Solutions and its Axiomatic Results under Fuzzy Behavior and Multicriteria Situations

Hui-Chuan Wei, Ai-Tzu Li, Wei-Ni Wang, and Yu-Hsien Liao

Abstract—**Based on the supreme marginal contributions among fuzzy activity level (or decision) vectors, we introduce two allocation methods to investigate allocation rules under fuzzy behavior and multicriteria situations simultaneously. Further, we adopt several axiomatic results to analyze the rationality for these allocation methods. In order to modify the discrimination among the players and their activity levels respectively, two weighted allocation methods are introduced. Furthermore, some more interpretations for these axioms and axiomatic results are also provided throughout this paper.**

Index Terms—**The supreme marginal contribution, fuzzy behavior, multicriteria situation, axiomatic result.**

I. Introduction

In a traditional transferable-utility (TU) game, each player is either fully involved or totally out of participation with some other players. In a *fuzzy* TU *game* (Aubin [1], [2]), each player is permitted to participate with infinite various activity levels (or decisions). Several allocations on fuzzy TU games could be always adopted to many fields. Related results have been investigated in Branzei et al. [4], Nishizaki and Sakawa [14], Muto et al. [13], Hwang [6], Li and Zhang [9], Meng and Zhang [11], Khorram et al. [8], Borkotokey and Mesiar [3], Hwang and Liao [7], Masuya and Inuiguchi [10] and so on.

Consistency is a crucial property of allocations in the axiomatic techniques on traditional games. Consistency states the independence of a value with respect to fixing some players with their assigned payoffs. It asserts that the recommendation made for any problem should always agree with the recommendation made in the subproblem that appears when the payoffs of some players are settled on. It has been introduced in different ways depending upon how the payoffs of the players that "leave the bargaining" are defined. This fundamental property has been always investigated in various topics by applying *reduced games*, such as bargaining problems, cost allocation problems and so on.

In different fields, from the sciences to industry, engineering, and the social sciences, managers face an increasing need to focus on multiple aims efficiently in their operational processes. Related situations include analyzing distribution tradeoffs, selecting optimal decision or process designs, or any other condition where you need an efficient solution with tradeoffs between two or more aims. In many cases, these real-world efficient situations could be formulated as multicriteria mathematical optimization models. The allocations of such situations require appropriate techniques to offer optimal results that - unlike traditional viewpoints or methods - take several properties of the aims into account. Here we would like to offer a mathematical foundation for multicriteria optimal allocations to analyze situations with multiple aims.

By applying the marginal contributions, the equal allocation of non-separable costs (EANSC, Ransmeier [15]) and the marginal index are proposed on traditional TU games respectively. Moulin [12] defined the *complement-reduction* to show that the EANSC could provide a fair rule for allocating utilities. The above mentioned results raise one motivation:

- whether the marginal contributions and related results could be extended under fuzzy behavior and multicriteria situations.

Here we aim to provide different necessary mathematical foundations of multicriteria optimal allocations to analyze problems with multiple aims and fuzzy behavior simultaneously. Different from the frameworks of traditional and fuzzy TU games, we consider the framework of *multicriteria fuzzy TU games*. Two new allocations, the *supreme equal allocation of non-separable costs* (SEANSC) and the *normalized fuzzy index*, are introduced in Section 2. Based on the notion of the SEANSC, all players firstly receive their supreme marginal contributions from the grand coalition, and further allocate the remaining utilities equally. Based on the notion of the normalized fuzzy index, all players allocate the utility of the grand fuzzy coalition proportionally by applying the supreme marginal contributions of all players. These two allocations are generalizations of the marginal contributions in fuzzy behavior and multicriteria situations. To present the rationality for these two allocations, we introduced an extended reduction and related properties of consistency to provide several axiomatic results in Sections 3 and 4:

- The SEANSC is the only allocation satisfying the properties of *multicriteria standard for games* and *multicriteria bilateral consistency*.
- The SEANSC is the only allocation satisfying the properties of *multicriteria efficiency*, *multicriteria zero-independence*, *multicriteria symmetry* and *multicriteria bilateral consistency*.
- Since the normalized fuzzy index violates multicriteria bilateral consistency, we define the *revised bilateral consistency* to show that the normalized fuzzy index is the only allocation satisfying the axioms of *normalized-standard of games* and *revised bilateral consistency*.

Based on the notion of the SEANSC, all players first receive their supreme marginal contributions from the grand coalition and further allocate the remaining utilities equally.

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That is, any additional fixed utility (e.g.,., the cost of a common facility) should be distributed equally among the players who are concerned. However, the players and their activity levels may not be fixed in different situations. In many applications, however, the SEANSC seems unrealistic for the situation that is being modeled. Players might represent constituencies of different sizes or they might have different bargaining abilities. Also, the lack of symmetry may arise when different bargaining abilities for different players and activity levels are modeled. In line with the above interpretations, we now aim to ensure that any additional fixed utility could be distributed among the players and their activity levels in proportion to their weights.

To modify the discrimination among the players and their activity levels, a reasonable step is for weights to be assigned to the "players" and the "levels," respectively. In Section 5, we adopt the *weight function for players* and the *weight function for levels* to propose two weighted extensions of the SEANSC and related axiomatic results. Additional interpretations and discussions for these axioms and axiomatic results are provided throughout this study.

II. Preliminaries

Let U be the universe of players. For $i \in U$ and $f_i \in (0,1]$, we set $F_i = [0, f_i]$ to be the action (decision) space of player i and $F_i^+ = (0, f_i]$, where 0 denotes no participation. Let $F^N = \prod_{i\in N} F_i$ be the product set of the action spaces for players in N. For all $T \subseteq N$, a player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T \in F^N$, which is the vector with $e_i^T = 1$ if $i \in T$, and $e_i^T = 0$ if $i \in N \setminus T$. Denote 0_N the zero vector in $\mathbb{R}^N$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, 2, \cdots, m\}$.

A **fuzzy transferable-utility (TU) game**[1] is a triple (N, f, v), where N is a non-empty and finite set of players, $f = (f_i)_{i\in N} \in (0,1]^N$ is the vector that describes the highest levels of activity for each player, and $v : F^N \to \mathbb{R}$ is a function with $v(0_N) = 0$ which assigns to each action vector $\alpha = (\alpha_i)_{i\in N} \in F^N$ the worth that the players can obtain when each player i plays at level α_i. A **multicriteria fuzzy TU game** is a triple (N, f, V^m), where $m \in \mathbb{N}$, $V^m = (v^t)_{t\in\mathbb{N}_m}$ and (N, f, v^t) is a fuzzy TU game for all $t \in \mathbb{N}_m$. Denote the class of all multicriteria fuzzy TU games by Γ.

An **allocation** is a map σ assigning to each $(N, f, V^m) \in \Gamma$ an element

$$\sigma\big(N, f, V^m\big) = \big(\sigma^t\big(N, f, V^m\big)\big)_{t\in\mathbb{N}_m},$$

where $\sigma^t\big(N, f, V^m\big) = \big(\sigma_i^t\big(N, f, V^m\big)\big)_{i\in N} \in \mathbb{R}^N$ and $\sigma_i^t\big(N, f, V^m\big)$ is the payoff of the player i when i participates in $\big(N, f, v^t\big)$. Let $(N, f, V^m) \in \Gamma$, $T \subseteq N$ and $\alpha \in \mathbb{R}^N$, we denote $S(\alpha) = \{i \in N | \alpha_i \neq 0\}$, and denote $\alpha_T \in \mathbb{R}^T$ to be the restriction of α to T. Given $i \in N$, we introduce the substitution notation α_{-i} to stand for $\alpha_{N\setminus\{i\}}$ and let $\gamma = (\alpha_{-i}, t) \in \mathbb{R}^N$ be defined by $\gamma_{-i} = \alpha_{-i}$ and $\gamma_i = t$.

Next, we provide different fuzzy generalizations of the equal allocation of non-separable costs (EANSC) and the normalized marginal index under multicriteria situation.

[1] A fuzzy TU game, which is defined by Aubin [1], [2], is a pair (N, v^a), where N is a coalition and v^a is a mapping such that $v^a : [0,1]^N \longrightarrow \mathbb{R}$ and $v^a(0_N) = 0$.

Definition 1:

1) The **supreme EANSC (SEANSC)**, $\overline{\beta}$, is defined by

$$\begin{aligned} &\overline{\beta_i^t}(N, f, V^m)\\ = \;& \beta_i^t(N, f, V^m) + \tfrac{1}{|N|} \cdot \Big[v^t(f) - \sum_{k\in N} \beta_k^t(N, f, V^m)\Big] \end{aligned}$$

for all $(N, f, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$ and for all $i \in N$. The value $\beta_i^t(N, f, V^m) = \sup_{j\in F_i^+}\{v^t(f_{-i}, j) - v^t(f_{-i}, 0)\}$ is the **supreme marginal contribution** of the player i in (N, f, v^t).[2] Under the notion of $\overline{\beta}$, all players firstly receive their supreme marginal contributions, and further allocate the remaining utilities equally.

2) The **normalized fuzzy index**, $\overline{\eta}$, is defined by

$$\overline{\eta_i^t}(N, f, V^m) = \frac{v^t(f)}{\sum\limits_{k\in N} \beta_k^t(N, f, V^m)} \cdot \beta_i^t(N, f, V^m)$$

for all $(N, f, V^m) \in \Gamma^*$, for all $t \in \mathbb{N}_m$ and for all $i \in N$, where $\Gamma^* = \{(N, f, V^m) \in \Gamma \mid \sum\limits_{i\in N} \beta_i^t(N, f, V^m) \neq 0 \text{ for all } t \in \mathbb{N}_m\}$. Under the notion of $\overline{\eta}$, all players allocate the utility of the grand fuzzy coalition proportionally by applying the supreme marginal contributions of all players.

Here we provide a brief application of multicriteria fuzzy TU games in the setting of "management". This kind of problem can be formulated as follows. Let $N = \{1, 2, \cdots, n\}$ be a set of all players of a grand management system (N, f, V^m). The function v^t could be treated as an utility function which assigns to each level vector $\alpha = (\alpha_i)_{i\in N} \in F^N$ the worth that the players can obtain when each player i participates at operation strategy $\alpha_i \in F_i$ in the sub-management system (N, f, v^t). Modeled in this way, the grand management system (N, f, V^m) could be considered as a multicriteria fuzzy TU game, with v^t being each characteristic function and F_i being the set of all operation strategies of the player i. In the following sections, we would like to show that the SEANSC and the normalized fuzzy marginal allocation could provide "*optimal allocation mechanisms*" among all players, in the sense that this organization can get payoff from each combination of operation strategies of all players under fuzzy behavior and multicriteria situations.

III. Axiomatic results for the SEANSC

In order to to analyze the rationality for the SEANSC, we adopt an extended reduction and some axioms to provide some axiomatic results. An allocation ψ satisfies **multicriteria efficiency (MEFF)** if for all $(N, f, V^m) \in \Gamma$ and for all $t \in \mathbb{N}_m$, $\sum_{i\in N} \psi_i^t(N, f, V^m) = v^t(f)$. An allocation ψ satisfies **multicriteria standard for games (MSFG)** if $\psi(N, f, V^m) = \overline{\beta}(N, f, V^m)$ for all $(N, f, V^m) \in \Gamma$ with $|N| \leq 2$. An allocation ψ satisfies **multicriteria symmetry (MSYM)** if $\psi_i(N, f, V^m) = \psi_k(N, f, V^m)$ for all $(N, f, V^m) \in \Gamma$ with $\beta_i^t(N, f, v) = \beta_k^t(N, f, v)$ for some $i, k \in N$ and for all $t \in \mathbb{N}_m$. An allocation ψ satisfies **multicriteria zero-independence (MZI)** if $\psi(N, f, V^m) =$

[2] From now on we restrict our attention to bounded fuzzy TU games, defined as those games (N, f, v^t) such that, there exists $M_v \in \mathbb{R}$ such that $v^t(\alpha) \leq M_v$ for all $\alpha \in F^N$. We adopt it to guarantee that $\beta_i(N, f, v^t)$ is well-defined.

$\psi(N, f, W^m) + (b^t)_{t\in\mathbb{N}_m}$ for all $(N, f, V^m), (N, f, W^m) \in \Gamma$ with $v^t(\alpha) = w^t(\alpha) + \sum_{i\in S(\alpha)} b_i^t$ for some $b^t \in \mathbb{R}^N$, for all $t \in \mathbb{N}_m$ and for all $\alpha \in F^N$.

Property MEFF asserts that all players allocate all the utility completely. Property MSFG is a generalization of the two-person standardness axiom of Hart and Mas-Colell [5]. Property MSYM asserts that if the supreme marginal contributions are the same, then the payoffs should be the same. Property MZI can be interpreted as an extremely weak kind of *additivity*. By Definition 1, it is easy to see that the SEANSC satisfies MEFF, MSFG, MSYM and MZI.

Moulin [12] defined the reduced game as that in which each coalition in the subgroup could attain payoffs to its members only if they are compatible with the initial payoffs to "all" the members outside of the subgroup. A natural extension of the Moulin reduction on multicriteria fuzzy TU games could be defined as follows.

Let $(N, f, V^m) \in \Gamma$, $S \subseteq N$ and ψ be an allocation. The **reduced game** $(S, f_S, V^m_{S,\psi})$ is defined by $V^m_{S,\psi} = (v^t_{S,\psi})_{t\in\mathbb{N}_m}$ and

$$
\begin{aligned}
& v^t_{S,\psi}(\alpha) \\
= & \begin{cases} 0 & \text{if } \alpha = 0_S, \\ v^t\big(\alpha, f_{N\setminus S}\big) - \sum\limits_{i\in N\setminus S} \psi_i^t(N, f, V^m) & \text{otherwise,} \end{cases}
\end{aligned}
$$

for all $\alpha \in F^S$. An allocation ψ satisfies **multicriteria bilateral consistency (MBCON)** if $\psi_i^t(S, f_S, V^m_{S,\psi}) = \psi_i^t(N, f, V^m)$ for all $(N, f, V^m) \in \Gamma$, for all $t \in \mathbb{N}_m$, for all $S \subseteq N$ with $|S| = 2$ and for all $i \in S$.

Lemma 1: The SEANSC $\overline{\beta}$ satisfies MBCON.

Proof: Let $(N, f, V^m) \in \Gamma$, $S \subseteq N$ and $t \in \mathbb{N}_m$. Assume that $|N| \geq 2$ and $|S| = 2$. Therefore,

$$
\begin{aligned}
& \overline{\beta_i^t}(S, f_S, V^m_{S,\overline{\beta}}) \\
= & \beta_i^t(S, f_S, V^m_{S,\overline{\beta}}) + \tfrac{1}{|S|} \cdot \big[v^t_{S,\overline{\beta}}(f_S) - \sum_{k\in S} \beta_k^t(S, f_S, V^m_{S,\overline{\beta}})\big]
\end{aligned} \tag{1}
$$

for all $i \in S$ and for all $t \in \mathbb{N}_m$. Furthermore,

$$
\begin{aligned}
& \beta_i^t(S, f_S, V^m_{S,\overline{\beta}}) \\
= & \sup_{j\in F_i^+} \{v^t_{S,\overline{\beta}}(f_{S\setminus\{i\}}, j) - v^t_{S,\overline{\beta}}(f_{S\setminus\{i\}}, 0)\} \\
= & \sup_{j\in F_i^+} \{v^t(f_{-i}, j) - v^t(f_{-i}, 0)\} \\
= & \beta_i^t(N, f, V^m).
\end{aligned} \tag{2}
$$

By equations (1), (2) and definitions of $v^t_{S,\overline{\beta}}$ and $\overline{\beta}$,

$$
\begin{aligned}
& \overline{\beta_i^t}(S, f_S, V^m_{S,\overline{\beta}}) \\
= & \beta_i^t(N, f, V^m) + \tfrac{1}{|S|} \cdot \big[v^t_{S,\overline{\beta^t}}(f_S) - \sum_{k\in S} \beta_k^t(N, f, V^m)\big] \\
= & \beta_i^t(N, f, V^m) + \tfrac{1}{|S|} \cdot \big[v^t(f) - \sum_{k\in N\setminus S} \overline{\beta_k^t}(N, f, V^m) \\
& \qquad - \sum_{k\in S} \beta_k^t(N, f, V^m)\big] \\
= & \beta_i^t(N, f, V^m) + \tfrac{1}{|S|} \cdot \big[\sum_{k\in S} \overline{\beta_k^t}(N, f, V^m) \\
& \qquad - \sum_{k\in S} \beta_k^t(N, f, V^m)\big] \\
& \big(\textbf{by MEFF of } \overline{\beta^t}\big) \\
= & \beta_i^t(N, f, V^m) + \tfrac{1}{|S|} \cdot \Big[\tfrac{|S|}{|N|} \cdot \big[v^t(f) \\
& \qquad - \sum_{k\in N} \beta_k^t(N, f, V^m)\big]\Big] \\
= & \beta_i^t(N, f, V^m) + \tfrac{1}{|N|} \cdot \big[v^t(f) - \sum_{k\in N} \beta_k^t(N, f, V^m)\big] \\
= & \overline{\beta^t}_i(N, f, V^m)
\end{aligned}
$$

for all $i \in S$ and for all $t \in \mathbb{N}_m$. So, the SEANSC satisfies MBCON. ■

Next, we characterize the SEANSC by means of multicriteria bilateral consistency.

Theorem 1: The SEANSC is the only allocation satisfying MSFG and MBCON.

Proof: By Lemma 1, $\overline{\beta}$ satisfies MBCON. Clearly, $\overline{\beta}$ satisfies MSFG.

To prove uniqueness, suppose ψ satisfies MSFG and MBCON. By MSFG and MBCON of ψ, it is easy to derive that ψ also satisfies MEFF, hence we omit it. Let $(N, f, V^m) \in \Gamma$. By MSFG of ψ, $\psi(N, f, V^m) = \overline{\beta}(N, f, V^m)$ if $|N| \leq 2$. The case $|N| > 2$: Let $i \in N$, $t \in \mathbb{N}_m$ and $S = \{i, k\}$ for some $k \in N \setminus \{i\}$.

$$
\begin{aligned}
& \psi_i^t(N, f, V^m) - \psi_k^t(N, f, V^m) \\
= & \psi_i^t(S, f_S, V^m_{S,\psi}) - \psi_k^t(S, f_S, V^m_{S,\psi}) \\
& \big(\textbf{by MBCON of } \psi^t\big) \\
= & \overline{\beta_i^t}(S, f_S, V^m_{S,\psi}) - \overline{\beta_k^t}(S, f_S, V^m_{S,\psi}) \\
& \big(\textbf{by MSFG of } \psi^t\big) \\
= & \beta_i^t(S, f_S, V^m_{S,\psi}) - \beta_k^t(S, f_S, V^m_{S,\psi}) \\
= & \sup_{j\in F_i^+} \{v^t_{S,\psi}(f_{S\setminus\{i\}}, j) - v^t_{S,\psi}(f_{S\setminus\{i\}}, 0)\} \\
& \qquad - \sup_{j\in F_k^+} \{v^t_{S,\psi}(f_{S\setminus\{k\}}, j) - v^t_{S,\psi}(f_{S\setminus\{k\}}, 0)\} \\
= & \sup_{j\in F_i^+} \{v^t(f_{-i}, j) - v^t(f_{-i}, 0)\} \\
& \qquad - \sup_{j\in F_k^+} \{v^t(f_{-k}, j) - v^t(f_{-k}, 0)\} \\
= & \beta_i^t(N, f, V^m) - \beta_k^t(N, f, V^m) \\
= & \overline{\beta_i^t}(N, f, V^m) - \overline{\beta_k^t}(N, f, V^m).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \psi_i^t(N, f, V^m) - \psi_k^t(N, f, V^m) \\
= & \overline{\beta_i^t}(N, f, V^m) - \overline{\beta_k^t}(N, f, V^m).
\end{aligned}
$$

By MEFF of ψ and $\overline{\beta}$,

$$
\begin{aligned}
& |N| \cdot \psi_i^t(N, f, V^m) - v^t(f) \\
= & \sum_{k\in N} [\psi_i^t(N, f, V^m) - \psi_k^t(N, f, V^m)] \\
= & \sum_{k\in N} [\overline{\beta_i^t}(N, f, V^m) - \overline{\beta_k^t}(N, f, V^m)] \\
= & |N| \cdot \overline{\beta_i^t}(N, f, V^m) - v^t(f).
\end{aligned}
$$

Hence, $\psi_i^t(N,f,V^m)=\overline{\beta_i^t}(N,f,V^m)$ for all $i\in N$ and for all $t\in\mathbb{N}_m$. ■

Next, we characterize the SEANSC by means of related properties of MEFF, MSYM, MZI and MBCON.

Lemma 2: If an allocation ψ satisfies MEFF, MSYM and MZI, then ψ satisfies MSFG.

Proof: Assume that a solution ψ satisfies MEFF, MSYM and MZI. Let $(N,f,V^m)\in\Gamma$. The proof is completed by MEFF of ψ if $|N|=1$. Let $(N,f,V^m)\in\Gamma$ with $N=\{i,k\}$ for some $i\neq k$. We define a game (N,f,W^m) to be that $w^t(\alpha)=v^t(\alpha)-\sum_{i\in S(\alpha)}\beta_i^t(N,f,V^m)$ for all $\alpha\in F^N$ and for all $t\in\mathbb{N}_m$. By definition of W^m,

$$\begin{aligned}
&\beta_i^t(N,f,W^m)\\
=&\sup_{j\in F_i^+}\{w^t(j,f_k)-w^t(0,f_k)\}\\
=&\sup_{j\in F_i^+}\{v^t(j,f_k)-v^t(0,f_k)-\beta_i^t(N,f,V^m)\}\\
=&\sup_{j\in F_i^+}\{v^t(j,f_k)-v^t(0,f_k)\}-\beta_i^t(N,f,V^m)\\
=&\beta_i^t(N,f,V^m)-\beta_i^t(N,f,V^m)\\
=&0.
\end{aligned}$$

Similarly, $\beta_k^t(N,f,W^m)=0$. Therefore, $\beta_i^t(N,f,W^m)=\beta_k^t(N,f,W^m)$. By MSYM of ψ, $\psi_i^t(N,f,W^m)=\psi_k^t(N,f,W^m)$. By MEFF of ψ,

$$w^t(f)=\psi_i^t(N,f,W^m)+\psi_k^t(N,f,W^m)=2\cdot\psi_i^t(N,f,W^m).$$

Therefore,

$$\begin{aligned}
&\psi_i^t(N,f,W^m)\\
=&\frac{w^t(f)}{2}\\
=&\frac{1}{2}\cdot\big[v^t(f)-\beta_i(N,f,V^m)-\beta_k(N,f,V^m)\big].
\end{aligned}$$

By MZI of ψ,

$$\begin{aligned}
&\psi_i^t(N,f,V^m)\\
=&\beta_i^t(N,f,V^m)+\frac{1}{2}\cdot\big[v^t(f)-\beta_i^t(N,f,V^m)\\
&\qquad-\beta_k^t(N,f,V^m)\big]\\
=&\overline{\beta_i^t}(N,f,V^m).
\end{aligned}$$

Similarly, $\psi_k^t(N,f,V^m)=\overline{\beta_k^t}(N,f,V^m)$. Hence, ψ satisfies MSFG. ■

Theorem 2: On Γ, the SEANSC is the only allocation satisfying MEFF, MSYM, MZI and MBCON.

Proof: By Definition 1, $\overline{\beta}$ satisfies MEFF, MSYM and MZI. The remaining proofs follow from Theorem 1 and Lemmas 1, 2. ■

The following examples are to show that each of the axioms used in Theorems 1 and 2 is logically independent of the remaining axioms.

Example 1: Define a solution ψ by for all $(N,f,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\psi_i^t(N,f,V^m)=\begin{cases}\overline{\beta_i^t}(N,f,V^m) & \text{if } |N|\leq 2,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, ψ satisfies MSFG, but it violates MBCON.

Example 2: Define a solution ψ to be that

$$\psi_i^t(N,f,V^m)=\beta_i^t(N,f,V^m)$$

for all $(N,f,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$. Clearly, ψ satisfies MSYM, MZI and MBCON, but it violates MEFF and MSFG.

Example 3: Define a solution ψ to be that

$$\psi_i^t(N,f,V^m)=\frac{v^t(f)}{|N|}$$

for all $(N,f,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$. Clearly, ψ satisfies MEFF, MSYM and MBCON, but it violates MZI.

Example 4: Define a solution ψ by for all $(N,f,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\begin{aligned}
\psi_i^t(N,f,V^m)\ &=\big[v^t(f)-v^t(f_{-i},0)\big]+\tfrac{1}{|N|}\cdot\Big[v^t(f)\\
&\qquad-\sum_{k\in N}\big[v^t(f)-v^t(f_{-k},0)\big]\Big].
\end{aligned}$$

Clearly, ψ satisfies MEFF, MZI and MBCON, but it violates MSYM.

Example 5: Define a solution ψ by for all $(N,f,v)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\begin{aligned}
&\psi_i^t(N,f,V^m)\\
=&\beta_i^t(N,f,V^m)+\frac{d^t(i)}{\sum\limits_{k\in N}d^t(k)}\cdot\big[v^t(f)-\sum_{k\in N}\beta_k^t(N,f,V^m)\big],
\end{aligned}$$

where for all $(N,f,V^m)\in\Gamma$, $d^t:N\to\mathbb{R}^+$ is defined by $d^t(i)=d^t(k)$ if $\beta_i^t(N,f,v)=\beta_k^t(N,f,v)$. Define a solution θ by for all $(N,f,V^m)\in\Gamma$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\theta_i^t(N,f,V^m)=\begin{cases}\overline{\beta_i^t}(N,f,V^m) & \text{if } |N|\leq 2,\\ \psi_i^t(N,f,V^m) & \text{otherwise.}\end{cases}$$

Clearly, θ satisfies MEFF, MSYM and MZI, but it violates MBCON.

IV. The axiomatic results for the normalized fuzzy index

Similar to Theorem 1, we would like to characterize the normalized fuzzy index by means of bilateral consistency. Unfortunately, it is easy to see that $(S,f_S,V^m_{S,\psi})$ does not exist if $\sum_{i\in S}\beta_i^t(N,f,V^m)=0$. Thus, we consider the **revised bilateral consistency** as follows. An allocation ψ satisfies **revised bilateral consistency (RBCON)** if $(S,f_S,V^m_{S,\psi})\in\Gamma^*$ for some $(N,f,V^m)\in\Gamma$ and for some $S\subseteq N$ with $|S|=2$, it holds that $\psi_i^t(S,f_S,V^m_{S,\psi})=\psi_i^t(N,f,V^m)$ for all $t\in\mathbb{N}_m$ and for all $i\in S$.

Lemma 3: The normalized fuzzy index satisfies RBCON on Γ^*.

Proof: Let $(N,f,V^m)\in\Gamma^*$. If $|N|\leq 2$, then the proof is completed. Assume that $|N|\geq 3$ and $S\subseteq N$ with $|S|=2$. Similar to equation (2),

$$\beta_i^t(S,f_S,V^m_{S,\overline{\eta}})=\beta_i^t(N,f,V^m). \tag{3}$$

for all $i\in S$ and for all $t\in\mathbb{N}_m$. Define that $a_t=$

$\frac{v^t(f)}{\sum_{p\in N}\beta_p^t(N,f,V^m)}$. For all $i\in S$ and for all $t\in\mathbb{N}_m$,

$$\begin{aligned}
&\overline{\eta_i^t}(S,f_S,V^m_{S,\overline{\eta}})\\
=\ &\frac{v^t_{S,\overline{\eta}}(f_S)}{\sum_{k\in S}\beta_k^t(S,f_S,V^m_{S,\overline{\eta}})}\cdot\beta_i^t(S,f_S,V^m_{S,\overline{\eta}})\\
=\ &\frac{v^t(f)-\sum_{h\in S^c}\overline{\eta_h^t}(N,f,V^m)}{\sum_{k\in S}\beta_k^t(N,f,V^m)}\cdot\beta_i^t(N,f,V^m)\\
&\big(\textbf{by equation (3) and definition of } V^m_{S,\overline{\eta}}\big)\\
=\ &\frac{\sum_{h\in S}\overline{\eta_h^t}(N,f,V^m)}{\sum_{k\in S}\beta_k^t(N,f,V^m)}\cdot\beta_i^t(N,f,V^m)\\
&\big(\textbf{by MEFF of } \overline{\eta}\big)\\
=\ &a_t\cdot\beta_i^t(N,f,V^m)\\
&\big(\textbf{by Definition 1}\big)\\
=\ &\overline{\eta_i^t}(N,f,V^m).\\
&\big(\textbf{by Definition 1}\big)
\end{aligned}\tag{4}$$

By equations (3), (4), the solution $\overline{\eta}$ satisfies RBCON. ■

An allocation ψ satisfies **normalized-marginal-standard for games (NM-SG)** if $\psi(N,f,v)=\overline{\eta}(N,f,v)$ for all $(N,f,v)\in\Gamma$, $|N|\le 2$.

Theorem 3: On Γ^*, the allocation $\overline{\eta}$ is the only allocation satisfying NM-SG and RBCON.

Proof: By Lemma 3, $\overline{\eta}$ satisfies RBCON. Clearly, $\overline{\eta}$ satisfies NM-SG.

To prove uniqueness, suppose ψ satisfies RBCON and NM-SG on Γ^*. By NM-SG and RBCON of ψ, it is easy to derive that ψ also satisfies MEFF, hence we omit it. Let $(N,f,V^m)\in G^*$. We will complete the proof by induction on $|N|$. If $|N|\le 2$, it is trivial that $\psi(N,f,V^m)=\overline{\beta}(N,f,V^m)$ by NM-SG. Assume that it holds if $|N|\le p-1$, $p\le 3$. The case $|N|=p$: Let $i,j\in N$ with $i\ne j$ and $t\in\mathbb{N}_m$. By Definition 1, $\overline{\beta_k^t}(N,f,V^m)=\frac{v^t(f)}{\sum_{h\in N}\beta_h^t(N,f,V^m)}\cdot\beta_k^t(N,f,V^m)$ for all $k\in N$. Assume that $\alpha_k^t=\frac{\beta_k^t(N,f,V^m)}{\sum_{h\in N}\beta_h^t(N,f,V^m)}$ for all $k\in N$. Therefore,

$$\begin{aligned}
&\psi_i^t(N,f,V^m)\\
=\ &\psi_i^t\big(N\setminus\{j\},f_{N\setminus\{j\}},V^m_{N\setminus\{j\},\psi}\big)\\
&\big(\textbf{by RBCON of } \psi\big)\\
=\ &\overline{\beta_i^t}\big(N\setminus\{j\},f_{N\setminus\{j\}},V^m_{N\setminus\{j\},\psi}\big)\\
&\big(\textbf{by NM-SG of } \psi\big)\\
=\ &\frac{v^t_{N\setminus\{j\},\psi}(f_{N\setminus\{j\}})}{\sum_{k\in N\setminus\{j\}}\beta_k^t\big(N\setminus\{j\},f_{N\setminus\{j\}},V^m_{N\setminus\{j\},\psi}\big)}\\
&\cdot\beta_i^t\big(N\setminus\{j\},f_{N\setminus\{j\}},V^m_{N\setminus\{j\},\psi}\big)\\
=\ &\frac{v^t(f)-\psi_j^t(N,f,V^m)}{\sum_{k\in N\setminus\{j\}}\beta_k^t(N,f,V^m)}\cdot\beta_i^t(N,f,V^m)\\
&\big(\textbf{by equation (2)}\big)\\
=\ &\frac{v^t(f)-\psi_j^t(N,f,V^m)}{-\beta_j^t(N,f,V^m)+\sum_{k\in N}\beta_k^t(N,f,V^m)}\cdot\beta_i^t(N,f,V^m).
\end{aligned}\tag{5}$$

By equation (5),

$$\psi_i^t(N,f,V^m)\cdot[1-\alpha_j^t]=[v^t(f)-\psi_j^t(N,f,V^m)]\cdot\alpha_j^t$$

$$\begin{aligned}
\implies\ &\sum_{i\in N}\psi_i^t(N,f,V^m)\cdot[1-\alpha_j^t]\\
&=[v^t(f)-\psi_j^t(N,f,V^m)]\cdot\sum_{i\in N}\alpha_j^t\\
\implies\ &v^t(f)\cdot[1-\alpha_j^t]=[v^t(f)-\psi_j^t(N,f,V^m)]\cdot 1\\
&\big(\textbf{by MEFF of } \psi\big)\\
\implies\ &v^t(f)-v^t(f)\cdot\alpha_j^t=v^t(f)-\psi_j^t(N,f,V^m)\\
\implies\ &\overline{\beta_j^t}(N,f,V^m)=\psi_j^t(N,f,V^m).
\end{aligned}$$

The proof is completed. ■

The following examples are to show that each of the axioms adopted in Theorem 3 is logically independent of the remaining axioms.

Example 6: Define a solution ψ to be that for all $(N,f,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\psi_i^t(N,f,V^m)=0.$$

Clearly, ψ satisfies RBCON, but it violates NM-SG.

Example 7: Define a solution ψ to be that for all $(N,f,V^m)\in\Gamma^*$, for all $t\in\mathbb{N}_m$ and for all $i\in N$,

$$\psi_i^t(N,f,V^m)=\begin{cases}\overline{\eta_i^t}(N,f,V^m) & \text{, if } |N|\le 2,\\ 0 & \text{, otherwise.}\end{cases}$$

Clearly, ψ satisfies NM-SG, but it violates RBCON.

Remark 1: It is easy to show that the normalized fuzzy index satisfies MEFF, MSYM and NM-SG, but it violates MZI.

V. Two weighted extensions

In different situations, all players and their activity levels in could be assigned different weights by weight functions. These weights could be interpreted as *a-priori measures of importance*; they are taken to reflect considerations not captured by the characteristic function. For example, we may be dealing with a problem of cost allocation among investment projects. Then the weights could be associated to the profitability of the different projects. In a problem of allocating travel costs among various institutions visited (cf. Shapley [16]), the weights may be the number of days spent at each one.

If $d:U\to\mathbb{R}^+$ be a positive function, then d is called a **weight function for players**. If $w:F^U\to\mathbb{R}^+$ be a positive function, then w is called a **weight function for levels**. By these two types of the weight function, two weighted revisions of the SEANSC is defined as follows.

Definition 2:

- The **1-supreme weighted allocation of nonseparable costs (1-SWANSC)**, η^d, is defined by for all $(N,f,V^m)\in\Gamma$, for all weight function for players d, for all $t\in\mathbb{N}_m$ and for all player $i\in N$,

$$\begin{aligned}\eta_i^{d,t}(N,f,V^m)\ &=\beta_i^t(N,f,V^m)+\frac{d(i)}{\sum_{k\in N}d(k)}\cdot\big[v^t(f)\\&\quad-\sum_{k\in N}\beta_k^t(N,f,V^m)\big].\end{aligned}\tag{6}$$

- The **2-supreme weighted allocation of nonseparable costs (2-SWANSC)**, η^w, is defined by for all

$(N, f, V^m) \in \Gamma$, for all weight function for players w, for all $t \in \mathbb{N}_m$ and for all player $i \in N$,

$$\eta_i^{w,t}(N, f, V^m) = \beta_i^{w,t}(N, f, V^m) + \frac{1}{N} \cdot \big[v^t(f) - \sum_{k \in N} \beta_k^{w,t}(N, f, V^m)\big], \quad (7)$$

where $\beta_i^{w,t}(N, f, V^m) = \sup_{j \in F_i^+} \{w(j) \cdot [v^t(f_{-i}, j) - v^t(f_{-i}, 0)]\}$.

Remark 2: By Definition 2, it is easy to show that the 1-SWANSC satisfies MEFF and MZI, but it violates MSYM. Similarly, the 2-SWANSC satisfies MEFF, but it violates MSYM and MZI.

An allocation ψ satisfies **1-weighted standard for games (1WSFG)** if $\psi(N, f, V^m) = \eta^d(N, f, V^m)$ for all $(N, f, V^m) \in \Gamma$ with $|N| \leq 2$ and for all weight function for players d. An allocation ψ satisfies **2-weighted standard for games (2WSFG)** if $\psi(N, f, V^m) = \eta^w(N, f, V^m)$ for all $(N, f, V^m) \in \Gamma$ with $|N| \leq 2$ and for all weight function for levels w. Similar to the proofs of Lemma 1 and Theorem 1, we propose the analogies of Lemma 1 and Theorem 1.

- The 1-SWANSC η^d and the 2-SWANSC η^w satisfy MBCON.
- On Γ, the 1-SWANSC η^d is the only allocation satisfying 1WSFG and MBCON.
- On Γ, the 2-SWANSC η^w is the only allocation satisfying 2WSFG and MBCON.

VI. Concluding Remarks

1) In many situations, each player is permitted to participate with infinite various activity levels (or decisions and strategies). Players also face an increasing need to efficiently focus on multiple goals in their operational processes. Thus, we simultaneously focus on fuzzy behavior and multicriteria situation. Weights are naturally a part of the framework of utility allocation. For example, we may be dealing with a problem of utility allocation among investment projects. Then the weights could be associated with the profitability of the different projects. Therefore, we also consider the weighted allocation notion. Differing from pre-existing investigations on traditional TU games and fuzzy TU games, some results of this paper are provided as follows.
    - Based on fuzzy behavior and multicriteria situation simultaneously, we consider the framework of multicriteria fuzzy TU games.
    - By applying the supreme marginal contributions under fuzzy behavior and multicriteria situation simultaneously, we propose the SEANSC, the normalized fuzzy index and related axiomatic results.
    - In order to modify the discrimination among the players and their activity levels respectively, we introduce two weighted extensions of the SEANSC and related axiomatic results.
    - All the allocation methods and related results are introduced initially in the frameworks of traditional TU games and fuzzy TU games.
2) Inspired by the main results of this study, a reasonable idea to consider is that some traditional allocations could be extended by simultaneously applying the supreme marginal contributions in the multicriteria situation and fuzzy behavior. This task is left to the readers.

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