Oscillation Properties for a Class of Delay Partial Difference Equations with Three Parameters

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Abstract—**The aim of this paper is to investigate the oscillatory properties of solutions for a class of delay partial difference equation with three parameters. In order to study the oscillation results, the regions of non-positive roots of its characteristic equation which is equivalent to the oscillation results are investigated. Some necessary and sufficient conditions by means of the envelope theory are derived.**

Index Terms—**delay partial difference equation, oscillation, envelope, characteristic equation.**

I. Introduction

Partial difference equations are types of difference equations that involve functions of two or more independent variables. Delay partial difference equations have numerous applications as in molecular orbits, population dynamic with spatial migrations, image processing, random walk problems, material mechanics, etc[1-7].In recent years, the study of the qualitative analysis for the oscillatory property of delay partial difference equation has attracted considerable attention, see [8-12] and the references therein.

In [8], by means of the z-transform, B. G. Zhang and R. P. Agarwal have investigated the following first order delay partial difference equation

$$A_{m+1,n} + A_{m,n+1} - pA_{m,n} + \sum_{i=1}^{\mu} q_i A_{m-k_i,n-l_i} = 0,$$

where p, q are real numbers, k_i and $l_i \in N_0$, $i = 1, 2, ..., \mu$, $N_t = \{t, t+1, ...\}$, and μ is a positive integer. They gave some sufficient conditions for the equation to be oscillatory.

In [12], Chunhua Yuan and Shutang Liu studied the following first order delay partial difference equation

$$u_{m+1,n} + au_{m,n+1} + bu_{m,n} + cu_{m-\sigma,n-\tau} = 0,$$

where a, b, c are real numbers with $a^2 + b^2 + c^2 \neq 0$, and m, n, σ, τ are nonnegative integers. By applying the envelope theory, they achieved the necessary and sufficient conditions for the equation to be oscillatory.

Motivated by the above research, this paper investigates the following second order delay partial difference equation

$$u_{m+2,n} + pu_{m,n+2} + qu_{m,n} + ru_{m-\sigma,n-\tau} = 0, \qquad (1)$$

where p, q, r are real numbers with $p^2 + q^2 + r^2 \neq 0$, and m, n, σ, τ are nonnegative integers.

The purpose of this paper is to apply the envelope theory of the family of planes, to derive necessary and sufficient conditions for the delay partial difference equation (1) to be oscillatory.

Before stating our main results, some definitions used in this paper are presented.

Definition 1 A solution of (1) is a real double sequence $\{u_{m,n}\}$ which is defined for $m \geq -\sigma$,$n \geq -\tau$ and satisfies (1) for $m \geq 0$ and $n \geq 0$.

Definition 2 A solution $\{u_{m,n}\}$ of (1) is said to be eventually positive (or negative) if $u_{m,n} > 0$ (or $u_{m,n} < 0$) for $m > M$ and $n > N$, where M and N are some large integers. It is said to be oscillatory if it is neither eventually positive nor eventually negative. (1) is called oscillatory if all of its nontrivial solutions are oscillatory.

II. Preliminaries

This section will give some lemmas that will be used in the proof of the main results in section 3.

Lemma 1 [11] The following statements are equivalent:

(i) Every solution of equation (1) is oscillatory.

(ii) The characteristic equation of equation (1)

$$\lambda^2 + p\mu^2 + q + r\lambda^{-\sigma}\mu^{-\tau} = 0$$

has no positive root.

Lemma 2 [12] Suppose that $f(x,y), g(x,y), h(x,y)$ and $v(x,y)$ are differentiable on $(-\infty,+\infty)\times(-\infty,+\infty)$. Let Γ be a two-parameter family of planes defined by the equation

$$f(\lambda,\mu)x + g(\lambda,\mu)y + h(\lambda,\mu)z = v(\lambda,\mu),$$

where λ and μ are parameters. Let Σ be the envelope of the family Γ. Then the equation

$$f(\lambda,\mu)a + g(\lambda,\mu)b + h(\lambda,\mu)c = v(\lambda,\mu)$$

has no real root if and only if there is no tangent plane of Σ passing through the point (a,b,c) in xyz-space.

Lemma 3 [13] Suppose that $f(x), g(x), h(x)$ and $v(x)$ are differentiable on $(-\infty,+\infty)$. Let Γ be the one-parameter family of planes defined by the equation

$$f(\lambda)x + g(\lambda)y + h(\lambda)z = v(\lambda),$$

where λ is a parameter. Let Σ be the envelope of the family Γ. Then the equation

$$f(\lambda)a + g(\lambda)b + h(\lambda)c = v(\lambda)$$

has no real root if and only if there is no tangent plane of Σ passing through the point (a,b,c) in xyz-space.

Lemma 4 [11] Suppose that $f(x)$ is differentiable on $(0,+\infty)$ such that $f(x)$ is not identically zero on $(0,+\infty)$ and $\lim_{x\to+\infty} f(x) > 0$ or $\lim_{x\to 0^+} f(x) > 0$. Then

$$F(x,y) = y + f(x) = 0$$

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has no positive root on $(0,+\infty)\times(0,+\infty)$ if and only if $f(x)=0$ has no positive root on $(0,+\infty)$.

Lemma 5 Assume that

$$f(\lambda,q,r)=\lambda^{\sigma+2}+q\lambda^{\sigma}+r,$$

where σ is a positive integer, q and r are real parameters. Then the equation $f(\lambda,q,r)=0$ has no positive root if and only if $q\geq 0$ and $r\geq 0$ or $q<0$ and

$$r>(-1)^{\frac{\sigma+2}{2}}\frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}}q^{\frac{\sigma+2}{2}}.$$

Proof Consider the family of straight lines defined by L_λ:$f(\lambda,x,y)=0$, where $\lambda\in(0,+\infty)$. Since

$$f_\lambda(\lambda,x,y)=(\sigma+2)\lambda^{\sigma+1}+\sigma\lambda^{\sigma-1}x,$$

the determinant of the system $f(\lambda,x,y)=0=f_\lambda(\lambda,x,y)$ is $-\sigma\lambda^{\sigma-1}$ which does not vanish for $\lambda>0$. The characteristic region of $f(\lambda,x,y)$ which belongs to $\mathbb{C}\setminus(0,+\infty)$ is just the multiplicity-2 set with order 0 of the envelope G for the family $\{L_\lambda|\lambda\in(0,+\infty)\}$ ([13],Theorem 2.6). The parametric functions of G could be given by ([13],Theorem 2.3)

$$x(\lambda)=-\frac{\sigma+2}{\sigma}\lambda^2,\quad y(\lambda)=\frac{2}{\sigma}\lambda^{\sigma+2},\lambda>0.$$

Actually, G can also be described by the graph of the function $y=G(x)$, where

$$G(x)=(-1)^{\frac{\sigma+2}{2}}\frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}}x^{\frac{\sigma+2}{2}}.$$

Since

$$G'(x)=(-1)^{\frac{\sigma+2}{2}}\frac{\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma}{2}}}x^{\frac{\sigma}{2}},$$

$$G''(x)=(-1)^{\frac{\sigma+2}{2}}\frac{\sigma^{\frac{\sigma+2}{2}}}{2(\sigma+2)^{\frac{\sigma}{2}}}x^{\frac{\sigma-2}{2}},$$

$G(x)$ is a positive and strictly decreasing, strictly convex function on $(-\infty,0)$ such that $G(0^-)=0,G(-\infty)=+\infty$. From the property of the function $G(x)$, we can see that the equation $f(\lambda,q,r)=0$ has no positive root if and only if $q\geq 0$ and $r\geq 0$ or $q<0$ and

$$r>(-1)^{\frac{\sigma+2}{2}}\frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}}q^{\frac{\sigma+2}{2}}.$$

The proof is complete.

III. Main results

In this section, some necessary and sufficient conditions for the oscillatory properties of equation (1) are established.

To facilitate discussions, we divide nonnegative integers σ and τ into four mutually exclusive cases: (i) $\sigma\geq 1$ and $\tau\geq 1$, (ii) $\sigma\geq 1$ and $\tau=0$, (iii) $\sigma=0$ and $\tau\geq 1$, (iv) $\sigma=0$ and $\tau=0$.

Theorem 1 Assume that $\sigma\geq 1$ and $\tau\geq 1$. Then every solution of equation (1) oscillates if and only if $p\geq 0,q\geq 0$ and $r\geq 0$ or $p>0,q<0$, and

$$r>(-1)^{\frac{\sigma+\tau+2}{2}}\frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}q^{\frac{\sigma+\tau+2}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}p^{\frac{\tau}{2}}}.$$

Proof When $\sigma\geq 1$ and $\tau\geq 1$, the characteristic equation of equation (1) is

$$\phi(p,q,r,\lambda,\mu)=\lambda^2+p\mu^2+q+r\lambda^{-\sigma}\mu^{-\tau}=0. \tag{2}$$

Let

$$\begin{aligned}F(p,q,r,\lambda,\mu)&=\lambda^{\sigma}\mu^{\tau}\phi(p,q,r,\lambda,\mu)\\&=\lambda^{\sigma+2}\mu^{\tau}+p\lambda^{\sigma}\mu^{\tau+2}+q\lambda^{\sigma}\mu^{\tau}+r=0.\end{aligned} \tag{3}$$

From (3), we can see that (2) has no positive root if and only if (3) has no positive root. Since we mainly discuss the oscillatory solutions of equation (1), by Lemma 1, attention will be restricted to the case where $\lambda>0$ and $\mu>0$. We will consider (p,q,r) as a point in xyz-space, and try to search for the exact regions containing points (p,q,r) in xyz-space such that (3) has no positive root. Actually, $F(x,y,z,\lambda,\mu)=0$ can be regarded as an equation describing a two-parameter family of planes in xyz-space, where x,y and z are the coordinates of point of the planes in xyz-space and λ,μ are parameters.

According to the envelop theory, the points of the envelope of the two-parameter family of planes defined by (3) satisfy the following equations

$$\begin{cases}F(x,y,z,\lambda,\mu)=0,\\F_\lambda(x,y,z,\lambda,\mu)=(\sigma+2)\lambda^{\sigma+1}\mu^{\tau}+\sigma\lambda^{\sigma-1}\mu^{\tau+2}x\\\qquad+\sigma\lambda^{\sigma-1}\mu^{\tau}y=0,\\F_\mu(x,y,z,\lambda,\mu)=\tau\lambda^{\sigma+2}\mu^{\tau-1}+(\tau+2)\lambda^{\sigma}\mu^{\tau+1}x\\\qquad+\tau\lambda^{\sigma}\mu^{\tau-1}y=0,\end{cases} \tag{4}$$

where $\lambda>0$ and $\mu>0$. Eliminating λ and μ from (4), we get the function of the envelope

$$z(x,y)=(-1)^{\frac{\sigma+\tau+2}{2}}\frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}y^{\frac{\sigma+\tau+2}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}x^{\frac{\tau}{2}}}, \tag{5}$$

where $x>0,y<0$. Consequently, we have

$$\begin{aligned}\frac{\partial z}{\partial x}&=(-1)^{\frac{\sigma+\tau}{2}}\frac{\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau+2}{2}}y^{\frac{\sigma+\tau+2}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}x^{\frac{\tau+2}{2}}},\\\frac{\partial z}{\partial y}&=(-1)^{\frac{\sigma+\tau+2}{2}}\frac{\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}y^{\frac{\sigma+\tau}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau}{2}}x^{\frac{\tau}{2}}},\\\frac{\partial^2 z}{\partial x^2}&=(-1)^{\frac{\sigma+\tau+2}{2}}\frac{(\tau+2)\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau+2}{2}}y^{\frac{\sigma+\tau+2}{2}}}{2(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}x^{\frac{\tau+4}{2}}},\\\frac{\partial^2 z}{\partial y^2}&=(-1)^{\frac{\sigma+\tau+2}{2}}\frac{(\sigma+\tau)\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}y^{\frac{\sigma+\tau-2}{2}}}{2(\sigma+\tau+2)^{\frac{\sigma+\tau}{2}}x^{\frac{\tau}{2}}},\\\frac{\partial^2 z}{\partial x\partial y}&=(-1)^{\frac{\sigma+\tau}{2}}\frac{\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau+2}{2}}y^{\frac{\sigma+\tau}{2}}}{2(\sigma+\tau+2)^{\frac{\sigma+\tau}{2}}x^{\frac{\tau+2}{2}}}.\end{aligned}$$

When $x>0,y<0$, we have $\partial^2z/\partial x^2>0,\partial^2z/\partial y^2>0,\partial^2z/\partial x^2\cdot\partial^2z/\partial y^2-(\partial^2z/\partial x\partial y)^2=\frac{\sigma^{\sigma+1}\tau^{\tau+1}y^{\sigma+\tau}}{2(\sigma+\tau+2)^{\sigma+\tau+2}x^{\tau+2}}>0$ and $z(x,y)>0$. Hence, $z(x,y)$ is a positive and strictly convex function on $(0,+\infty)\times(-\infty,0)$. Moreover, the envelope defined by (5) is a strictly convex surface S over $(0,+\infty)\times(-\infty,0)$ as described in Figure 1. Thus from Figure 1, it is clearly seen that when (p,q,r) is in the first closed octant, namely, $p\geq 0,q\geq 0$ and $r\geq 0$, or when (p,q,r) is vertically above the envelope S, namely, $p>0,q<0$ and

$$r>(-1)^{\frac{\sigma+\tau+2}{2}}\frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}q^{\frac{\sigma+\tau+2}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}p^{\frac{\tau}{2}}},$$

there cannot be any tangent plane of the envelope S which passes through the (p, q, r). Since (2) is the same as (3)

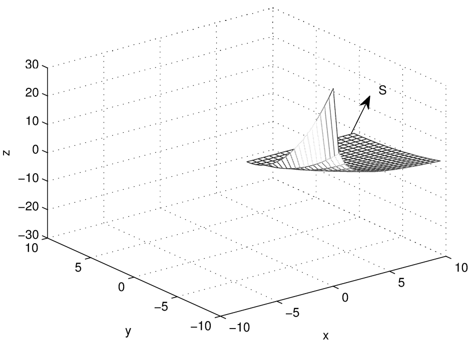

Fig. 1. Envelope surface for $\sigma = 1$ and $\tau = 1$

for the existence of positive solutions. Lemma 1 implies the statement of this theorem. The proof completes.

Theorem 2 Assume that $\sigma \geq 1$ and $\tau = 0$. Then every solution of equation (1) oscillates if and only if $p \geq 0, q \geq 0$ and $r \geq 0$ or $p \geq 0, q < 0$ and

$$r > (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}} q^{\frac{\sigma+2}{2}}.$$

Proof When $\sigma \geq 1$ and $\tau = 0$, the characteristic equation of equation (1) is

$$\phi(p, q, r, \lambda, \mu) = \lambda^2 + p\mu^2 + q + r\lambda^{-\sigma} = 0. \tag{6}$$

When $p < 0$, it is obvious that (6) has positive solutions. We just need to consider two cases: (a) $p = 0$ and (b) $p > 0$

Case (a). $p = 0$. (6) can be written as

$$\phi(p, r, \lambda) = \lambda^2 + q + r\lambda^{-\sigma} = 0. \tag{7}$$

Let

$$f(q, r, \lambda) = \lambda^{\sigma}\phi(p, r, \lambda) = \lambda^{\sigma+2} + q\lambda^{\sigma} + r = 0. \tag{8}$$

Since (8) is same as (7) for the existence of positive solutions, from Lemma 5, (6) has no positive root if and only if $q \geq 0$ and $r \geq 0$ or $q < 0$ and

$$r > (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}} q^{\frac{\sigma+2}{2}}.$$

Case (b). $p > 0$. (6) can be rewritten as

$$\phi(p, q, r, \lambda, \mu) = p\left(\frac{1}{p}\lambda^2 + \mu^2 + \frac{q}{p} + \frac{r}{p}\lambda^{-\sigma}\right) = 0.$$

Let

$$f\left(\frac{1}{p}, \frac{q}{p}, \frac{r}{p}, \lambda\right) = \frac{1}{p}\lambda^2 + \frac{q}{p} + \frac{r}{p}\lambda^{-\sigma}. \tag{9}$$

Since $\lim_{\lambda \to +\infty} f(1/p, q/p, r/p, \lambda) > 0$ with $p > 0$ and $f(1/p, q/p, r/p, \lambda)$ is differentiable with regard to $\lambda > 0$. By Lemma 4, (6) has no positive root if and only if (9) has no positive root. Let

$$F\left(\frac{1}{p}, \frac{q}{p}, \frac{r}{p}, \lambda\right) = \lambda^{\sigma} f\left(\frac{1}{p}, \frac{q}{p}, \frac{r}{p}, \lambda\right) = \frac{1}{p}\lambda^{\sigma+2} + \frac{q}{p}\lambda^{\sigma} + \frac{r}{p} = 0. \tag{10}$$

Then (9) has no positive root if and only if (10) has no positive root. Since we investigate oscillatory solutions of (1), by Lemma 1, attention will be restricted to the case where $\lambda > 0$. We will consider $(1/p, q/p, r/p)$ as a point in xyz-space and search for the exact regions including points $(1/p, q/p, r/p)$ in xyz-space such that (10) has no positive root. Actually, $F(x, y, z, \lambda) = 0$ can be regarded as an equation describing a one-parameter family of planes in xyz-space, where x, y and z are the coordinates of point of the plane in xyz-space and λ is a parameter.

According to the theory of envelope, the points of the envelope of the one-parameter family of planes described by (10) satisfy the following equations

$$\begin{cases} F(x, y, z, \lambda) = \lambda^{\sigma+2}x + \lambda^{\sigma}y + z = 0, \\ F_{\lambda}(x, y, z, \lambda) = (\sigma+2)\lambda^{\sigma+1}x + \sigma\lambda^{\sigma-1}y = 0, \end{cases} \tag{11}$$

where $\lambda > 0$. Eliminating $\lambda (> 0)$ from (11), we obtain the function of the envelope

$$z(x, y) = (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}} y^{\frac{\sigma+2}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}} x^{\frac{\sigma}{2}}}, \tag{12}$$

where $x > 0$ and $y < 0$. Consequently, we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= (-1)^{\frac{\sigma}{2}} \frac{\sigma^{\frac{\sigma+2}{2}} y^{\frac{\sigma+2}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}} x^{\frac{\sigma+2}{2}}}, \\ \frac{\partial z}{\partial y} &= (-1)^{\frac{\sigma+2}{2}} \frac{\sigma^{\frac{\sigma}{2}} y^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma}{2}} x^{\frac{\sigma}{2}}}, \\ \frac{\partial^2 z}{\partial x^2} &= (-1)^{\frac{\sigma+2}{2}} \frac{\sigma^{\frac{\sigma+2}{2}} y^{\frac{\sigma+2}{2}}}{2(\sigma+2)^{\frac{\sigma}{2}} x^{\frac{\sigma+4}{2}}}, \\ \frac{\partial^2 z}{\partial y^2} &= (-1)^{\frac{\sigma+2}{2}} \frac{\sigma^{\frac{\sigma+2}{2}} y^{\frac{\sigma-2}{2}}}{2(\sigma+2)^{\frac{\sigma}{2}} x^{\frac{\sigma}{2}}}, \\ \frac{\partial^2 z}{\partial x \partial y} &= (-1)^{\frac{\sigma}{2}} \frac{\sigma^{\frac{\sigma+2}{2}} y^{\frac{\sigma}{2}}}{2(\sigma+2)^{\frac{\sigma}{2}} x^{\frac{\sigma+2}{2}}}. \end{aligned}$$

When $x > 0$ and $y < 0$, we have $z(x, y) > 0$, $\partial^2 z/\partial x^2 > 0$, $\partial^2 z/\partial y^2 > 0$ and $\partial^2 z/\partial x^2 \cdot \partial^2 z/\partial y^2 - (\partial^2 z/\partial x \partial y)^2 = 0$. Hence, the envelope defined by (12) is a convex surface

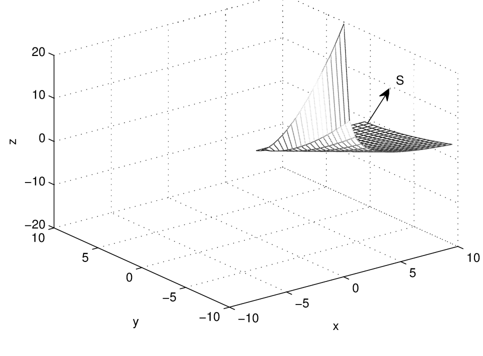

Fig. 2. Envelope surface for $\sigma = 1$ and $\tau = 0$

S over $(0, +\infty) \times (-\infty, 0)$ as depicted in Figure 2. Thus we can easily see that there are two cases for the point $(1/p, q/p, r/p)$ through which there cannot be any tangent plane of the envelope S which passes. The first case is that $(1/p, q/p, r/p)$ is in the first closed octant except on the positive coordinate axis $x = 0$, namely, $1/p > 0, q/p \geq 0$,and $r/p \geq 0$, which are equivalent to $p > 0, q \geq 0$ and $r \geq 0$.

The second case is that $(1/p, q/p, r/p)$ is vertically above the envelope S in the forth octant, namely, $1/p > 0, q/p < 0$ and $r/p > (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}} q^{\frac{\sigma+2}{2}}$, which are equivalent to $p > 0, q < 0$ and $r > (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}} q^{\frac{\sigma+2}{2}}$. Meanwhile, if $(1/p, q/p, r/p)$ is lied in somewhere else except the above two cases, such a tangent plane exists.

Since (6) is the same as (10) for the existence of positive solutions, combining case (a) and case (b), one can see that (6) does not have any positive root if and only if $p \geq 0, q \geq 0$ and $r \geq 0$ or $p \geq 0, q < 0$ and

$$r > (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}} q^{\frac{\sigma+2}{2}}.$$

Lemma 1 implies the statement of this theorem. The proof completes.

Theorem 3 Assume that $\sigma = 0$ and $\tau \geq 1$. Then every solution of equation (1) oscillates if and only if $p \geq 0, q \geq 0$ and $r \geq 0$ or $p > 0, q < 0$ and

$$r > (-1)^{\frac{\tau+2}{2}} \frac{2\tau^{\frac{\tau}{2}} q^{\frac{\tau+2}{2}}}{(\tau+2)^{\frac{\tau+2}{2}} p^{\frac{\tau}{2}}}.$$

Proof When $\sigma = 0$ and $\tau \geq 1$, the characteristic equation of equation (1) is

$$\phi(p, q, r, \lambda, \mu) = \lambda^2 + p\mu^2 + q + r\mu^{-\tau} = 0. \tag{13}$$

When $p < 0$, it is obvious that (13) has positive solutions. We just need to consider two cases: (a) $p = 0$ and (b) $p > 0$.

Case (a). $p = 0$. In this case, it is easy that (13) has no positive root if and only if $q \geq 0$ and $r \geq 0$.

Case (b). $p > 0$. In this case, let

$$f(p, q, \lambda, \mu) = p\mu^2 + q + r\mu^{-\tau} = 0. \tag{14}$$

Since $\lim_{\mu\to+\infty} f(p, q, r, \mu) > 0$ for $p > 0$ and $f(p, q, r, \mu)$ is differentiable with respect to $\mu > 0$, from Lemma 4, we can see that (13) has no positive root if and only if (14) has no positive root. Let

$$F(p, q, r, \mu) = \mu^\tau f(p, q, r, \mu) = p\mu^{\tau+2} + q\mu^\tau + r = 0. \tag{15}$$

It is clear that (14) has no positive root if and only if (15) has no positive root. Since we investigate oscillatory solutions of (1), by Lemma 1, attention will be restricted to the case where $\mu > 0$. We will consider (p, q, r) as a point in xyz-space and search for the exact regions including (p, q, r) in xyz-space such that (15) has no positive root. Actually, $F(x, y, z, \mu) = 0$ can be regarded as an equation describing a one-parameter family of planes in xyz-space, where x, y and z are the coordinates of point of the plane in xyz-space and μ is a parameter.

From the theory of envelope, the points of the envelope of the one-parameter family of planes described by (15) satisfy the follow equations

$$\begin{cases} F(x, y, z, \mu) = \mu^{\tau+2}x + \mu^\tau y + z = 0, \\ F_\mu(x, y, z, \mu) = (\tau+2)\mu^{\tau+1}x + \tau\mu^{\tau-1}y = 0, \end{cases} \tag{16}$$

where $\mu > 0$. Eliminating $\mu(> 0)$ from (16), we obtain the function of the envelope

$$z(x, y) = (-1)^{\frac{\tau+2}{2}} \frac{2\tau^{\frac{\tau}{2}} y^{\frac{\tau+2}{2}}}{(\tau+2)^{\frac{\tau+2}{2}} x^{\frac{\tau}{2}}}, \tag{17}$$

where $x > 0$ and $y < 0$. From (17), we have

$$\begin{aligned} \frac{\partial z}{\partial x} &= (-1)^{\frac{\tau}{2}} \frac{\tau^{\frac{\tau+2}{2}} y^{\frac{\tau+2}{2}}}{(\tau+2)^{\frac{\tau+2}{2}} x^{\frac{\tau+2}{2}}}, \\ \frac{\partial z}{\partial y} &= (-1)^{\frac{\tau+2}{2}} \frac{\tau^{\frac{\tau}{2}} y^{\frac{\tau}{2}}}{(\tau+2)^{\frac{\tau}{2}} x^{\frac{\tau}{2}}}, \\ \frac{\partial^2 z}{\partial x^2} &= (-1)^{\frac{\tau+2}{2}} \frac{\tau^{\frac{\tau+2}{2}} y^{\frac{\tau+2}{2}}}{2(\tau+2)^{\frac{\tau}{2}} x^{\frac{\tau+4}{2}}}, \\ \frac{\partial^2 z}{\partial y^2} &= (-1)^{\frac{\tau+2}{2}} \frac{\tau^{\frac{\tau+2}{2}} y^{\frac{\tau-2}{2}}}{2(\tau+2)^{\frac{\tau}{2}} x^{\frac{\tau}{2}}}, \\ \frac{\partial^2 z}{\partial x \partial y} &= (-1)^{\frac{\tau}{2}} \frac{\tau^{\frac{\tau+2}{2}} y^{\frac{\tau}{2}}}{2(\tau+2)^{\frac{\tau}{2}} x^{\frac{\tau+2}{2}}}. \end{aligned}$$

When $x > 0$ and $y < 0$, we have $z(x, y) > 0$, and

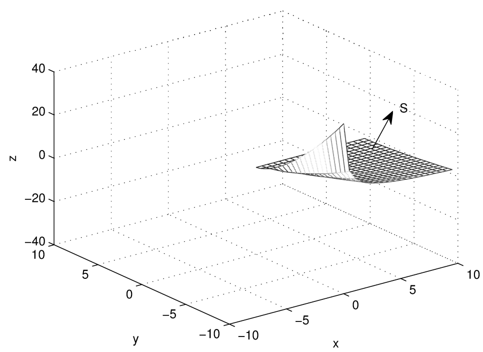

Fig. 3. Envelope surface for $\sigma = 0$ and $\tau = 1$

$\partial^2 z/\partial x^2 > 0, \partial^2 z/\partial y^2 > 0$ and $\partial^2 z/\partial x^2 \cdot \partial^2 z/\partial y^2 - (\partial^2 z/\partial x\partial y)^2 = 0$. Hence, the envelope defined by (17) is a convex surface S over $(0, +\infty) \times (-\infty, 0)$ as depicted in Figure 3. Thus we can easily see that there are two cases for the point (p, q, r) through which there cannot be any tangent plane of the envelope S which passes. The first case is that (p, q, r) is in the first closed octant except on the positive coordinate axis $x = 0$, namely, $p > 0, q \geq 0$, and $r \geq 0$. The second case is that (p, q, r) is vertically above the envelope S in the forth octant, namely, $p > 0, q < 0$ and $r > (-1)^{\frac{\tau+2}{2}} \frac{2\tau^{\frac{\tau}{2}}}{(\tau+2)^{\frac{\tau+2}{2}}} q^{\frac{\tau+2}{2}}$. Meanwhile, if (p, q, r) is lied in somewhere else except the above two cases, such a tangent plane exists.

Since (13) is the same as (15) for the existence of positive solutions, it follows from case (a) and case (b) that (13) does not have any positive root if and only if $p \geq 0, q \geq 0$ and $r \geq 0$ or $p > 0, q < 0$ and

$$r > (-1)^{\frac{\tau+2}{2}} \frac{2\tau^{\frac{\tau}{2}} q^{\frac{\tau+2}{2}}}{(\tau+2)^{\frac{\tau+2}{2}} p^{\frac{\tau}{2}}}.$$

By lemma 1, the proof completes.

Theorem 4 Assume that $\sigma = 0$ and $\tau = 0$. Then every solution of equation (1) oscillates if and only if $p > 0$ and $q + r = 0$.

Proof When $\sigma = 0$ and $\tau = 0$, one can rewrite equation (1) as

$$u_{m+2,n} + pu_{m,n+2} + (q + r)u_{m,n} = 0. \tag{18}$$

The characteristic equation of equation (18) is

$$\lambda^2 + p\mu^2 + (q + r) = 0. \tag{19}$$

It can be clearly seen that (19) does not have any positive root if and only if $p > 0$ and $q + r = 0$. From Lemma 1, we can see that every solution of equation (1) oscillates if and only if $p > 0$ and $q + r = 0$. This completes the proof.

IV. Illustrative Examples

In this section, we give some examples to illustrate the results obtained in Section 3.

Example 1 Consider the delay partial difference equation

$$u_{m+2,n} + 0.5u_{m,n+2} + 0.2u_{m,n} + 0.3u_{m-1,n-1} = 0. \tag{20}$$

Clearly, $\sigma = 1$, $\tau = 1$, $p = 0.5$, $q = 0.2$ and $r = 0.3$. Since $p = 0.5 > 0$, $q = 0.2 > 0$ and $r = 0.3 > 0$, by 1, every solution of equation (20) is oscillatory. The oscillatory behavior of equation (20) is demonstrated by Figure 4.

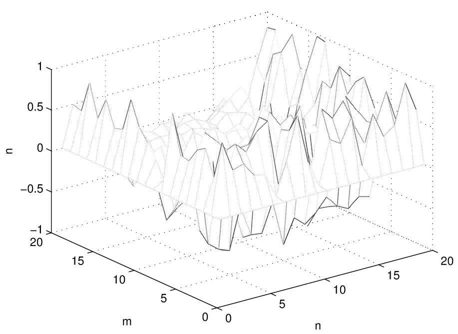

Fig. 4. Oscillatory behavior of equation (20)

Example 2 Consider the delay partial difference equation

$$u_{m+2,n} + 0.81u_{m,n+2} - 0.1u_{m,n} + 0.15u_{m-1,n-1} = 0. \tag{21}$$

In this case, $\sigma = 1$, $\tau = 1$, $p = 0.81$, $q = -0.1$ and $r = 0.15$. Since $p = 0.81 > 0$, $q = -0.1 < 0$ and

$$r = 0.15 > \frac{1}{72} = (-1)^{\frac{\sigma+\tau+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}\tau^{\frac{\tau}{2}}q^{\frac{\sigma+\tau+2}{2}}}{(\sigma+\tau+2)^{\frac{\sigma+\tau+2}{2}}p^{\frac{\tau}{2}}},$$

by Theorem 1, every solution of equation (21) oscillats. The oscillatory behavior of equation (21) is demonstrated by Figure 5.

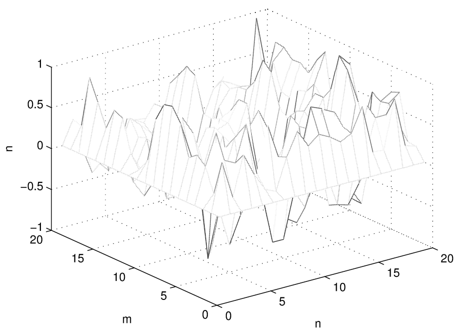

Fig. 5. Oscillatory behavior of equation (21)

Example 3 Consider the delay partial difference equation

$$u_{m+2,n} + 0.7u_{m,n+2} + 0.3u_{m,n} + 0.1u_{m-1,n} = 0. \tag{22}$$

Obviously, $\sigma = 1$, $\tau = 0$, $p = 0.7$, $q = 0.3$ and $r = 0.1$. Since $p = 0.7 > 0$, $q = 0.3 > 0$ and $r = 0.1 > 0$, according to Theorem 2, every solution of equation (22) is oscillatory. The oscillatory behavior of equation (22) is demonstrated by Figure 6.

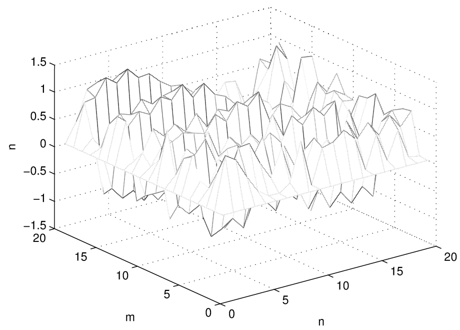

Fig. 6. Oscillatory behavior of equation (22)

Example 4 Consider the delay partial difference equation

$$u_{m+2,n} + 0.64u_{m,n+2} - 0.1u_{m,n} + 0.18u_{m-1,n} = 0. \tag{23}$$

In this case, $\sigma = 1$, $\tau = 0$, $p = 0.64$, $q = -0.1$ and $r = 0.18$. Since $p = 0.64 > 0$, $q = -0.1 < 0$ and

$$r = 0.18 > \frac{\sqrt{30}}{450} = (-1)^{\frac{\sigma+2}{2}} \frac{2\sigma^{\frac{\sigma}{2}}}{(\sigma+2)^{\frac{\sigma+2}{2}}} q^{\frac{\sigma+2}{2}},$$

according to Theorem 2, every solution of equation (23) oscillats. The oscillatory behavior of equation (23) is demonstrated by Figure 7.

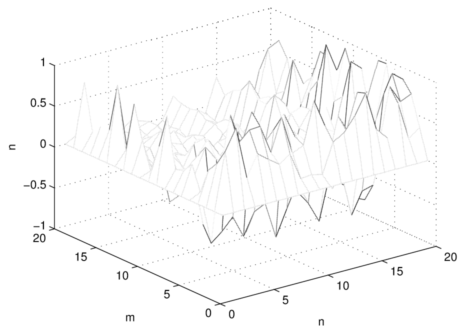

Fig. 7. Oscillatory behavior of equation (23)

Example 5 Consider the delay partial difference equation

$$u_{m+2,n} + 0.8u_{m,n+2} = 0. \tag{24}$$

Clearly, $\sigma = 0$, $\tau = 0$, $p = 0.8$, $q = -0.02$ and $r = 0.02$. Since $p = 0.8 > 0$, and $p + r = 0$, according to Theorem 4, every solution of equation (24) is oscillatory. The oscillatory behavior of equation (24) is demonstrated by Figure 8.

Acknowledgment

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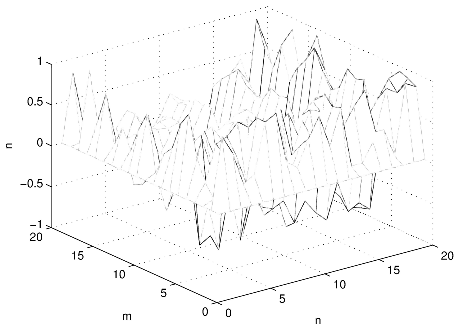

Fig. 8. Oscillatory behavior of equation (24)

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