Dynamics in a Discrete-Time Three Dimensional Cancer System

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Abstract—**We investigate the dynamics in a discrete-time cancer system. Firstly, we give necessary and sufficient conditions for the existence and stability of the fixed points. Secondly, we demonstrate that the system is chaotic in the sense of marotto when the parameters of this system satisfy some conditions. And third, we present numerical simulations not only to show the consistency with our theoretical analysis but also to exhibit the complex but interesting dynamical systems. Numerical simulations included bifurcation diagrams, Lyapunov exponents, Lyapunov dimension and attractor strange.**

Index Terms—**Discrete-Time, Cancer-system, Stability, Lyapunov-exponents, Marotto's-Chaos, Bifurcation.**

I. Introduction

CANCER is one of the main causes of morbidity and mortality in the world. There are several different stages in the growth of a tumor before it becomes so large that is the healthy body produces millions of cells a day and millions of other cells die. However, failure of the cell to perform this process normally causes many diseases, including cancer see [10] and [21-23]. Many laboratories across the world are spending large sums of money on cancer research in order to find cures and improve existing treatments. In comparison to molecular biology, cell biology, Mathematical systems can be used to understand and design new experiments, by formulating hypotheses regarding the potential biological mechanisms that could lead to delays cancer relapses or to the permanent elimination. Many authors have used mathematical models to describe the interactions of tumor cells with healthy host cells and immune system cells are the main components of these models and these interactions may yield different outcomes. There are many existing reviews of mathematical systems of tumor see [2], [3], [12] and [19].

Actually discrete-time systems described are more reasonable than the continuous-time systems when populations have nonoverlapping generations. Moreover, using discrete-time models is more efficient for computation and numerical simulations. By analysis, it is proved that the discrete-time system has different properties and structures compared with the continuous one and these results reveal far richer dynamical behaviors of the discrete-time system compared to the continuous one see [6], [20] and [29].

In this paper, we consider the following discrete-time cancer system:

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$$\begin{cases} x_{n+1} = s_1 x_n \left(1 - \dfrac{x_n}{q_1}\right) - p_{12} x_n y_n - p_{13} x_n z_n, \\ y_{n+1} = s_2 y_n \left(1 - \dfrac{y_n}{q_2}\right) - p_{21} x_n y_n, \\ z_{n+1} = s_3 \left(\dfrac{x_n z_n}{x_n + q_3}\right) - p_{31} x_n z_n - \eta z_n, \end{cases} \tag{1}$$

where x denotes the number of cancer cells, y denotes the healthy host cells and z denotes effector immune cells, and $s_1, s_2, s_3, q_1, q_2, q_3, s_2, p_{12}, p_{21}, p_{13}, p_{31}$ are positive parameters see [9], [14] and [16]. Here s_1 represents the growth rate of cancer cells in the absence of any effect from other cell populations with maximum carrying capacity q_1, p_{12} and p_{13} refers to the cancer cells killing rate by the healthy host cells and effector cells respectively, s_2 represents the growth rate of healthy host cells with maximum carrying capacity q_2, p_{21} represents the rate of inactivation of the healthy cells by cancer cells. The rate of recognition of the cancer cells by the immune system depends on the antigenicity of the cancer cells. Since this recognition process is very complex, in order to keep the model simple, assume the stimulation of the immune system depends directly on the number of cancer cells with positive constants s_3 and q_3. The effector cells are inactivated by the cancer cells at the rate p_{31} as well as they die naturally at the rate η.

II. Existence and stability of fixed points

We nondimensionalize our system (1) by using the following rescaling for the continuous-time t see [7-9].

$$\overline{u} = \frac{x}{q_1}, \overline{v} = \frac{y}{q_2}, \overline{w} = \frac{z}{q_3}, \tau = s_1 t$$

where the new parameters:

$$a_{12} = \frac{p_{12} q_2}{s_1}, r_2 = \frac{s_1}{s_2}, a_{21} = \frac{p_{21} q_1}{s_1}, r_3 = \frac{s_3}{s_1}, k_3 = \frac{q_3}{q_1},$$

$$a_{31} = \frac{p_{31} q_1}{s_1}, a_{31} = \frac{p_{13} q_3}{s_1}, d_3 = \frac{\eta}{s_1}$$

then the system (1) is converted to

$$\begin{cases} \overline{u}_{n+1} = \overline{u}_n (1 - \overline{u}_n) - a_{12} \overline{u}_n \overline{v}_n - a_{13} \overline{u}_n \overline{w}_n, \\ \overline{v}_{n+1} = r_2 \overline{v}_n (1 - \overline{v}_n) - a_{21} \overline{u}_n \overline{v}_n, \\ \overline{w}_{n+1} = r_3 \left(\dfrac{\overline{u}_n \overline{w}_n}{\overline{u}_n + k_3}\right) - a_{31} \overline{u}_n \overline{w}_n - d_3 \overline{w}_n. \end{cases} \tag{2}$$

For simplicity, we will still use x and y instead of $\overline{u}, \overline{v}$ and $\overline{w}$. Thus, the system (1) can be rewritten as:

$$\begin{cases} x_{n+1} = x_n (1 - x_n) - a_{12} x_n y_n - a_{13} x_n z_n, \\ y_{n+1} = r_2 y_n (1 - y_n) - a_{21} x_n y_n, \\ z_{n+1} = r_3 \left(\dfrac{x_n z_n}{x_n + k_3}\right) - a_{31} x_n z_n - d_3 z_n. \end{cases} \tag{3}$$

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Below we consider the above modified system (3).
It is clear that the fixed points of system (3) satisfy the following equations:

$$\begin{cases} x = x(1-x) - a_{12}xy - a_{13}xz, \\ y = r_2 y(1-y) - a_{21}xy, \\ z = r_3\left(\dfrac{xz}{x+k_3}\right) - a_{31}xz - d_3 z. \end{cases}$$

In order to obtain the fixed points of the system (3), we set

$$\begin{cases} x = 0, \\ x = -a_{12}y - a_{13}z. \end{cases} \tag{4}$$

$$\begin{cases} y = 0, \\ y = \dfrac{r_2-1}{r_2} - \dfrac{a_{21}}{r_2}x. \end{cases} \tag{5}$$

$$\begin{cases} z = 0, \\ x^2 + \left(k_3 + \dfrac{d_3 - r_2 + 1}{a_{31}}\right)x + \dfrac{k_3}{a_{31}} = 0. \end{cases} \tag{6}$$

The solution of Equations (4)-(6) together yields to five fixed points. We discuss their local behavior according to their biological relevance. Now, we study the stability of these fixed points.

(1) For the first fixed point is trivial and given as $v_1 = (0,0,0)$, the corresponding characteristic equation is $\lambda^3 - (1 + r_2 - d_3)\lambda^2 + r_2 d_3 = 0$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = r_2$ and $\lambda_3 = -d_3$. Since all the parameters are positive this equilibrium has two unstable and one stable eigenvalue. Therefore, we have a saddle at this fixed point.

(2) For the second fixed point is obtained as $v_2 = \left(0, \dfrac{r_2-1}{r_2}, 0\right)$ when $r_2 \neq 1$, the Jacobian matrix evaluated at v_2 is given by

$$J(v_2) = \begin{pmatrix} -a_{12}\left(\dfrac{r_2-1}{r_2}\right) + 1 & 0 & 0 \\ -a_{21}\left(\dfrac{r_2-1}{r_2}\right) & 2 - r_2 & 0 \\ 0 & 0 & -d_3 \end{pmatrix}$$

Clearly, $J(v_2)$ has eigenvalues $\lambda_1 = 1 - a_{12}\left(\dfrac{r_2-1}{r_2}\right)$, $\lambda_2 = 2 - r_2$ and $\lambda_3 = -d_3$. $|\lambda_i| < 1$ $(i = 2,3)$ holds iff $1 < r_2 < 3$, $d_3 < 1$ and $|\lambda_1| < 1$ if $r_2 > 1$, $0 \le a_{12} < \dfrac{2r_2}{r_2-1}$.
Stability of v_2 depends on the values of r_2, d_3 and a_{12}.

Lemma 1. For the cancer system (3), the following statements are true:

(i) v_2 is asymptotically stable if $1 < r_2 < 3$, $d_3 < 1$ and $0 \le a_{12} < \dfrac{2r_2}{r_2-1}$.

(ii) v_2 is unstable if one of the following conditions holds:

(a) $0 < r_2 < 1$ and $a_{12} > 0$

(b) $d_3 > 1$

(c) $r_2 > 1$ and $a_{12} > \dfrac{2r_2}{r_2-1}$.

(3) The third fixed point is $v_3 = \left(\dfrac{a_{12}(r_2-1)}{a_{12}a_{21}-r_2}, \dfrac{1-r_2}{a_{12}a_{21}-r_2}, 0\right)$, provided that $a_{12}a_{21} - r_2 \neq 0$ and $r_2 \neq 1$. If $r_2 = 1$ then is $v_3 = v_1$ and if $a_{12}a_{21} - r_2 = 0$ then is $v_3 = v_2$.
The eigenvalues of the Jacobian matrix at fixed point v_3 are

$$\lambda_1 = a_{12}(r_2 - 1)\left(\frac{r_3}{a_{12}a_{21}k_3 + a_{12}r_2 - k_3 r_2 - a_{12}} - \frac{a_{31}}{a_{12}a_{21} - r_2}\right)$$

$$\lambda_{2,3} = \frac{1}{2}\left(\frac{2a_{12}a_{21} - a_{12}r_2 + r_2^2 + a_{12} - 3r_2 \mp \sqrt{\Delta}}{a_{12}a_{21} - r_2}\right)$$

where
$\Delta = (r_2-1)^2\left(r_2^2 + 2a_{12} + a_{12}^2 - 4a_{21}a_{12}^2\right)$,

(i) If $r_2^2 + 2a_{12} + a_{12}^2 > 4a_{21}a_{12}^2$ we have three real eigenvalues.

(ii) If $r_2^2 + 2a_{12} + a_{12}^2 < 4a_{21}a_{12}^2$ we have one real and two complex eigenvalues.

v_3 is asymptotically stable if $|\lambda_i| < 1$ where $(i = 1,2,3)$.

(4) The fourth fixed point of the system is $v_4 = (-a_{13}z^*, 0, z^*)$, where $z^* \neq 0$. The Jacobian matrix evaluated at v_4 is given by

$$J(v_4) = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ 0 & L_{22} & 0 \\ L_{31} & 0 & L_{33}, \end{pmatrix},$$

where

$$L_{11} = a_{13}z^* + 1, L_{12} = a_{21}a_{13}z^*, \quad L_{13} = a_{13}^2 z^*,$$

$$L_{22} = a_{21}a_{13}z^* + r_2,$$

$$L_{31} = \frac{r_3 z^*}{-a_{13}z^* + k_3} + \frac{r_3 a_{13} z^{*\,2}}{(-a_{13}z^* + k_3)^2} - a_{13}z^*,$$

$$L_{33} = \frac{r_3 a_{13} z^*}{a_{13}z^* - k_3} + a_{31}a_{13}z^* - d_3.$$

The eigenvalues of the Jacobian matrix at this point are

$$\lambda_1 = L_{22} = a_{21}a_{13}z^* + r_2, \tag{7}$$

$$\lambda_{2,3} = \frac{1}{2}\left[(L_{11} + L_{33}) \mp \sqrt{(L_{11} - L_{33})^2 + 4L_{31}L_{13}}\right] \tag{8}$$

(i) If $(L_{11} - L_{33})^2 + 4L_{31}L_{13} > 0$ we have three real eigenvalues.

(ii) If $(L_{11} - L_{33})^2 + 4L_{31}L_{13} < 0$ we have at this point has one real and two complex eigenvalues with stable real parts with the selected parameter sets.

And the characteristic equation of the Jacobian matrix $J(v_4)$ can be written as

$$P(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0, \tag{9}$$

where

$$A_0 = -L_{33}L_{22}L_{11} + L_{31}L_{13}L_{22},$$

$$A_1 = L_{11}L_{22} + L_{11}L_{33} - L_{13}L_{31} + L_{33}L_{22},$$

$$A_2 = -L_{33} - L_{22} - L_{11}.$$

According to the Jury conditions [10], in order to find the asymptotically stable region of v_4, we

need to find the region that satisfy the following conditions:
P$(1) > 0, P(-1) < 0, \mid A_0 \mid < A_n, \mid B_0 \mid > \mid B_{n-1} \mid$
where $B_k = \begin{vmatrix} A_0 & A_{n-k} \\ A_n & A_k \end{vmatrix}$.
Since

$$P(1) = 1 + A_2 + A_1 + A_0,$$

$$P(-1) = -1 + A_2 - A_1 + A_0,$$

According the relations $P(1) > 0, P(-1) < 0, \mid \mid A_0 \mid < A_n, \mid B_0 \mid > \mid B_{n-1} \mid$, we have that
$\mid A_0 \mid < 1, \mid A_0 + 1 \mid > \mid A_1 \mid$ and $\mid A_0 - 1 \mid \mid A_0 + A_1 + 1 \mid > \mid A_0 A_1 - A_2 \mid$.

(5) The fifth fixed point is a nontrivial $v_5 = (x^*, y^*, z^*)$. The Jacobian matrix of the system (1) at v_5 is given by

$$J(v_5) = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & 0 \\ S_{31} & 0 & S_{33} \end{pmatrix}, \tag{10}$$

where

$$S_{11} = -a_{12} y^* - a_{13} z^* - 2x^* + 1,$$

$$S_{12} = -a_{12} x^*, \quad S_{21} = -a_{21} y^*,$$

$$S_{13} = -a_{13} x^*, S_{22} = r_2 (1 - 2y^*) - a_{21} x^*,$$

$$S_{31} = \frac{r_3 x^*}{x^* + k_3} - \frac{r_3 x^* z^*}{(x^* + k_3)^2} - a_{31} x^*,$$

$$S_{33} = \frac{r_3 x^*}{x^* + k_3} - a_{31} x^* - d_3.$$

And the characteristic equation of the Jacobian matrix $J(v_4)$ can be written as
P$^* (\lambda) = \lambda^3 + C_2 \lambda^2 + C_1 \lambda + C_0 = 0$.
According to the Jury conditions [10], in order to find the asymptotically stable region of v_5, we need to find the region that satisfy the following conditions:
P$^*(1) > 0, \quad P^*(-1) < 0, \quad \mid C_0 \mid < C_n, \quad \mid D_0 \mid > \mid D_{n-1} \mid$,
where $D_k = \begin{vmatrix} C_0 & C_{n-k} \\ C_n & C_k \end{vmatrix}$.
Since

$$P^*(1) = 1 + C_2 + C_1 + C_0,$$

$$P^*(-1) = -1 + C_2 - C_1 + C_0,$$

from the relations $P^*(1) > 0, P^*(-1) < 0, \mid A_0 \mid < A_n, \mid B_0 \mid > \mid B_{n-1} \mid$, we have that
$\mid C_0 \mid < 1, \mid C_0 + 1 \mid > \mid C_1 \mid$ and $\mid C_0 - 1 \mid \mid C_0 + C_1 + 1 \mid > \mid C_0 C_1 - C_2 \mid$.

III. Chaotic dynamics for three-dimensional discrete cancer system

Li and Yorke (1975) introduced the first mathematical definition of discrete chaos see [11] and established a simple criterion for chaos in one-dimensional dynamical system. Then Marotto see [4-5] generalized the result to higher-dimensional dynamical systems, there exists an error in the condition of the original Marotto theorem, it has been corrected and modified this important theorem by Shi and Chen see [25]. In this section, we shall prove that the system (3) exhibit chaotic dynamics with the selected parameter set.

Lemma 2. Let $I \subset R$ be an interval and $F : I \to I$ be a continuous map see [24]. Assume that there is a point $a \in I$, satisfying

$$F^3(a) \le a < F(a) \le F^2(a) \text{ or } F^3(a) \ge a > F(a) \ge F^2(a)$$

then:

(1) For every $i = 1, 2, ...$, there is a periodic point of F^i with period n in I.

(2) There are an uncountable set $S \subset I$ (containing no periodic points) and an uncountable subset $S_0 \subset S$, such that

(A) for every $p, q \in S_0$ with $p \neq q$

$$\lim_{n\to\infty} \sup \mid F^n(p) - F^n(q) \mid > 0$$

and

$$\lim_{n\to\infty} \inf \mid F^n(p) - F^n(q) \mid = 0,$$

(B) for every $p \in S$ and periodic point $q \in I$ with $p \neq q$

$$\lim_{n\to\infty} \sup \mid F^n(p) - F^n(q) \mid > 0.$$

The one dimensional dynamical system $v_{i+1} = F(v_i)$ that satisfies the above conditions is said to be chaotic in the sense of Li and Yorke .

Marotto (1978) generalized the work of Li and Yorke (1975) to the n-dimensional consider the following theorem.

Theorem 1. Marotto theorem given in [24-27]. consider the following n-dimensional discrete system:

$$v_{n+1} = F(v_n), \quad n = 0, 1, 2, ..., \tag{11}$$

where $v_n \in R^n$ and $F : R^n \to R^n$ is continuous. Let $B_r(v)$ denote the ball in R^n of radius r centred at point v and $\overline{B}_r(v)$ its interior. Also, let $\parallel v \parallel$ be the usual euclidean norm of v in R^n. Then, (1) $\Rightarrow$ (2)

(1) All eigenvalues of the Jacobian $DF(v)$ of map (11) at the fixed point v are greater than one with euclidean norm.

(2) There exist some $s > 1$ and $r > 0$, such that for all $u, v \in B_r(v)$,

$$\parallel F(u) - F(v) \parallel > s \parallel u - v \parallel .$$

Shi and Chen (2004b), proved that there exists an error in the condition of the original Marotto theorem which has been corrected and a modified version of this theorem is given as follows:

Theorem 2. (A Modified Version of the Marotto Theorem see [24])

Consider the n-dimensional discrete dynamical system:

$$v_{n+1} = F(v_n), \quad n = 0, 1, 2, ..., \tag{12}$$

where $v_n \in R^n$ and $F : R^n \to R^n$, suppose that the system (12) has a fixed point v^*.

Assume that

(1) F is continuously differentiable in some neighbourhood of v^* and all the eigenvalues of $DF(v^*)$ have

absolute values larger than 1, which implies that there exists a positive constant r and a euclidean norm, such that F is expanding in $B_r(v^*)$ in euclidean norm, and

(2) v^* is a snap-back repeller of F with $F^m(v_0) = v^*$ for some $v_0 \in B_r(v^*)$, $v_0 \neq v^*$ and some positive integer m. Furthermore, F is continuously differentiable in some neighbourhoods of $v_0, v_1, ..., v_{m-1}$, respectively, and $\det[DF(v_j)] \neq 0$ for $0 \leq j \leq m-1$, where $v_j = F(v_{j-1})$.

Then, all the results of the Marotto Theorem hold.

A. A Proof of spatial chaotic behavior for the 3-D discrete cancer system

In this subsection, using Theorem 3, to obtain the following results of system (3).

Theorem 3. *If* $d_3 > 1, 0 < r_2 < 1, r_2 \neq a_{12} \neq 0$ *and* $a_{12} \neq \frac{r_2}{r_2 - 1}$,
or $d_3 > 1, r_2 > 3, a_{12} > \frac{2r_2}{r_2-1}$ *and* $r_2 \neq a_{12}$,
then the cancer discrete system given in equations (3) is chaotic.

Proof.

Step 1. Let $v^* = \left(0, 1 - \dfrac{1}{r_2}, 0\right) = y^*.e_2 \in R^3$, where $y^* = 1 - \dfrac{1}{r_2}$, $e_2 = (0,1,0)$.

The fixed point v^* of the system (3) can be written $v^* = F(v^*)$.

$F(v)$ given in Theorem 3 of system (3), its continuously differentiable in $B_r(v^*)$ for some $r > 0$. The Jacobian evaluated at the fixed point v^* is given by

$$D(F(v^*)) = \begin{pmatrix} -a_{12}y^* + 1 & 0 & 0 \\ -a_{21}y^* & r_2(1-2y^*) & 0 \\ 0 & 0 & -d_3 \end{pmatrix}.$$

Clearly, $D(F(v^*))$ has eigenvalues $\lambda_1 = 1 - a_{12}(r_2 - 1)/r_2$, $\lambda_2 = 2 - r_2$ and $\lambda_3 = -d_3$.

$\mid \lambda_i \mid > 1$, $(i = 1, 2, 3)$ holds iff

(i) $d_3 > 1$, $0 < r_2 < 1$ and $a_{12} > 0$,

(ii) $d_3 > 1$, $r_2 > 3$ and $a_{12} > \dfrac{2r_2}{r_2 - 1}$.

Remark 1.

If $a_{12} = r_2$ and $0 < r_2 < 1$ then $\lambda_1 = -r_2$, $\mid \lambda_1 \mid < 1$, and if $a_{12} = \frac{2r_2}{r_2-1}$ then $\lambda_1 = 0$.

That is with conditions of Theorem 2, all the eigenvalues of $D(F(v^*))$ are larger than 1 in absolute value, so v^* is an expanding fixed point of F given of system (3).

Therefore, there exist some $r > 0$ and euclidean norm $\| \,.\, \|$, such that F is expanding in $B_r(v^*)$.

That is, for any two distinct points $u, v \in \overline{B}_r(v^*)$, we have $\| F(u) - F(v) \| > s \| u - v \|$,

where $s > 1$ and u, v are sufficiently close to v^*.

Since $F(u) - F(v) = DF(v)(u - v) + \alpha$, where $\| \alpha \| / \| u - v \| \rightarrow 0$ as $\| u - v \| \rightarrow 0$, specially, $\| F(v) - F(v^*) \| = \| DF(v^*)(v - v^*) + \alpha \|$, with euclidean norm for (m, n) real matrices $A = (a_{ij})$ is $\| A \| = \left(\sum_i^m \sum_j^n a_{ij}^2 \right)^{\frac{1}{2}}$.

Since $F(v)$ is continuously differentiable, $DF(v^*)$ is also expanding for $v \in B_r(v^*)$. The norm of $DF(v^*)$ is given by

$$\| DF(v^*) \| = \sqrt{\frac{(a_{12}r_2 - a_{12} - r_2)^2}{r_2^2} + \frac{a_{12}^2\,(r_2-1)^2}{{r_2}^2} + (-r_2+2)^{\,2} + d_3^2} > 1.$$

Thus, condition (1) and (2) of Theorem 2 is satisfied.

Step 2. According to, Definition of (Theorem 2) snap-back repeller, we need to find one point $u \in \overline{B}_r(v^*)$, such that $u \neq v^*$, $F^M(u) = v^*$, and $\det\left[DF^M(u)\right] \neq 0$, for some positive integer M.

In fact, we have

$$\begin{cases} x(1-x) - a_{12}xy - a_{13}xz = x_1 \\ r_2y(1-y) - a_{21}xy = y_1 \\ \frac{r_3xz}{x+k_3} - a_{31}xz - d_3z = z_1 \end{cases} \tag{13}$$

$$\begin{cases} x_1(1-x_1) - a_{12}x_1y_1 - a_{13}x_1z_1 = x^* \\ r_2y_1(1-y_1) - a_{21}x_1y_1 = y^* \\ \frac{r_3x_1z_1}{x_1+k_3} - a_{31}x_1z_1 - d_3z_1 = z^* \end{cases} \tag{14}$$

If $x = 0$ the solution of (13) and (14) is $(x, y, z) = (0, \frac{1}{r_2}, 0)$.

Now, a map F^2 has been constructed to map the point $u = (x, y, z)$ to the fixed point $v^* = (x^*, y^*, z^*)$, after two iterations if there are solutions different from v^* for (13) and (14). By calculating, the solutions different from v^* for system (3) are:

If $x = x^- = \dfrac{-a_{31}k_3 - \sqrt{a_{31}k_3r_3}}{a_{31}}$, we have three solutions is given by
$(x^-, y^-, z^-), (x^-, y^+, z^+)$ and $\left(x^-, \dfrac{1}{2}, \dfrac{2 - 2x^- - a_{12}}{2a_{13}}\right)$,

where $y^- = \dfrac{r_2 - \sqrt{\Delta_1}}{2r_2}$, $y^+ = \dfrac{r_2 + \sqrt{\Delta_1}}{2r_2}$, $z^- = \dfrac{1 - x^- - a_{12}y^-}{a_{13}}$, $z^+ = \dfrac{1 - x^- - a_{12}y^+}{a_{13}}$ and $\Delta_1 = r_2^2 + 4(1 - a_{12}x^-)r_2 - 4$.

If $x = x^+ = \dfrac{-a_{31}k_3 + \sqrt{a_{31}k_3r_3}}{a_{31}}$, we have three solutions is given by
$(x^+, y'^-, z'^-), (x^+, y'^+, z'^+)$ and $\left(x^+, \dfrac{1}{2}, \dfrac{2 - 2x^+ - a_{12}}{2a_{13}}\right)$,

where $y'^- = \frac{r_2 - \sqrt{\Delta_2}}{2r_2}$, $y'^+ = \dfrac{r_2 + \sqrt{\Delta_2}}{2r_2}$, $z'^- = \dfrac{1 - x^+ - a_{12}y'^-}{a_{13}}$, $z'^+ = \dfrac{1 - x^+ - a_{12}y^+}{a_{13}}$
and $\Delta_2 = r_2^2 + 4(1 - a_{12}x^+)r_2 - 4$.

Next, we accept the solution $u = (0, \dfrac{1}{r_2}, 0)$, where $u \neq v^*$ and $F^2(u) = v^*$,

the Jacobian matrix of F^2 evaluated at u is given by

$$DF^2(u) = \begin{pmatrix} \dfrac{a_{12}^2r_2 - a_{12}r_2^2 - a_{12}^2 + r_2^2}{r_2^2} & 0 & 0 \\ \dfrac{a_{21}\,(a_{12}r_2 - a_{12} - r_2)}{r_2^2} & -\,(r_2-2)^2 & 0 \\ 0 & 0 & d_3^2 \end{pmatrix}$$

and

$$\det[DF^2(u)] = -\frac{\left(a_{12}^2r_2 - a_{12}r_2^2 - a_{12}^2 + r_2^2\right)(r_2-2)^2\, d_3^2}{r_2^2} \neq 0.$$

The conditions of theorem 2 is satisfied. ■

IV. Numerical Simulations

Example 1. In this example, numerical simulations are shown for verifying the condition in Theorem 2.

For $a_{12} = 3.5$, $a_{13} = 2.5$, $a_{21} = 1.15$, $a_{31} = 0.2$, $d_3 = 1.001$, $k_3 = 3.9$, $r_2 = 3.79$, $r_3 = 0.5$, $x_0 = 0$, $y_0 = 0,1$ and $z_0 = 0,1$and the eigenvalues associated with v^* is $\lambda_1 = -1.5765$, $\lambda_2 = -1.79$ and $\lambda_3 = -1.001$, then the system (3) is unstable. $\mid \lambda_i \mid > 1$, $i = 1, 2, 3$. And the parameters satisfies the conditions of theorem 3, and there exists a point $u = (0, 0.2638, 0)$, satisfies that $F^2(u) = v^*$ and $\det(F^2(u)) = 0.3879 \neq 0$. Thus, v^* is a snap-back repeller.

A. Lyapunov exponents

In this subsection we calculated the Lyapunov exponents and the Kaplan-Yorke dimension [1], [11] and [13]. The Lyapunov exponents for a discrete n-dimensional systems is given in [28] the following definition:

Definition 1. Consider the n-dimensional discrete dynamical system:

$$v_{k+1} = F(v_k), \quad v_k \in R^n, k = 0, 1, 2, ... \qquad (15)$$

where $F : R^n \to R^n$, is the vector field associated with map (15), let $J(v)$be its Jacobian evaluated at v, also define the matrix: $T_p(v_0) = J(v_{p-1})J(v_{p-2})...J(v_1)J(v_0)$.

Moreover, let $J_i(v_0, l)$ be the modulus of the i^{th} eigenvalue of the l^{th} matrix $T_p(v_0)$ where $i = 1, 2, ..., n$ and $p = 0, 1, 2, ...$.

Now, the Lyapunov exponents of a n-dimensional discrete time models are defined by: $\lambda_i(v_0) = \ln\left(\lim_{p\to+\infty}\left(J_i(v_0, p)^{\frac{1}{p}}\right)\right)$.

Kaplan-Yorke dimension D_{KY} defined as: $D_{KY} = k + \sum_{i=1}^{k} \frac{\lambda_i}{|\lambda_{k+1}|}$,

Where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$ and where k is the large integer such that $\lambda_1 + \lambda_2 + ... + \lambda_p > 0$. particular Kaplan-Yorke suggest that D_{KY} is a lower bound of capacity dimension, that is, $D_{KY} \leq D_c$.

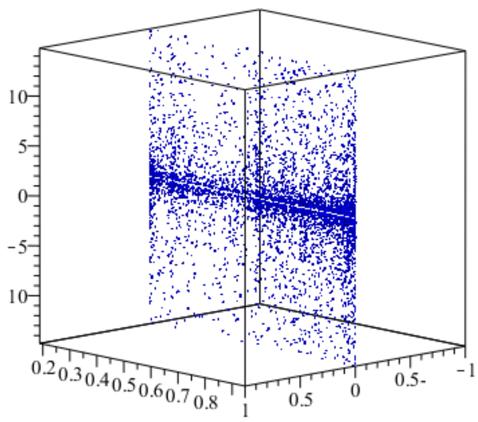

Fig. 1. The attractor strange of system (3) with parameters $a_{12} = 3.5$, $a_{13} = 2.5$, $a_{21} = 1.15$, $a_{31} = 0.2$, $d_3 = 1.001$, $k_3 = 3.9$, $r_2 = 3.79$, $r_3 = 0.5$ and $x_0 = 0$, $y_0 = 0.1$, $z_0 = 0.1$

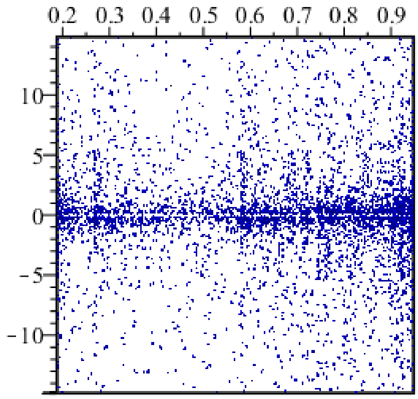

Fig. 2. Projection of attractor of system (3) on (y, z)-plane

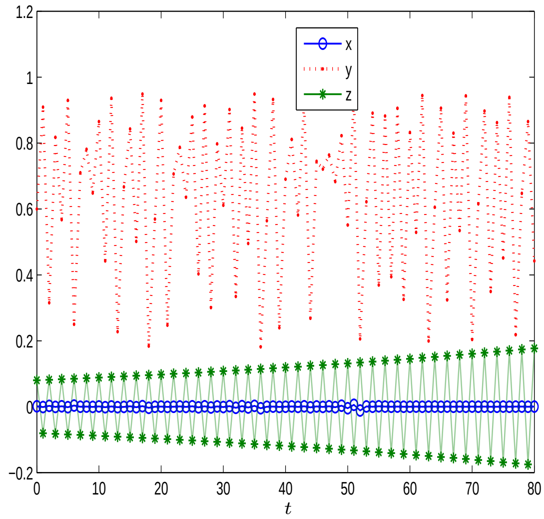

Fig. 3. Time responses of the system (3) with parameters $a_{12} = 3.5$, $a_{13} = 2.5$, $a_{21} = 1.15$, $a_{31} = 0.2$, $d_3 = 1.01$, $k_3 = 3.9$, $r_2 = 3.79$, and $r_3 = 0.5$ with $x_0 = 0.001$, $y_0 = 0.6$ and $z_0 = 0.08$,

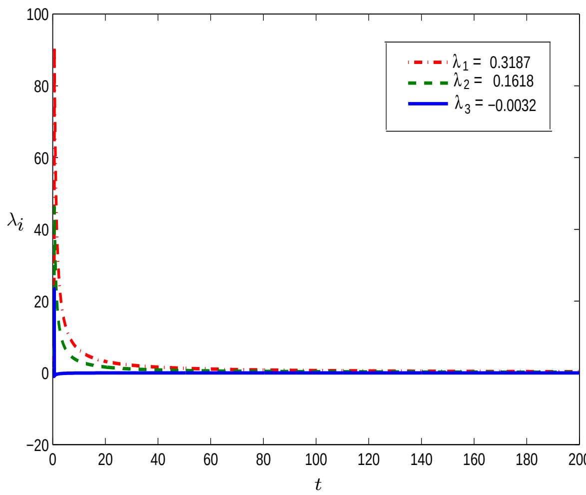

Fig. 4. The Lyapunov exponents of system (3) with parameters $a_{12} = 3.5$, $a_{13} = 2.5$, $a_{21} = 1.15$, $a_{31} = 0.2$, $d_3 = 1.001$, $k_3 = 3.9$, $r_2 = 3.79$, and $r_3 = 0.5$ with $x_0 = 0$, $y_0 = -0.1$ and $z_0 = 0.1$,

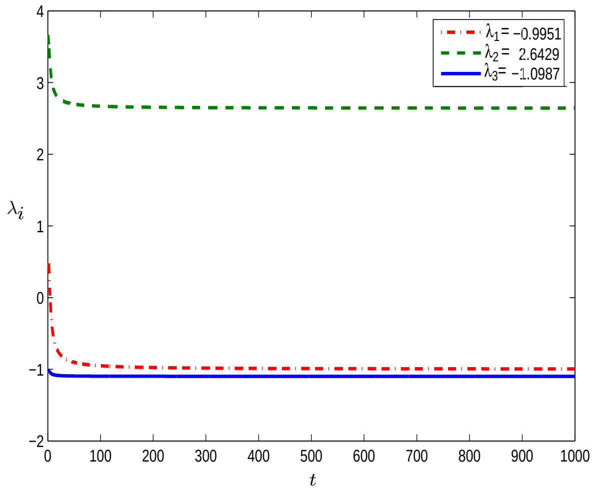

Fig. 5. The Lyapunov exponents of system (3) with parameters $a_{12} = 3.5$, $a_{13} = 2.5$, $a_{21} = 1.15$, $a_{31} = 0.2$, $d_3 = 1.001$, $k_3 = 3.9$, $r_2 = 3.79$, and $r_3 = 0.5$ with $x_0 = 0.1$, $y_0 = 0$ and $z_0 = 0.1$.

Therefore, the Lyapunov exponents are computed to be $\lambda_1 = 0.1618$, $\lambda_2 = -0.0032$ and $\lambda_3 = 0.3187$ where $\lambda_1 + \lambda_2 + \lambda_3 > 0$, the Lyapunov exponents are illustrated in Figure. 4, and the Kaplan-Yorke dimension is $D_{KY} = 2 + \frac{\lambda_1+\lambda_2}{|\lambda_3|} \simeq 2.5$ where $D_{KY} \leq D_c$.

If at least one Lyapunov exponent is positive for some control parameters value, then the system (3) is chaotic at that control parameters.

B. The bifurcation diagram

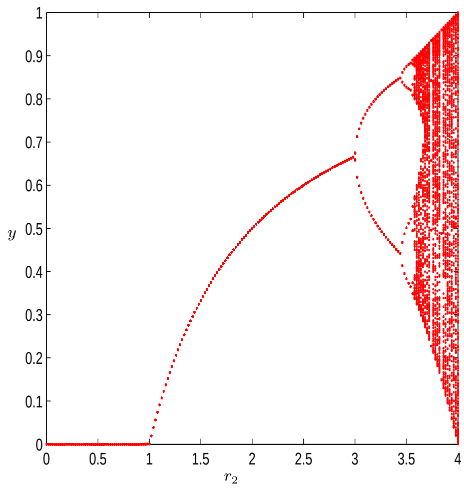

Fig. 6. Bifurcation diagram of system (3) for $(r_2, y)-$plane and $a_{12} = 0.62$, $a_{13} = 1.39$, $a_{21} = 1.13$, $r_3 = 0.2$, $k_3 = 3.8$, $a_{31} = 0.56$, $d_3 = 1$ and $x_0 = 0.15$, $y_0 = 0.2$ and $z_0 = 0.08$.

V. Conclusion

In this paper, we have presented the existence and stability of fixed points for three dimensional discrete system, we have proved that the system is Chaotic with Marotto theorem and we calculated the Lyapunov exponents and the Kaplan-Yorke dimension. We have presented some numerical results.

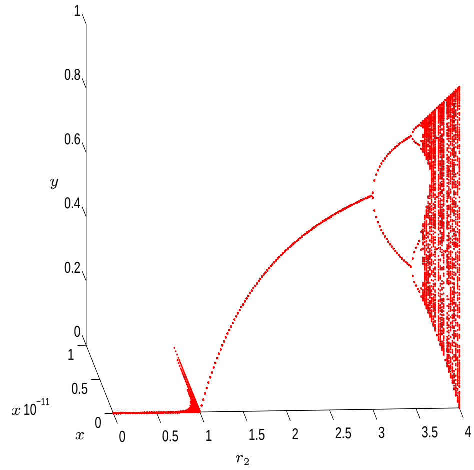

Fig. 7. Bifurcation diagram of system (3) in $(r_2, x, y)-$space for $a_{12} = 0.62$, $a_{13} = 1.39$, $a_{21} = 1.13$, $a_{31} = 0.56$, $d_3 = 1$, $k_3 = 3.8$ and $r_3 = 0.2$ and $x_0 = 0.15$, $y_0 = 0.2$ and $z_0 = 0.08$.

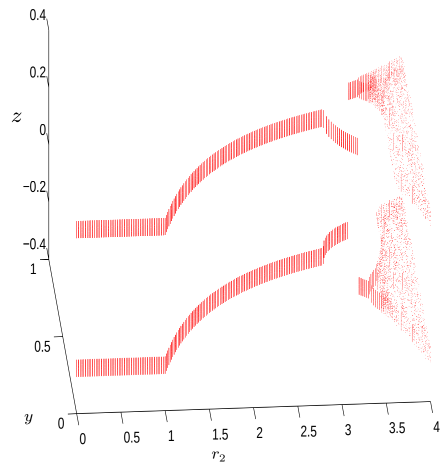

Fig. 8. Bifurcation diagram of system (3) in $(r_2, y, z)-$space for $a_{12} = 3.5$, $a_{13} = 2.5$, $a_{21} = 1.15$, $a_{31} = 0.2$, $d_3 = 1.001$, $k_3 = 3.9$ and $r_3 = 0.5$ and $x_0 = 0$, $y_0 = 0.1$ and $z_0 = 0.1$.

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