conditions. In this paper, an extension of [14] is proposed with better stability characteristics, higher orders of accuracy and smaller error constants than the methods in [14]. Numerical examples are given to illustrate the accuracy of the proposed methods.**

Keywords: ***Linear Multistep Formulae, Boundary Value Methods,*** $0_{k_1,k_2}$***-stable,*** A_{k_1,k_2}***-stable.***

AMS subject classification: 65L04, 65L05

1 Introduction

Our interest is on the approximate numerical integration of the stiff initial value problem

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0 \tag{1}$$

on the finite interval $I = [x_0, x_N]$ where $y : I \to R^m$ and $f : I \times R^m \to R^m$ is continuous and differentiable.

In recent years a wide variety of approaches have been proposed for the development of more advanced and efficient methods for stiff problems (1). A potentially good numerical method for the solution of stiff systems of ordinary differential equations (ODEs) must have good accuracy and some reasonably wide region of absolute stability ([5]). A-Stability which was proposed in ([7]) is one of the first and most important stability requirement particularly for linear multi-step methods . However A-Stability requirement puts a severe limitation on the choice of suitable methods for stiff problems. This is proved in the so-called Dahlquist second barrier ([5]) which says, among other things, that the order of an A-stable linear multistep method must be ≤ 2 and that an A-stable linear multistep method must be implicit. This famous theorem of Dahlquist has opened a new research direction in the development of numerical algorithms for the solution of stiff initial value problems (IVPs). The search for higher order A-stable multistep methods to improve the accuracy and extend the stability region is carried out in the two main directions:

- Use higher order derivatives of the solutions.

- Throw in additional stages, off-step points, super-future points and the likes. This leads into the large field of general linear methods [9].

In a recent paper [14], Nwachukwu and Mokwunyei considered a family of second derivative generalized Adams-type methods (SDGAMs) for the numerical solution of the stiff IVPs (1) in ODEs. These formulas of order $p = 2k + 2$ which are all $0_{v,k-v}$-stable and $A_{v,k-v}$-stable with $(v, k-v)$-boundary conditions for all values of the step-length $k \geq 1$ are of the form:

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\gamma_i g_{n+i} \tag{2}$$

where

$$v = \begin{cases} \frac{k+1}{2} & for \quad odd \quad k \\ \frac{k}{2} & for \quad even \quad k \end{cases} \tag{3}$$

Table 1: The Error Constant (EC) and Order p of the class of methods in [14] for $k = 1(1)10$

k	EC	p
1	$\frac{1}{720}$	4
2	$\frac{-1}{9450}$	6
3	$\frac{103}{25401600}$	8
4	$\frac{-89}{314344800}$	10
5	$\frac{379397}{28768836096000}$	12
6	$\frac{-3901}{4382752374000}$	14
7	$\frac{1964407}{43597116186624000}$	16
8	$\frac{724523791}{242582188319190720000}$	18
9	$\frac{22424299416863}{141590371678145239449600000}$	20
10	$\frac{346654620623}{33392651133078198561600000}$	22

The error constants and order of convergence of the class of methods in [14] for various values of the step number k are given in table 1. This may be compared with the error constants (EC) of the proposed BVM in (6). See that the EC of (6) are smaller in magnitude than that from (2) in table 1

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The BVM in (2) are used with the following set of additional

initial methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k}\beta_i f_i + h^2\sum_{i=0}^{k}\gamma_i g_i, \qquad (4)$$

$$j = 1, \ldots, v-1$$

and final methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k}\beta_{k-i} f_{N-i} + h^2\sum_{i=0}^{k}\gamma_{k-i} g_{N-i}, \qquad (5)$$

$$j = N-k+v+1, \ldots, N.$$

These methods (2) with higher order than the SDGBDF of Nwachukwu and Okor [13] generalize the second derivative methods of Jator and Sahi [10] and extends the GAMs proposed by Brugnano and Trigiante [1] to second derivative methods.

In this paper, we shall derive a class of extended generalized Adams-Type second derivative boundary value methods (EGASDBVMs). This class of methods is developped as an extension of the SDGAMs of Nwachukwu and Mokwunyei [14]. The new methods which shall contain a super future point in the first and second order derivative terms of the SDGAMs will be seen to have better stability characteristics, higher orders of accuracy and smaller error constant than the SDGAMs. These methods shall be derived using the Taylor's series approach and shall be implemented as BVMs in the sense of [1, 2, 3, 4, 10, 12, 13, 14].

In section 2 we derive the new methods. In section 3 we consider the implementation details of the BVMs. Some numerical experiments are given in section 4. In section 5, we have the conclusion of the paper.

2 Derivation of the EGASDBVMs

We are going to introduce a new class of extended generalized Adams-Type second derivative boundary value methods (EGASDBVMs) with the following general form

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k+1}\beta_i f_{n+i} + h^2\sum_{i=0}^{k+1}\gamma_i g_{n+i}, \qquad (6)$$

where

$$v = \begin{cases} \frac{k+1}{2} & for \quad odd \quad k \\ \frac{k+2}{2} & for \quad even \quad k, \end{cases} \qquad (7)$$

where $g(x,y) = y'' = f_x + f_y f$ and the coefficients β_i and γ_i are chosen so that (6) has order $p = 2k+4$. The class of methods (6) is of order p if and only if

$$v^q - (v-1)^q = q\sum_{i=1}^{k+1} i^{q-1}\beta_i + q(q+1)\sum_{i=1}^{k+1} i^{q-2}\gamma_i,$$

$0 \le q \le p$. The coefficients, the order and the error constants of the k-step methods (6) are given in Table 2, for $k = 1, 2, ..., 10$. Following the approach in Nwachukwu et al [12], Nwachukwu and Okor [13] and Nwachukwu and Mokwunyei [14], for odd values of k, the boundary loci of the EGASDBVMs are given in Figures 1. For even values of k, the boundary loci of the methods (6) coincide with the imaginary axis. The new methods are $0_{v,(k+1)-v}$-stable and $A_{v,(k+1)-v)}$-stable and are used with $(v,(k+1)-v)$-boundary conditions.

3 Implementation Details of the BVMs

The EGASDBVMs (6) are used with the following set of additional methods

initial methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k+1}\beta_i f_i + h^2\sum_{i=0}^{k+1}\gamma_i g_i, \qquad (8)$$

$$j = 1, \ldots, v-1$$

and final methods

$$y_j - y_{j-1} = h\sum_{i=0}^{k+1}\beta_{k-i} f_{N-i} + h^2\sum_{i=0}^{k+1}\gamma_{k-i} g_{N-i}, \qquad (9)$$

$$j = N-(k+1)+v+1, \ldots, N$$

For $k = 2$ we have

the main method,

$$\begin{aligned} y_{n+2} - y_{n+1} &= h(\frac{3}{224}f_n + \frac{109}{224}f_{n+1} + \frac{109}{224}f_{n+2} \\ &+ \frac{3}{224}f_{n+3}) + h^2(\frac{31}{10080}g_n + \frac{113}{1120}g_{n+1} \\ &- \frac{113}{1120}g_{n+2} - \frac{31}{10080}g_{n+3}), \end{aligned}$$

the initial additional method

$$\begin{aligned} y_1 - y_0 &= h(\frac{6893}{18144}f_0 + \frac{313}{672}f_1 + \frac{89}{672}f_2 + \frac{397}{18144}f_3) \\ &+ h^2(\frac{1283}{30240}g_0 - \frac{851}{3360}g_1 - \frac{269}{3360}g_2 - \frac{163}{30240}g_3) \end{aligned}$$

and the final additional method

$$\begin{aligned} y_{N+1} - y_N &= h(\frac{397}{18144} f_{N-2} + \frac{89}{672} f_{N-1} + \frac{313}{672} f_N \\ &+ \frac{6893}{18144} f_{N+1}) + h^2(\frac{163}{30240} g_{N-2} \\ &+ \frac{269}{3360} g_{N-1} + \frac{851}{3360} g_N - \frac{1283}{30240} g_{N+1}) \end{aligned}$$

For $k = 3$ we have

the main method,

$$\begin{aligned} y_{n+2} - y_{n+1} &= h(\frac{26081}{4354560} f_n + \frac{122341}{272160} f_{n+1} \\ &+ \frac{313}{630} f_{n+2} + \frac{12091}{272160} f_{n+3} \\ &+ \frac{14111}{4354560} f_{n+4}) + h^2(\frac{893}{725760} g_n \\ &+ \frac{6887}{90720} g_{n+1} - \frac{47}{320} g_{n+2} \\ &- \frac{1721}{90720} g_{n+3} - \frac{103}{145152} g_{n+4}), \end{aligned}$$

the initial additional method,

$$\begin{aligned} y_1 - y_0 &= h(\frac{1539551}{4354560} f_0 + \frac{89371}{272160} f_1 + \frac{103}{630} f_2 \\ &+ \frac{38341}{272160} f_3 + \frac{59681}{4354560} f_4) + h^2(\frac{26051}{725760} g_0 \\ &- \frac{31207}{90720} g_1 - \frac{81}{320} g_2 - \frac{1243}{18144} g_3 - \frac{2237}{725760} g_4), \end{aligned}$$

the first final additional method,

$$\begin{aligned} y_N - y_{N-1} &= h(\frac{14111}{4354560} f_{N-3} + \frac{12091}{272160} f_{N-2} \\ &+ \frac{313}{630} f_{N-1} + \frac{122341}{272160} f_N + \frac{26081}{4354560} f_{N+1}) \\ &+ h^2(\frac{103}{145152} g_{N-3} + \frac{1721}{90720} g_{N-2} \\ &+ \frac{47}{320} g_{N-1} - \frac{6887}{90720} g_N - \frac{893}{725760} g_{N+1}) \end{aligned}$$

and the second final additional method,

$$\begin{aligned} y_{N+1} - y_N &= h(\frac{59681}{4354560} f_{N-3} + \frac{38341}{272160} f_{N-2} \\ &+ \frac{103}{630} f_{N-1} + \frac{89371}{272160} f_N + \frac{1539551}{4354560} f_{N+1}) \\ &+ h^2(\frac{2237}{725760} g_{N-3} + \frac{1243}{18144} g_{N-2} \\ &+ \frac{81}{320} g_{N-1} + \frac{31207}{90720} g_N - \frac{26051}{725760} g_{N+1}) \end{aligned}$$

4 Numerical experiments

This section deals with some numerical experiments carried out in MATLAB. The performance of the EGASDBVMs is examined on five tests of initial value problems, particularly stiff systems.

Example 1: Consider the mildly stiff problem composed of two first order equations which has been solved by Yakubu and Markus [16]

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the exact solution is given by the sum of two decaying exponential components

$$\begin{cases} y_1(x) = 4e^{-x} - 3e^{-1000x} \\ y_2(x) = -2e^{-x} + 3e^{-1000x} \end{cases}$$

The stiffness ratio is 1:1000.

The results of Yakubu and Markus [16] are reproduced in Table 7 and compared with that obtained using the EGASDBVMs of order $p = 8$. It can be seen in Table 7 that the result obtained for the EGASDBVMs is superior to those of Yakubu and Markus [16] of orders $p = 8$ and $p = 11$.

Example 2: We consider the Singularly Perturbed Problem proposed by Hairer and Wanner [9] which has been solved by Nwachukwu and Okor [13] and Nwachukwu and Mokwunyei [14]

$$y_1' = -(2 + 10^4) y_1 + 10^4 y_2^2, \qquad y_2' = y_1 - y_2 - y_2^2,$$
$$y_1(0) = 1, \quad y_2(0) = 1$$

The exact solution is $y_1 = e^{-2t}, \quad y_2 = e^{-t}$

The EGASDBVM for $k = 3$ is applied to this problem and the absolute errors are compared with the second derivative generalized backward differentiation formulae (SDGBDF) proposed by Nwachukwu and Okor [13] and the second derivative generalized Adams-type methods (SDGAMs) developed by Nwachukwu and Mokwunyei [14]. The results in Table 8 show that the newly derived method (6) performs better than the SDGBDF and the SDGAMs for the same step number, $k = 3$.

Example 3: We consider the moderately stiff problem solved by Jia-Xiang and Jiao-Xun [11],

$$y' = -y - 10z, \quad y(0) = 1; \quad y(x) = e^{-x} cos 10x$$
$$z' = 10y - z, \quad z(0) = 0; \quad z(x) = e^{-x} sin 10x$$

The results obtained for Problem 3 using step sizes $h = \{0.04, 0.1, 0.4\}$ are given in Table 9. The method is implemented with these step sizes to be able to compare the newly developed method (6) with the existing methods. The maximum errors ($Max||y_i - y(x_i)||$) in the interval $0 < x < 10$ are computed. x_T are some points on the range of integration. It is observed that the EGASD-BVM is superior to the methods of Ehigie et al. [6], Gear [8] and Jia-Xiang and Jiao-Xun [11] and highly competitive with the method of Nwachukwu and Mokwunyei [14]. The details of the numerical results are presented in Table 9.

Example 4: We consider the following nonlinear IVP considered by Wu and Xia [15]

$$y_1' = -1002y_1 + 1000y_2^2, \qquad y_1(0) = 1,$$

$$y_2' = y_1 - y_2(1 + y_2), \qquad y_2(0) = 1.$$

The exact solution of the system is given by

$$y_1(x) = e^{-2x}, \qquad y_2(x) = e^{-x} \ .$$

From the numerical results in Table 10 it is obvious that our method for step sizes $h = \{0.024, 0.01\}$ compares favourably with the method of Jator and Sahi [10] with step sizes $h = \{0.008, 0.006\}$ and it is more accurate than the method of Wu and Xia [15] where step sizes $h = \{0.002, 0.001\}$ are used. Table 10 contains the details of the numerical results.

Example 5: Van der Pol equations (nonlinear problem), [9]

$$y_1' = y_2, \qquad y_2' = -y_1 + 10y_2(1 - y_1^2),$$

$$y_1(0) = 2, \quad y_2(0) = 0$$

The results of Example 5 using the EGASDBVMs for $k = 2$ are presented in Figure 2. The results are compared with the solutions from the Ode15s in MATLAB. The solid lines are the solutions of the EGASDBVMs. The figure shows that the new method coincides with the Ode15s in MATLAB.

5 Conclusion

A newly derived class of extended generalized Adams-type second derivative boundary value methods (EGAS-DBVMs) has been developed for the solution of stiff systems of ordinary differential equations and implemented as boundary value methods. The efficiency of the EGAS-DBVMs has been demonstrated on some standard numerical examples. Details of the numerical results are displayed in Tables (7 -10) and Figure 2 .

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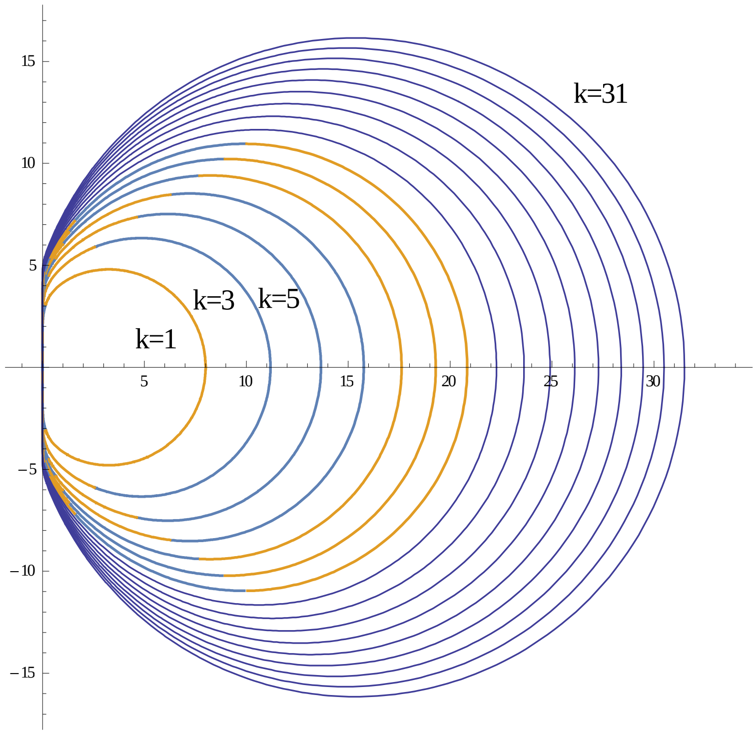

Figure 1: Stability region (exterior of closed curves) of (6), k=1 (2) 31

Table 2: The Coefficients, Error Constant (EC) and Order p of the class of methods (6) for $k = 1(1)10$

k	v	β_0	β_1	β_2	β_3	β_4
1	1	$\frac{101}{240}$	$\frac{8}{15}$	$\frac{11}{240}$	0	0
2	2	$\frac{3}{224}$	$\frac{109}{224}$	$\frac{109}{224}$	$\frac{3}{224}$	0
3	2	$\frac{26081}{4354560}$	$\frac{122341}{272160}$	$\frac{313}{630}$	$\frac{12091}{272160}$	$\frac{14111}{4354560}$
4	3	$\frac{4001}{4561920}$	$\frac{3581}{168960}$	$\frac{136267}{285120}$	$\frac{136267}{285120}$	$\frac{3581}{168960}$
5	3	$\frac{18227803}{56609280000}$	$\frac{156486943}{12972960000}$	$\frac{758335087}{1660538880}$	$\frac{587192}{1216215}$	$\frac{68960401}{1660538880}$
6	4	$\frac{217426757}{3736212480000}$	$\frac{72234599}{29889699840}$	$\frac{384031751}{15375360000}$	$\frac{1569368687}{3321077760}$	$\frac{1569368687}{3321077760}$
7	4	$\frac{275618952431}{14227497123840000}$	$\frac{97292813749}{88921857024000}$	$\frac{13323334967}{814302720000}$	$\frac{1942806486353}{4234374144000}$	$\frac{314527667}{661620960}$
8	5	$\frac{27673701304843}{7224981721251840000}$	$\frac{50714562811}{200610349056000}$	$\frac{139894823711}{35416577064960}$	$\frac{3534206761517}{131650541568000}$	$\frac{34298478405773}{73139189760000}$
9	5	$\frac{331823317063312681}{275891087727230976000000}$	$\frac{970724650372992181}{965618807045308416000000}$	$\frac{218383591648975999}{105966398578360320000}$	$\frac{107659928227440097}{5588071800030720000}$	$\frac{478957870612778899}{1042672289218560000}$
10	6	$\frac{1708782140591030611}{683361001908987494400000}$	$\frac{2047389963572905829}{843655557912330240000000}$	$\frac{37741522652639971}{66968273210572800000}$	$\frac{18501000613885421}{35213449373732044800}$	$\frac{30831424889569155}{111007310251622400000}$

Table 3: Table 2 continued

k	β_5	β_6	β_7	β_8	β_9
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	0
4	$\frac{4001}{4561920}$	0	0	0	0
5	$\frac{82799329}{12972960000}$	$\frac{4173367}{18869760000}$	0	0	0
6	$\frac{384031751}{15375360000}$	$\frac{72234599}{29889699840}$	$\frac{217426757}{3736212480000}$	0	0
7	$\frac{827205144427}{21171870720000}$	$\frac{4146181343}{488581632000}$	$\frac{331496225143}{444609285120000}$	$\frac{14004257503}{948499808256000}$	0
8	$\frac{34298478405773}{731391897600000}$	$\frac{3534206761517}{131650541568000}$	$\frac{139894823711}{35416577064960}$	$\frac{50714562811}{200610349056000}$	$\frac{27673701304843}{7224981721251840000}$
9	$\frac{1387127254973}{29462808375000}$	$\frac{193548663362377577}{5213361446092800000}$	$\frac{3916362537630989}{399147985716480000}$	$\frac{1094574383125943}{784936285765632000}$	$\frac{147407778627852047}{1931237614090616832000}$
10	$\frac{614536471476768239}{13176627830784000000}$	$\frac{614536471476768239}{13176627830784000000}$	$\frac{308314248895691557}{11100731025162240000}$	$\frac{18501000613885421}{3521344937373204480}$	$\frac{3774152265263971}{66968273210572800000}$

Table 4: Table 2 continued

k	β_{10}	β_{11}	γ_0	γ_1	γ_2
1	0	0	$\frac{13}{240}$	$\frac{-1}{6}$	0
2	0	0	$\frac{31}{10080}$	$\frac{113}{1120}$	$\frac{-1}{80}$
3	0	0	$\frac{893}{725760}$	$\frac{6887}{90720}$	$\frac{-47}{320}$
4	0	0	$\frac{313}{1774080}$	$\frac{39517}{5322240}$	$\frac{143471}{1330560}$
5	0	0	$\frac{3777757}{62270208000}$	$\frac{59197}{16016000}$	$\frac{24151013}{276756480}$
6	0	0	$\frac{3972713}{373621248000}$	$\frac{4203911}{5748019200}$	$\frac{153317011}{13837824000}$
7	0	0	$\frac{343925311}{101624979456000}$	$\frac{386887999}{1270312243200}$	$\frac{85580953}{13571712000}$
8	0	0	$\frac{369733300393}{567677135241216000}$	$\frac{367112146447}{5406448907059200}$	$\frac{3607240110641}{2365321396838400}$
9	$\frac{1877160528485018593}{1931237614090616832000000}$	0	$\frac{43303804162621}{218961180735897600000}$	$\frac{195229126044109}{7663641325756416000}$	$\frac{548860751149819}{756902846988288000}$
10	$\frac{2047389963572905829}{84365555791233024000000}$	$\frac{1708782140591030611}{6833610019089874944000000}$	$\frac{2833553100322723}{705055001969590272000000}$	$\frac{46586581380730333}{7833944466328780800000}$	$\frac{77093601173942957}{402888572554051584000}$

Table 5: Table 2 continued

k	γ_3	γ_4	γ_5	γ_6	γ_7
1	0	0	0	0	0
2	$\frac{-31}{10080}$	0	0	0	0
3	$\frac{-1721}{90720}$	$\frac{-103}{145152}$	0	0	0
4	$\frac{-143471}{1330560}$	$\frac{-39517}{5322240}$	$\frac{-313}{1774080}$	0	0
5	$\frac{-37969}{272160}$	$\frac{-6228251}{276756480}$	$\frac{-309979}{144144000}$	$\frac{-379397}{8895744000}$	0
6	$\frac{1666591847}{14944849920}$	$\frac{-1666591847}{14944849920}$	$\frac{-153317011}{13837824000}$	$\frac{-4203911}{5748019200}$	$\frac{-3972713}{373621248000}$
7	$\frac{119713951991}{1270312243200}$	$\frac{-56714731}{418037760}$	$\frac{-17444761861}{705729024000}$	$\frac{-90058579}{24429081600}$	$\frac{-1364044741}{6351561216000}$
8	$\frac{96373826609}{6895980748800}$	$\frac{109868350967579}{965437304832000}$	$\frac{-109868350967579}{965437304832000}$	$\frac{-96373826609}{6895980748800}$	$\frac{-3607240110641}{2365321396838400}$
9	$\frac{46346126414417}{5321973142886400}$	$\frac{687900349648999}{6951148594790400}$	$\frac{-383141112401}{2874009600000}$	$\frac{-911785775029757}{34755742973952000}$	$\frac{-19412103395197}{3801409387776000}$
10	$\frac{151502944631310641}{62671555730630246400}$	$\frac{424992278449301977}{26113148221095936000}$	$\frac{4610664347356591493}{39969104420044800000}$	$\frac{-4610664347356591493}{39969104420044800000}$	$\frac{-424992278449301977}{26113148221095936000}$

Table 6: Table 2 continued

k	γ_8	γ_9	γ_{10}	γ_{11}	EC	p
1	0	0	0	0	$\frac{1}{9450}$	6
2	0	0	0	0	$\frac{103}{25401600}$	8
3	0	0	0	0	$\frac{89}{314344800}$	10
4	0	0	0	0	$\frac{379397}{28768836096000}$	12
5	0	0	0	0	$\frac{3901}{4382752374000}$	14
6	0	0	0	0	$\frac{1964407}{43597116186624000}$	16
7	$\frac{-1964407}{752777625600}$	0	0	0	$\frac{724523791}{242582188319190720000}$	18
8	$\frac{-367112146447}{5406448907059200}$	$\frac{-369733300393}{567677135241216000}$	0	0	$\frac{22424299416863}{141590371678145239449600000}$	20
9	$\frac{-15510407670221}{30276113879531520}$	$\frac{-30149869152983}{1532728265151283200}$	$\frac{-22424299416863}{139338933195571200000}$	0	$\frac{346654620623}{33392651133078198561600000}$	22
10	$\frac{-151502944631310641}{62671555730630246400}$	$\frac{-77093601173942957}{402888572554051584000}$	$\frac{-46586581380730333}{7833944466328780800000}$	$\frac{-2833553100322723}{705055001969590272000000}$	$\frac{214052065409772559}{377502585353403711610945536000000}$	24

Table 7: Absolute errors in the numerical integration of example 1

x	y_i	EGASDBVM (p=8)	Method (3.2) [16] (p=8)	Method (3.4) [16] (p=11)
5	y_1	$3.2024170123130\times10^{-2}$	$1.96006208591687\times10^{-2}$	$1.58384223934188\times10^{-2}$
	y_2	$3.2602729149065\times10^{-2}$	$9.80025491509760\times10^{-1}$	$7.91952513657462\times10^{-3}$
40	y_1	$7.198139407653299\times10^{-15}$	$3.81292881577727\times10^{-7}$	$1.02234876430633\times10^{-7}$
	y_2	$7.198139407651458\times10^{-15}$	$1.90646440788863\times10^{-7}$	$5.11174382153563\times10^{-8}$
70	y_1	$8.848025628743198\times10^{-26}$	$8.90990527186305\times10^{-12}$	$9.16017987660008\times10^{-13}$
	y_2	$8.848025628742016\times10^{-26}$	$4.45495263593152\times10^{-12}$	$4.58008993830191\times10^{-13}$
100	y_1	$1.087608242814579\times10^{-36}$	$2.08203236381127\times10^{-18}$	$6.66853658595783\times10^{-18}$
	y_2	$1.087608242814395\times10^{-36}$	$1.04101618190563\times10^{-18}$	$3.33426829297979\times10^{-18}$

Table 8: Absolute error in example 2, $h = 0.01$, Error y_i=$|y_i - y(x_i)|$, $i = 1, 2$

x	y_i	Error in EGASDBVM (k=3)	Error in SDGAM [14] (k=3)	Error in SDGBDF [13] (k=3)
1.0	y_1	2.77556×10^{-17}	1.18313×10^{-10}	3.06126×10^{-11}
	y_2	0.0	4.04676×10^{-13}	4.22623×10^{-11}
2.0	y_1	1.73472×10^{-17}	1.54753×10^{-13}	1.03235×10^{-11}
	y_2	8.32667×10^{-17}	1.47077×10^{-13}	3.96899×10^{-11}
3.0	y_1	7.80626×10^{-18}	6.23676×10^{-15}	2.39019×10^{-12}
	y_2	7.63278×10^{-17}	5.19376×10^{-14}	2.40044×10^{-11}
4.0	y_1	1.24683×10^{-18}	7.85992×10^{-16}	4.31932×10^{-13}
	y_2	3.46945×10^{-17}	2.08930×10^{-14}	1.20298×10^{-11}
5.0	y_1	1.76183×10^{-19}	9.34040×10^{-17}	8.00396×10^{-14}
	y_2	1.47451×10^{-17}	7.37951×10^{-15}	5.82196×10^{-12}
6.0	y_1	6.26804×10^{-20}	1.15866×10^{-17}	1.27167×10^{-14}
	y_2	1.38778×10^{-17}	2.60686×10^{-15}	2.56518×10^{-12}
7.0	y_1	1.33408×10^{-20}	1.87491×10^{-18}	2.18299×10^{-15}
	y_2	8.02310×10^{-18}	1.04864×10^{-15}	1.15005×10^{-12}
8.0	y_1	2.47492×10^{-21}	2.34152×10^{-19}	3.27871×10^{-16}
	y_2	4.06576×10^{-18}	3.70580×10^{-16}	4.79014×10^{-13}
9.0	y_1	4.20208×10^{-22}	2.94063×10^{-20}	4.83562×10^{-17}
	y_2	1.89735×10^{-18}	1.31676×10^{-16}	1.95919×10^{-13}
10.0	y_1	6.96900×10^{-23}	4.78731×10^{-21}	7.87909×10^{-18}
	y_2	8.40257×10^{-19}	5.32479×10^{-17}	8.33725×10^{-14}

Table 9: Maximum error, Max$||y_i - y(x_i)||$, for example 3

Method	h	N	x_T	y_1 (Max$\|y_i - y(x_i)\|$)	y_2 (Max$\|y_i - y(x_i)\|$)
EGASDBVM ($k = 2$)	0.04	125	5	1.21×10^{-9}	7.45×10^{-10}
	0.1	50	5	1.07×10^{-6}	1.96×10^{-7}
	0.4	25	10	9.04×10^{-5}	1.63×10^{-4}
EGASDBVM ($k = 3$)	0.04	125	5	4.20×10^{-12}	1.50×10^{-11}
	0.1	50	5	4.59×10^{-8}	3.65×10^{-8}
	0.4	25	10	7.46×10^{-6}	2.46×10^{-6}
SDGAM ($k = 2$) [14]	0.04	125	5	1.28×10^{-7}	2.85×10^{-8}
	0.1	50	5	9.11×10^{-6}	1.60×10^{-5}
	0.4	25	10	1.93×10^{-6}	2.01×10^{-6}
SDGAM ($k = 3$) [14]	0.04	125	5	1.17×10^{-9}	2.96×10^{-10}
	0.1	50	5	2.33×10^{-7}	8.57×10^{-7}
	0.4	25	10	9.60×10^{-5}	7.73×10^{-5}
BVM2 [6]	0.04	125	5	7.45×10^{-6}	4.07×10^{-5}
	0.1	50	5	6.5×10^{-5}	1.50×10^{-3}
	0.4	25	10	3.20×10^{-5}	3.04×10^{-6}
BVM3[6]	0.04	125	5	8.33×10^{-6}	1.32×10^{-6}
	0.1	50	5	7.45×10^{-4}	9.5×10^{-5}
	0.4	25	10	7.90×10^{-4}	5.84×10^{-3}
DBDF [11]	0.1	47	5	4.4×10^{-4}	
	0.4	85	10	1.0×10^{-4}	
GEAR [8]	0.04	122	5	3.8×10^{-4}	

Table 10: Absolute errors, $|y - y(x)|$, for example 4

Method	t	h	N	y	*Error*
EGASDBVM ($k = 2$)	1	0.024	42	y_1	0.0000
				y_2	0.0000
	10	0.01	950	y_1	4.9631×10^{-24}
				y_2	4.0658×10^{-20}
SDAM ($k = 2$) [10]	1	0.008	120	y_1	1.6348×10^{-14}
				y_2	0.0000
	10	0.006	1500	y_1	2.4815×10^{-24}
				y_2	2.0329×10^{-20}
WU-XIA [15]	1	0.002	500	y_1	2.5606×10^{-7}
				y_2	8.0150×10^{-8}
	10	0.001	10000	y_1	5.5468×10^{-16}
				y_2	6.0936×10^{-12}

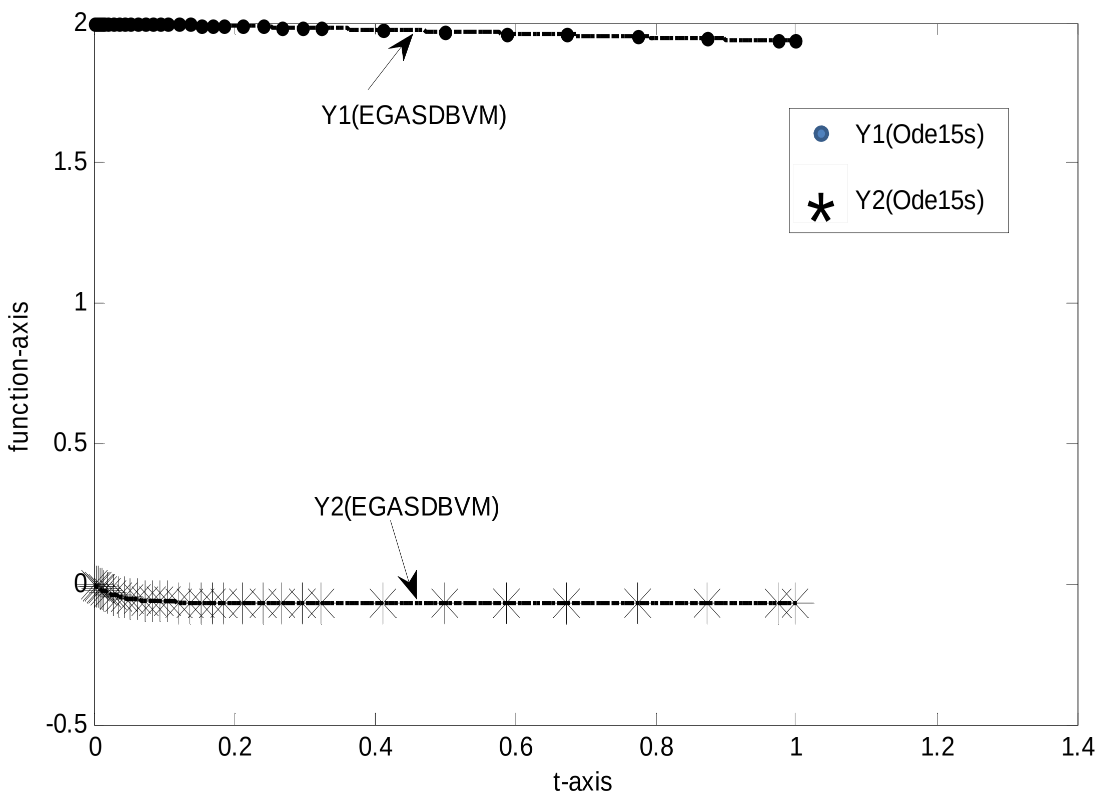

Figure 2: Numerical Results for Example 5 using the EGASDBVM for $k = 2$

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