Direct Integration of Fourth Order Initial and Boundary Value Problems using Nyström Type Methods

S. N. Jator[a], E. O. Adeyefa[b] *

Abstract— **Nyström type methods are widely used for the numerical integration of initial value problems (IVPs) in ordinary differential equations (ODEs). Specifically, they are extensively used for directly solving second order IVPs. Nevertheless, they are not normally used for the numerical integration of boundary value problems (BVPs). This paper focuses on the formulation of a family of block Nyström type methods (BNM (η, p)) for the numerical solution of fourth order IVPs and BVPs, where $\eta$is the number of off-grid points and p is the order of the method. The family of BNM (η, p) is formulated from continuous schemes obtained via collocation and interpolation techniques and applied in a block-by-block manner as numerical integrators for fourth order ODEs. The convergence properties of this family of methods are discussed via zero-stability and consistency. Numerical examples are included and comparisons are made with existing methods in the literature.**

Keywords: Fourth Order Ordinary Differential Equations, Zero-stability and consistency, Block Nyström Method, Convergence
AMS Subject Classification: 65L05, 65L06

1 Introduction

The theory of differential equations (DEs) has connections with several fields such as engineering, science, and management. In particular, DEs have applications in fluid dynamics (see Alomari et al. [3]), beam theory (see Jator [21]), electric circuits (see Boutayeb and Chetouani [8]), ship dynamics (see Wu et al. [35], Twizell [32], Cortell [11]) and neural networks (see Malek and Beidokhti [27]). They are also applied to the reaction and diffusion of chemicals, the dynamics of populations in biology, the development and treatment of diseases in medicine, molecular dynamics, the motion of rocket, and several other areas. So, the demand for the solution of DEs is on the increase as the quest for numerical methods has increasingly been of much interest to researchers owing to the fact that most of these DEs are difficult to solve or their analytical solutions do not exist. Thus, the focus of this paper is to develop a family of block Nyström type methods for the numerical solution of the ODE of the form

$$y^{iv} = f(x, y, y', y'', y'''), \qquad x \in [x_0, x_N], \tag{1}$$

subject to initial conditions
$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \quad y'''(x_0) = y'''_0$
which are not a restriction on the proposed method, since the method can also handle ODEs with Dirichlet, Neumann or Robin boundary conditions with only minor modifications in the boundary conditions.

The IVP of the form (1) is solved by reduction to an equivalent system of first order ODE and an appropriate numerical method is employed to solve the resultant system (see Adesanya et al. [1], Butcher [10], Lambert [25], Hairer et al. [17], and Dormand [13]). This approach has been reported to increase the number of equations four times and thereby more function evaluations need to be evaluated and hence resulting in a longer execution time and more computational effort (see Jator [21], Awoyemi ([4],[6]), Waeleh et al. [34], Mehrkanoon [28]). Moreover, Bun and Vasil'yer [9] reported that the in some instances the system of equations to be solved when the method of reduction is applied cannot be solved explicitly with respect to the derivatives of the highest order.

A successful application of numerical algorithm for directly solving a general fourth order initial value problems of the form (1) has been demonstrated in the literature (see Awoyemi ([5],[6]), Kayode ([22], [23])). However, all these methods were implemented in predictor-corrector mode and hence, according to Jator, (Jator [21]) the implementation of such schemes is more costly since the subroutines for incorporating the starting values lead to lengthy computational time. Besides, they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution model. To address the setback of the predictor-corrector method, Vigo-Aguiar and Ramos [33], Jator [20], Mohammed [29], Kayode et al. [24], Awoyemi et al. [7], Yap and Ismail [36], Hussain et al. [19], Adeyefa [2] among others independently proposed block methods for solving higher order ordinary differential equation which do not require the development of separate predictors, but simultaneously generate approximations at different grid points within the interval

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of integration without overlapping of sub-intervals experienced in the predictor-corrector method. Furthermore, the BNM (η, p) is superior to those mentioned above since it is equipped with block extension for solving BVPs.

The aim of developing new methods has always been to improve on the efficiency and convergence of existing methods with the ultimate aim of reducing the error of approximation. Thus, in what immediately follows in Section 2, we formulate the proposed methods for directly integrating fourth order ODEs. The analysis of the BNM (η, p) is discussed in Section 4, the implementation of the methods is given in Section 5 and numerical examples are given to show the efficiency of the methods in Section 6. Finally, the conclusion of the paper is discussed in Section 7.

2 Development of the BNM (η, p)

In order to numerically integrate (1), we consider the partition Q_N given by $Q_N := \{x_n = x_0 + nh\}$, $n = 0, 1, \ldots, N$, $h = \frac{x_N - x_0}{N}$,
and then use the proposed BNM (η, p) consisting of the following discrete formulas

$$y_{n+1} = \alpha_0 y_n + h\alpha'_0 y'_n + h^2\alpha''_0 y''_n + h^3\alpha'''_0 y'''_n + h^4\left(\sum_{j=0}^{1}\beta_j f_{n+j} + \sum_{j=1}^{\eta}\beta_{c_j} f_{n+c_j}\right)$$

$$y_{n+c_i} = \alpha_{i,0} y_n + h\alpha'_{i,0} y'_n + h^2\alpha''_{i,0} y''_n + h^3\alpha'''_{i,0} y'''_n + h^4\left(\sum_{j=0}^{1}\beta_{i,j} f_{n+j} + \sum_{j=1}^{\eta}\beta_{i,c_j} f_{n+c_j}\right),$$

$$hy'_{n+1} = \alpha_1 y_n + h\alpha'_1 y'_n + h^2\alpha''_1 y''_n + h^3\alpha'''_1 y'''_n + h^4\left(\sum_{j=0}^{1}\gamma_j f_{n+j} + \sum_{j=1}^{\eta}\gamma_{c_j} f_{n+c_j}\right)$$

$$hy'_{n+c_i} = \alpha_{i,1} y_n + h\alpha'_{i,1} y'_n + h^2\alpha''_{i,1} y''_n + h^3\alpha'''_{i,1} y'''_n + h^4\left(\sum_{j=0}^{1}\gamma_{i,j} f_{n+j} + \sum_{j=1}^{\eta}\gamma_{i,c_j} f_{n+c_j}\right),$$

$$h^2y''_{n+1} = \alpha_2 y_n + h\alpha'_2 y'_n + h^2\alpha''_2 y''_n + h^3\alpha'''_2 y'''_n + h^4\left(\sum_{j=0}^{1}\delta_j f_{n+j} + \sum_{j=1}^{\eta}\delta_{c_j} f_{n+c_j}\right)$$

$$h^2y''_{n+c_i} = \alpha_{i,2} y_n + h\alpha'_{i,2} y'_n + h^2\alpha''_{i,2} y''_n + h^3\alpha'''_{i,2} y'''_n + h^4\left(\sum_{j=0}^{1}\delta_{i,j} f_{n+j} + \sum_{j=1}^{\eta}\delta_{i,c_j} f_{n+c_j}\right),$$

$$h^3y'''_{n+1} = \alpha_3 y_n + h\alpha'_3 y'_n + h^2\alpha''_3 y''_n + h^3\alpha'''_3 y'''_n + h^4\left(\sum_{j=0}^{1}\kappa_j f_{n+j} + \sum_{j=1}^{\eta}\kappa_{c_j} f_{n+c_j}\right)$$

$$h^3y'''_{n+c_i} = \alpha_{i,3} y_n + h\alpha'_{i,3} y'_n + h^2\alpha''_{i,3} y''_n + h^3\alpha'''_{i,3} y'''_n + h^4\left(\sum_{j=0}^{1}\kappa_{i,j} f_{n+j} + \sum_{j=1}^{\eta}\kappa_{i,c_j} f_{n+c_j}\right)$$

where

$$\begin{cases} y_{n+1}, y_{n+c_i}, hy'_{n+1}, hy'_{n+c_i}, h^2y''_{n+1}, h^2y''_{n+c_i}, \\ h^3y'''_{n+1}, h^3y'''_{n+c_i} i = 1, \ldots, \eta \end{cases} \tag{2}$$

to integrate (1) over the partition considered, where $\alpha_k, \alpha_{i,k}, \alpha'_k, \alpha'_{i,k}, \alpha''_k, \alpha''_{i,k}, \alpha'''_k, \alpha'''_{i,k}$, $\beta_j, \beta_{i,j}, \beta_{c_j}, \beta_{i,c_j}$, $\gamma_j, \gamma_{i,j}, \gamma_{c_j}, \gamma_{i,c_j}$, $\delta_j, \delta_{i,j}, \delta_{c_j}, \delta_{i,c_j}$, $\kappa_j, \kappa_{i,j}, \kappa_{c_j}, \kappa_{i,c_j}$, $k = 1, \ldots, 3$ are coefficients and c_j are the off-grid points. The coefficients of the methods are chosen so that the method integrates the ODE (1) exactly, where the solutions are members of the linear space $\langle 1, x, \ldots, x^{m+w-1}\rangle$, where m is the number of interpolation points and w is the number of collocation points. We note that y_{n+j} and y_{n+c_j} denote the numerical approximations to the analytical solutions $y(x_{n+j})$ and $y(x_{n+c_j})$ respectively; and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}, y'''_{n+j})$ and $f_{n+c_j} = f(x_{n+c_j}, y_{n+c_j}, y'_{n+c_j}, y''_{n+c_j}, y'''_{n+c_j})$. The coefficients of (2) are provided by the continuous scheme derived next.

2.1 Continuous approximation for the BNM

In general, the coefficients of (2) are chosen so that the method integrates the ODE (1) exactly, where the solutions are members of the linear space $U(x) = (U_0(x), U_1(x), ..., U_{m+w-1}(x))$, $U_j(x) = x^j, j = 0, 1, \ldots, m+w-1$ are basis functions. In order to obtain the coefficients in (2), we begin by seeking an approximate solution of the form

$$Y(x) = \sum_{j=0}^{m+w-1} a_j U_j(x) = U(x)\Upsilon^T \tag{3}$$

on the interval $[x_n, x_{n+1}]$, where T is the transpose, $\Upsilon = (a_0, a_1, ..., a_{m+w-1})$ is a vector of coefficients to be determined, m and w are number of interpolation and collocation points respectively. The continuous scheme is constructed by demanding that the function $Y(x)$ passes through the points $(x_n, y_n), (x_{n+c_1}, y_{n+c_1}), \ldots, (x_{n+c_\eta}, y_{n+c_\eta}), (x_{n+1}, y_{n+1})$ and satisfies $m+w$ equations obtained by imposing that the following conditions hold.

$$\begin{cases} Y(x_n) = y_n, Y'(x_n) = y'_n, Y''(x_n) = y''_n, \\ \qquad Y'''(x_n) = y'''_n, \\ Y(x_{n+j}) = y_{n+j}, \quad j = 0, 1, \\ Y^{iv}(x_{n+c_j}) = f_{n+c_j}, j = 1, \ldots, \eta. \end{cases} \tag{4}$$

We note that equation (4) leads to a system of $(m+w)$ equations which is solved with the aid of Mathematica to obtain the coefficients $a'_j s$, given by the vector Υ. Specifically, we proceed as follows:

$$\mathbf{M}\Upsilon^T = \mathbf{V} \tag{5}$$

$$\Upsilon^T = \mathbf{M}^{-1}\mathbf{V} \tag{6}$$

where $\mathbf{M}$ is a matrix given as

$$M = \left\{ \begin{array}{ccc} U_0(x_n) & \dots & U_{m+w}(x_n) \\ U_0^{'}(x_n) & \dots & U_{m+w}^{'}(x_n) \\ U_0^{''}(x_n) & \dots & U_{m+w}^{''}(x_n) \\ U_0^{'''}(x_n) & \dots & U_{m+w}^{'''}(x_n) \\ U_0^{''''}(x_n) & \dots & U_{m+w}^{''''}(x_n) \\ U_0^{''''}(x_{n+c_1}) & \dots & U_{m+w}^{''''}(x_{n+c_1}) \\ \vdots & & \vdots \\ U_0^{''''}(x_{n+c_\eta}) & \dots & U_{m+w}^{''''}(x_{n+c_\eta}) \\ U_0^{''''}(x_{n+1}) & \dots & U_{m+w}^{''''}(x_{n+1}) \end{array} \right\},$$

and $\mathbf{V}$ is a vector defined as

$$V = \left[y_n, y_n^{'}, y_n^{''}, y_n^{'''}, f_n, f_{n+c_1}, \dots, f_{n+c_j}, f_{n+1}\right]^T.$$

The coefficients $a_j^{'}s$, given by the vector Υ are now known and given by (6). Our continuous scheme which is used to provide the coefficients in (2) is obtained by substituting (6) into (3) to yield

$$Y(x) = U(x)(M^{-1}V). \tag{7}$$

The continuous method (7) and additional methods provided by differentiating (7) are used to specify the proposed BNM (η, p) given by (2). In particular, (2) is specified by evaluating (8) at $x = x_{n+1}$ and $x = x_{n+c_j}$ and imposing that $Y(x_{n+1}) = y_{n+1}$, $Y(x_{n+c_j}) = y_{n+c_j}$, $y'_{n+1} = Y'(x)|_{x=x_{n+1}}$, $y'_{n+c_j} = Y'(x)|_{x=x_{n+c_j}}$, $y''_{n+1} = Y''(x)|_{x=x_{n+1}}$, $y''_{n+c_j} = Y''(x)|_{x=x_{n+c_j}}$, $y'''_{n+1} = Y'''(x)|_{x=x_{n+1}}$, and $y'''_{n+c_j} = Y'''(x)|_{x=x_{n+c_j}}, j = 1, \dots, \eta$.

$$\left\{ \begin{array}{l} Y(x) = U(x)(M^{-1}V), \\ Y^{'}(x) = \frac{d}{dx}(U(x)(M^{-1}V)), \\ Y^{''}(x) = \frac{d^2}{dx^2}(U(x)(M^{-1}V)), \\ Y^{'''}(x) = \frac{d^3}{dx^3}(U(x)(M^{-1}V)). \end{array} \right. \tag{8}$$

3 Specification of the Method

3.1 BNM with three off-grid points

In this section, we choose m = 4, w = 5, the number of off-grid points $\eta = 3$, and BNM (η, p) = BNM $(3, p)$. The BNM $(3, p)$ is then specified by evaluating (8) at $x = x_{n+1}$ and $x = x_{n+c_j}, j = 1, 2, 3$, where $\{c_1, c_2, c_3\} = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. In Figure 1, we give the coefficients of BNM $(3, p)$.

3.2 BNM with five off-grid points

In this subsection, m and w are respectively 4 and 7. We choose the number of off-grid points $\eta = 5$, and BNM (η, p) = BNM $(5, p)$. The BNM $(5, p)$ is then specified by evaluating (8) at $x = x_{n+1}$ and $x = x_{n+c_j}, j = 1, \dots, 5$, where $\{c_1, c_2, c_3, c_4, c_5\} = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$. In Figure 2, we give the coefficients of BNM $(5, p)$. where

$W = (y_{n+\frac{1}{6}} + y_{n+\frac{1}{3}} + y_{n+\frac{1}{2}} + y_{n+\frac{2}{3}} + y_{n+\frac{5}{6}} + y_{n+1}),$

$W' = (y'_{n+\frac{1}{6}} + y'_{n+\frac{1}{3}} + y'_{n+\frac{1}{2}} + y'_{n+\frac{2}{3}} + y'_{n+\frac{5}{6}} + y'_{n+1}),$

$W'' = (y''_{n+\frac{1}{6}} + y''_{n+\frac{1}{3}} + y''_{n+\frac{1}{2}} + y''_{n+\frac{2}{3}} + y''_{n+\frac{5}{6}} + y''_{n+1}),$

$W''' = (y'''_{n+\frac{1}{6}} + y'''_{n+\frac{1}{3}} + y'''_{n+\frac{1}{2}} + y'''_{n+\frac{2}{3}} + y'''_{n+\frac{5}{6}} + y'''_{n+1}),$

$g = (1,1,1,1,1,1)^T, g' = g'' = g''' = (0,0,0,0,0,0,)^T,$

$d = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1)^T$, $d' = (1,1,1,1,1,1)^T,$

$d'' = d''' = (0,0,0,0,0,0,)^T, c' = (\frac{1}{72}, \frac{1}{18}, \frac{1}{8}, \frac{2}{9}, \frac{25}{72}, \frac{1}{2})^T,$

$e = (\frac{1}{72}, \frac{1}{18}, \frac{1}{8}, \frac{2}{9}, \frac{25}{72}, \frac{1}{2})^T$, $e' = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1)^T,$

$e'' = (\frac{1}{72}, \frac{1}{18}, \frac{1}{8}, \frac{2}{9}, \frac{25}{72}, \frac{1}{2})^T$, $e''' = (1,1,1,1,1,1)^T,$

$e''' = (0,0,0,0,0,0,)^T, c = (\frac{1}{1296}, \frac{1}{162}, \frac{1}{48} \frac{4}{81}, \frac{125}{1296}, \frac{1}{6})^T,$

$c'' = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1)^T$, $c''' = (1,1,1,1,1,1)^T,$

$k = (\frac{95929}{4702924800}, \frac{4127}{18370800}, \frac{5471}{6451200}, \frac{488}{229635}, \frac{807125}{188116992}, \frac{191}{25200})^T,$

$k' = (\frac{343801}{783820800}, \frac{6887}{3061800}, \frac{1959}{358400}, \frac{3863}{382725}, \frac{505625}{31352832}, \frac{33}{1400}),$

$k'' = (\frac{28549}{4354560}, \frac{1027}{68040}, \frac{253}{10752}, \frac{272}{8505}, \frac{35225}{870912}, \frac{41}{840}),$

$k''' = (\frac{19087}{362880}, \frac{1139}{22680}, \frac{137}{2688}, \frac{143}{2835}, \frac{3715}{72576}, \frac{41}{840}),$

$m = (\frac{4001}{167961600}, \frac{4391}{9185400}, \frac{423}{179200}, \frac{7808}{1148175}, \frac{701875}{47029248}, \frac{39}{1400}),$

$m' = (\frac{6031}{9331200}, \frac{1499}{255150}, \frac{1599}{89600}, \frac{4664}{127575}, \frac{162125}{2612736}, \frac{33}{350}),$

$m'' = (\frac{275}{20736}, \frac{97}{1890}, \frac{165}{1792}, \frac{376}{2835}, \frac{8375}{48384}, \frac{3}{14}),$

$m''' = (\frac{2713}{15120}, \frac{47}{189}, \frac{27}{112}, \frac{232}{945}, \frac{725}{3024}, \frac{9}{35}),$

$n = (\frac{-23033}{940584960}, \frac{-199}{524880}, \frac{-39}{28672}, \frac{-632}{229635}, \frac{-790625}{188116992}, \frac{-3}{560}),$

$n' = (\frac{-32981}{52254720}, \frac{-233}{58320}, \frac{-537}{71680}, \frac{-226}{25515}, \frac{-85625}{10450944}, \frac{-3}{560}),$

$n'' = (\frac{-5717}{483840}, \frac{-2}{81}, \frac{-267}{17920}, \frac{-2}{945}, \frac{3125}{290304}, \frac{3}{140}),$

$n''' = (\frac{-15487}{120960}, \frac{11}{7560}, \frac{387}{4480}, \frac{64}{945}, \frac{2125}{24192}, \frac{9}{280}),$

$t = (\frac{811}{39191040}, \frac{97}{306180}, \frac{29}{23040}, \frac{256}{76545}, \frac{59375}{7838208}, \frac{19}{1260}),$

$t' = (\frac{5177}{9797760}, \frac{52}{15309}, \frac{1}{120}, \frac{272}{15309}, \frac{66875}{1959552}, \frac{2}{35}),$

$t'' = (\frac{10621}{1088640}, \frac{197}{8505}, \frac{5}{128}, \frac{656}{8505}, \frac{25625}{217728}, \frac{17}{105}),$

$t''' = (\frac{293}{2835}, \frac{166}{2835}, \frac{17}{105}, \frac{752}{2835}, \frac{125}{567}, \frac{34}{105}),$

$D = (\frac{-10693}{940584960}, \frac{-127}{734832}, \frac{-99}{143360}, \frac{-58}{32805}, \frac{-653125}{188116992}, \frac{-3}{560}),$

$D' = (\frac{-15107}{52254720}, \frac{-379}{204120}, \frac{-327}{71680}, \frac{-31}{3645}, \frac{-119375}{10450944}, \frac{-3}{280}),$

$D'' = (\frac{-7703}{1451520}, \frac{-97}{7560}, \frac{-363}{17920}, \frac{-2}{81}, \frac{-625}{96768}, \frac{3}{280}),$

$D''' = (\frac{-6737}{120960}, \frac{-269}{7560}, \frac{-243}{4480}, \frac{29}{945}, \frac{3875}{24192}, \frac{9}{280}),$

$E = (\frac{4219}{1175731200}, \frac{499}{9185400}, \frac{39}{179200}, \frac{128}{229635}, \frac{7625}{6718464}, \frac{3}{1400}),$

$E' = (\frac{5947}{65318400}, \frac{149}{255150}, \frac{129}{89600}, \frac{344}{127575}, \frac{1625}{373248}, \frac{3}{350}),$

$E'' = (\frac{403}{241920}, \frac{23}{5670}, \frac{57}{8960}, \frac{8}{945}, \frac{275}{20736}, \frac{3}{70}),$

$E''' = (\frac{263}{15120}, \frac{11}{945}, \frac{9}{560}, \frac{8}{945}, \frac{235}{3024}, \frac{9}{35}),$

$G = (\frac{-2323}{4702924800}, \frac{-137}{18370800}, \frac{-193}{6451200}, \frac{-88}{1148175}, \frac{-29375}{188116992}, \frac{-1}{3600}),$

$G' = (\frac{-9809}{783820800}, \frac{-491}{6123600}, \frac{-71}{358400}, \frac{-142}{382725}, \frac{-18625}{31352832}, \frac{-1}{1200}),$

$G'' = (\frac{-199}{870912}, \frac{-19}{34020}, \frac{-47}{53760}, \frac{-2}{1701}, \frac{-1375}{870912}, 0),$

and

$G''' = (\frac{-863}{362880}, \frac{-37}{22680}, \frac{-29}{13440}, \frac{-4}{2835}, \frac{-275}{72576}, \frac{41}{840})$

	y_n	y'_n	y''_n	y'''_n	f_n	$f_{n+\frac{1}{4}}$	$f_{n+\frac{1}{2}}$	$f_{n+\frac{3}{4}}$	f_{n+1}
$y_{n+\frac{1}{4}}$	1	$\frac{1}{4}$	$\frac{1}{32}$	$\frac{1}{384}$	$\frac{3373}{30965760}$	$\frac{139}{1548288}$	$\frac{-284}{5160960}$	$\frac{179}{7741440}$	$\frac{-131}{30965760}$
$y_{n+\frac{1}{2}}$	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{48}$	$\frac{37}{30240}$	$\frac{59}{30240}$	$\frac{-1}{1152}$	$\frac{11}{30240}$	$\frac{-1}{15120}$
$y_{n+\frac{3}{4}}$	1	$\frac{3}{4}$	$\frac{9}{32}$	$\frac{9}{128}$	$\frac{5319}{1146880}$	$\frac{2889}{286720}$	$\frac{-1539}{573440}$	$\frac{81}{57344}$	$\frac{-297}{1146880}$
y_{n+1}	1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{11}{945}$	$\frac{4}{135}$	$\frac{-1}{315}$	$\frac{4}{945}$	$\frac{-4}{945}$
$y'_{n+\frac{1}{4}}$		1	$\frac{1}{4}$	$\frac{1}{32}$	$\frac{113}{71680}$	$\frac{107}{64512}$	$\frac{-103}{107520}$	$\frac{43}{107520}$	$\frac{-47}{645120}$
$y'_{n+\frac{1}{2}}$		1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{321}{40320}$	$\frac{83}{5040}$	$\frac{-1}{168}$	$\frac{13}{5040}$	$\frac{-19}{40320}$
$y'_{n+\frac{3}{4}}$		1	$\frac{3}{4}$	$\frac{9}{32}$	$\frac{1431}{71680}$	$\frac{1863}{35840}$	$\frac{-243}{3580}$	$\frac{45}{7168}$	$\frac{81}{71680}$
y'_{n+1}		1	1	$\frac{1}{2}$	$\frac{31}{840}$	$\frac{34}{315}$	$\frac{1}{120}$	$\frac{2}{105}$	$\frac{-1}{504}$
$y''_{n+\frac{1}{4}}$			1	$\frac{1}{4}$	$\frac{367}{23040}$	$\frac{3}{128}$	$\frac{47}{3840}$	$\frac{29}{5760}$	$\frac{-7}{7680}$
$y''_{n+\frac{1}{2}}$			1	$\frac{1}{2}$	$\frac{53}{1440}$	$\frac{1}{10}$	$\frac{-1}{48}$	$\frac{1}{90}$	$\frac{-1}{480}$
$y''_{n+\frac{3}{4}}$			1	$\frac{3}{4}$	$\frac{147}{2560}$	$\frac{117}{640}$	$\frac{27}{1280}$	$\frac{3}{128}$	$\frac{-9}{2560}$
y''_{n+1}			1	1	$\frac{7}{90}$	$\frac{4}{15}$	$\frac{1}{15}$	$\frac{4}{45}$	0
$y'''_{n+\frac{1}{4}}$				1	$\frac{251}{2880}$	$\frac{323}{1440}$	$\frac{-11}{120}$	$\frac{53}{1440}$	$\frac{-19}{2880}$
$y'''_{n+\frac{1}{2}}$				1	$\frac{29}{360}$	$\frac{31}{90}$	$\frac{1}{15}$	$\frac{1}{90}$	$\frac{-1}{360}$
$y'''_{n+\frac{3}{4}}$				1	$\frac{27}{320}$	$\frac{51}{160}$	$\frac{9}{40}$	$\frac{21}{160}$	$\frac{-3}{320}$
y'''_{n+1}				1	$\frac{7}{90}$	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$

Fig 1: The coefficients of BNM (3,p)

	y_n	y'_n	y''_n	y'''_n	f_n	$f_{n+\frac{1}{6}}$	$f_{n+\frac{1}{3}}$	$f_{n+\frac{1}{2}}$	$f_{n+\frac{2}{3}}$	$f_{n+\frac{5}{6}}$	f_{n+1}
W	g	d	e	c	k	m	n	t	D	E	G
W'	g'	d'	e'	c'	k'	m'	n'	t'	D'	E'	G'
W''	g''	d''	e''	c''	k''	m''	n''	t''	D''	E''	G''
W'''	g'''	d'''	e'''	c'''	k'''	m'''	n'''	t'''	D'''	E'''	G'''

Fig 2: The coefficients of BNM (5,p)

4 Analysis of the methods

4.1 BNM (3,p)

4.1.1 Block form

Thus, the proposed BNM $(3, p)$ can be given in block form as follows:

$$\mathbf{A}^{(0)}Y_\mu = \mathbf{A}^{(1)}Y_{\mu-1} + h^4(\mathbf{B}^{(1)}F_{\mu-1} + \mathbf{B}^{(0)}F_\mu), \qquad (9)$$

where $\mu = 1, \ldots, N,\ n = 0, \ldots, N-1,\ \mathbf{A}^{(i)}, \mathbf{B}^{(i)}, i = 0,\ 1$ are matrices whose entries are given by the coefficients in Figure 1 and $\mathbf{A}^{(0)}$ is an identity matrix. We also define the vectors $Y_\mu,\ Y_{\mu-1},\ F_\mu$, and $F_{\mu-1}$ for BNM $(3, p)$ as follows:

$$Y_\mu = (y_{n+c_j},\ y_{n+1},\ hy'_{n+c_j},\ hy'_{n+1},\ h^2y''_{n+c_j},\ h^2y''_{n+1},\ h^3y'''_{n+c_j},\ h^3y'''_{n+1})^T,$$

$$F_\mu = (f_{n+c_j},\ f_{n+1},\ hf'_{n+c_j},\ hf'_{n+1},\ hf''_{n+c_j},\ hf''_{n+1},\ hf'''_{n+c_j},\quad hf'''_{n+1})^T,$$

$$Y_{\mu-1} = (\ y_{n-c_j},\ y_n,\ hy'_{n-c_j},\ hy'_n,\ h^2y''_{n-c_j},\ h^2y''_n,\ h^3y'''_{n-c_j},\quad h^3y'''_n)^T,$$

$$F_{\mu-1} = (f_{n-c_j},\ f_n,\ hf'_{n-c_j},\ hf'_n,\ hf''_{n-c_j},\ hf''_n,\ hf'''_{n-c_j},\ hf'''_n\ )^T,$$

where $j = 1, 2, 3$ and $\{c_1, c_2, c_3\} = \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. The additional $y_{n-c_j},\ y'_{n-c_j},\ y''_{n-c_j},\ y'''_{n-c_j},\ f_{n-c_j},\ f'_{n-c_j},\ f''_{n-c_j},\ f'''_{n-c_j}$ introduced are to augment the zero entries of the vector notations.

4.1.2 Local truncation error and Order

We define the local truncation error of the BNM using (9) as

$$L[Y(x); h] = \mathbf{A}^{(0)}Y_\mu - \mathbf{A}^{(1)}Y_{\mu-1} - h^4(B^{(1)}F_{\mu-1} + \mathbf{B}^{(0)}F_\mu), \qquad (10)$$

where L[Y(x);h] is a linear difference operator and impose that $Y(x_{n+1}) = y_{n+1} = y(x_n + jh)$, $Y(x_{n+c_j}) = y_{n+c_j} = y(x_n + c_jh)$, $y'_{n+1} = Y'(x)|_{x=x_{n+1}} = y'(x_n + jh)$, $y'_{n+c_j} = Y'(x)|_{x=x_{n+c_j}} = y'(x_n + c_jh)$, $y''_{n+1} = Y''(x)|_{x=x_{n+1}} = y''(x_n + jh)$, $y''_{n+c_j} = Y''(x)|_{x=x_{n+c_j}} = y''(x_n + c_jh)$, $y'''_{n+1} = Y'''(x)|_{x=x_{n+1}} = y'''(x_n + jh)$, $y'''_{n+c_j} = Y'''(x)|_{x=x_{n+c_j}} = y'''(x_n + c_jh)$, $f_{n+1} = y''''(x_n + jh)$, and $f_{n+c_j} = y''''(x_n + c_jh)$, $j = 1, \ldots, \eta$. Suppose that Y(x) is sufficiently differentiable, then, the expansion of L[Y(x);h] about point x using Taylor series gives
$L[Y(x); h] = C_0Y(x) + C_1hY'(x) + ... + C_ph^pY^p(x) + ... + C_{p+4}hY^{p+4}(x) + ...$
where $C_i, i = 0, 1, ...$ are column vectors whose entries comprise the error constants.

Definition
The BNM have algebraic order at least p≥1 provided there exists a constant $C_{p+4} \neq 0$ such that the local truncation error E_μ satisfies $\|E_\mu\| = C_{p+4}h^{p+4} + O(h^{p+5})$, where $\|\cdot\|$ is the maximum norm.
According to this definition, the local truncation error constants C_{p+4} of $(y_{n+k}, hy'_{n+k})^T$ for BNM (3,p) are given as Figure 3.

where $C_0 = C_1 = C_2 = \ldots C_p = ...C_{p+3} = 0$.

The order p of the BNM (3,p) has been obtained from the computation of the local truncation error constant as five.

4.1.3 Consistency and Zero-stability of BNM (3, 5)

The consistency of the method is established by the fact that the order of BNM $(3, 5)$ is greater than one (see Jator [20], Henrici [18]).

The zero-stability of a numerical method reveals the behavior of the method with a given value of $h > 0$ i.e. the stability of the difference system in the limit as h tends to zero. Thus, as h→ 0, equation (10) tends to the difference system $\mathbf{A}^{(1)}Y_{\mu-1} - \mathbf{A}^{(0)}Y_\mu = 0$ whose first characteristic polynomial is given by

$$\rho(R) = \det(R\mathbf{A}^{(0)} - \mathbf{A}^{(1)}) \qquad (11)$$

The block BNM $(3, 5)$ has its $\mathbf{A}^{(0)}$ and $\mathbf{A}^{(1)}$ as 16 by 16 matrices where $\mathbf{A}^{(0)}$ is identity matrix and $\mathbf{A}^{(1)}$ has its fourth, eighth, twelfth and sixteenth columns as Figure 4.

respectively while other entries are zero.
Substituting $\mathbf{A}^{(0)}$ and $\mathbf{A}^{(1)}$ in (11), we obtain $\rho(R) = R^{12}$ $(R^4 - 1)$.

According to Fatunla ([14], [15]), the method is zero-stable since $\rho(R) = 0$ satisfies $|R_j| \leq 1,\ j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed four.

$$C_{p+4}=\begin{bmatrix} \frac{41}{33973862400} & \frac{551}{142851340800} & \frac{11}{619315200} & \frac{1}{7257600} & \frac{53}{2642411520} & \frac{1}{22044960} & \frac{1}{7741440} & & \\ \frac{-1}{14515200} & \frac{59}{247726080} & \frac{13}{35271936} & \frac{1}{1548288} & \frac{-1}{241920} & \frac{7}{4423680} & \frac{13}{8398080} & \frac{1}{552960} & \frac{-1}{34560} \end{bmatrix}^T$$

Fig 3: Error constants of BNM (3,p)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{32} & \frac{1}{8} & \frac{9}{32} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{384} & \frac{1}{48} & \frac{9}{128} & \frac{1}{6} & \frac{1}{32} & \frac{1}{8} & \frac{9}{32} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

Fig 4: The 4th, 8th, 12th and 16th columns of $A^{(1)}$

4.2 BNM $(5, p)$

4.2.1 Block form

The proposed BNM $(5, p)$ can as well be given in block form as

$$\overline{\mathbf{A}}^{(0)}Y_\mu = \overline{\mathbf{A}}^{(1)}Y_{\mu-1} + h^4(\overline{\mathbf{B}}^{(1)}F_{\mu-1} + \overline{\mathbf{B}}^{(0)}F_\mu), \quad (12)$$

where $\mu = 1, \ldots, N,\ n = 0, \ldots, N-1, \overline{\mathbf{B}}^{(i)}, \overline{\mathbf{B}}^{(i)}, i = 0,\ 1$ are matrices whose entries are given by the coefficients in Figure 2, and $\overline{\mathbf{A}}^{(0)}$ is 24 by 24 identity matrix. We also define the vectors Y_μ, $Y_{\mu-1}$, F_μ, and $F_{\mu-1}$ for BNM (5,p) as follows:

$$Y_\mu = (y_{n+c_j},\ y_{n+1},\ hy'_{n+c_j},\ hy'_{n+1},\ h^2y''_{n+c_j},\ h^2y''_{n+1},\ h^3y'''_{n+c_j},\ h^3y'''_{n+1})^T,$$

$$F_\mu = (f_{n+c_j},\ f_{n+1},\ hf'_{n+c_j},\ hf'_{n+1},\ hf''_{n+c_j},\ hf''_{n+1},\ hf'''_{n+c_j},\ hf'''_{n+1})^T,$$

$$Y_{\mu-1} = (\ y_{n-c_j},\ y_n,\ hy'_{n-c_j},\ hy'_n,\ h^2y''_{n-c_j},\ h^2y''_n,\ h^3y'''_{n-c_j},\ h^3y'''_n)^T,$$

$$F_{\mu-1} = (f_{n-c_j},\ f_n,\ hf'_{n-c_j},\ hf'_n,\ hf''_{n-c_j},\ hf''_n,\ hf'''_{n-c_j},\ hf'''_n\ )^T,$$

where $j = 1, 2, 3, 4, 5$ and $\{c_1, c_2, c_3, c_4, c_5\} = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$. The additional y_{n-c_j}, y'_{n-c_j}, y''_{n-c_j}, y'''_{n-c_j}, f_{n-c_j}, f'_{n-c_j}, f''_{n-c_j}, f'''_{n-c_j} introduced are to augment the zero entries of the vector notations.

4.2.2 Local truncation error and Order

The local truncation error of BNM (5,p) is also defined using (12) as

$$L[Y(x); h] = \overline{\mathbf{A}}^{(0)}Y_\mu - \overline{\mathbf{A}}^{(1)}Y_{\mu-1} - h^4(\overline{\mathbf{B}}^{(1)}F_{\mu-1} + \overline{\mathbf{B}}^{(0)}F_\mu), \quad (13)$$

where L[Y(x);h] is a linear difference operator and impose that $Y(x_{n+1}) = y_{n+1} = y(x_n + jh)$, $Y(x_{n+c_j}) = y_{n+c_j} = y(x_n + c_jh)$, $y'_{n+1} = Y'(x)|_{x=x_{n+1}} = y'(x_n + jh)$, $y'_{n+c_j} = Y'(x)|_{x=x_{n+c_j}} = y'(x_n + c_jh)$, $y''_{n+1} = Y''(x)|_{x=x_{n+1}} = y''(x_n + jh)$, $y''_{n+c_j} = Y''(x)|_{x=x_{n+c_j}} = y''(x_n + c_jh)$, $y'''_{n+1} = Y'''(x)|_{x=x_{n+1}} = y'''(x_n + jh)$, $y'''_{n+c_j} = Y'''(x)|_{x=x_{n+c_j}} = y'''(x_n + c_jh)$, $f_{n+1} = y''''(x_n + jh)$, and $f_{n+c_j} = y''''(x_n + c_jh)$, $j = 1, \ldots, \eta$. Suppose that Y(x) is sufficiently differentiable, then, the expansion of L[Y(x);h] about point x using Taylor series gives
$L[Y(x); h] = C_0Y(x) + C_1hY'(x) + ... + C_ph^pY^p(x) + ... + C_{p+4}hY^{p+4}(x) + ...$
where $C_i, i = 0, 1, ...$ are column vectors whose entries comprise the error constants.

Thus, according to the definition given in section 4.1, the local truncation error constant $C_{p+4} \neq 0$ of BNM $(5, p)$ has been obtained as Figure 5

where the order, p = 7.

4.2.3 Consistency and Zero-stability of BNM $(5, 7)$

As earlier discussed, the order p > 1 of BNM (5, 7) established its consistency (see Jator [21], Henrici [18]).

The first characteristic polynomial of BNM $(5, 7)$ is given by $\rho(R) = \det(R\overline{\mathbf{A}}^{(0)} - \overline{\mathbf{A}}^{(1)})$ and its $\overline{\mathbf{A}}^{(1)}$ is a 24 by 24 matrix whose sixth, twelfth, eighteenth and twenty-fourth columns are given as Figure 6

$$C_{p+4} = \begin{bmatrix} \frac{15739}{10861273143705600} & \frac{733}{33941478574080} & \frac{1}{11496038400} & \frac{37}{165729875850} & \frac{198125}{434450925748224} & \frac{1}{1231718400} \\ \frac{4001}{109709829734400} & \frac{199}{857108044800} & \frac{29}{50164531200} & \frac{29}{26784626400} & \frac{7625}{4388393189376} & \frac{1}{391910400} \\ \frac{6031}{9142485811200} & \frac{233}{142851340800} & \frac{1}{391910400} & \frac{31}{8928208800} & \frac{1625}{365699432448} & \frac{1}{195955200} & \frac{275}{40633270272} \\ \frac{1}{198404640} & \frac{1}{167215104} & \frac{1}{198404640} & \frac{275}{40633270272} & \frac{-1}{1567641600} \end{bmatrix}^T$$

Fig 5: Error constants of BNM (5,p)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{72} & \frac{1}{18} & \frac{1}{8} & \frac{2}{9} & \frac{25}{72} & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{1296} & \frac{1}{62} & \frac{1}{48} & \frac{4}{81} & \frac{125}{1296} & \frac{1}{6} & \frac{1}{72} & \frac{1}{18} & \frac{1}{8} & \frac{2}{9} & \frac{25}{72} & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

Fig 6: The 6th, 12th, 18th and 24th columns of $\overline{A}^{(1)}$

respectively while other entries are zero and $\overline{\mathbf{A}}^{(0)}$ is 24 by 24 identity matrix.
Substituting $\overline{\mathbf{A}}^{(0)}$ and $\overline{\mathbf{A}}^{(1)}$ in $\rho(R) = \det(R\overline{\mathbf{A}}^{(0)} - \overline{\mathbf{A}}^{(1)})$, we obtain $\rho(R) = R^{20}\ (R^4 - 1)$.

According to Fatunla ([14], [15]), the methods are zero-stable since $\rho(R) = 0$ satisfies $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed four.

5 Implementation of BNM

We implement the BNM using a written code in Mathematica 10.0 enhanced by the features *NSolve*[ ] for linear problems and *FindRoot*[ ] for nonlinear problems respectively. In what follows, we summarize how BNM (η, p) is applied to solve initial value problems (IVPs) in a block-by-block fashion as well as applied to solve boundary value problems (BVPs) via a block unification technique.

5.1 IVPs-Block-by-block algorithm

- Step 1: Choose $N, h = (x_N - x_0)/N$, on the partition Q_N.
- Step 2: Using (6), $n = 0, \mu = 1$, for BNM $(3,5)$, solve for the values of
  $$[y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_1, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_1, y''_{\frac{1}{4}}, y''_{\frac{1}{2}}, y''_{\frac{3}{4}}, y''_1, y'''_{\frac{1}{4}}, y'''_{\frac{1}{2}}, y'''_{\frac{3}{4}}, y'''_1]^T$$
  simultaneously on the sub-interval $[x_0, x_1]$, as y_0, y'_0, y''_0 and y'''_0 are known from the IVP (1).
- Step 3: Next, for $n = 1, \mu = 2$ the values of
  $$[y_{\frac{5}{4}}, y_{\frac{3}{2}}, y_{\frac{7}{4}}, y_1, y'_{\frac{5}{4}}, y'_{\frac{3}{2}}, y'_{\frac{7}{4}}, y'_2, y''_{\frac{5}{4}}, y''_{\frac{3}{2}}, y''_{\frac{7}{4}}, y''_2, y'''_{\frac{5}{4}}, y'''_{\frac{3}{2}}, y'''_{\frac{7}{4}}, y'''_2]^T$$
  are simultaneously obtained over the sub-interval $[x_1, x_2]$, as y_1, y'_1, y''_1 and y'''_1 are known from the previous block.
- Step 4: The process is continued for $n = 2, \ldots, N-1$ and $\mu = 3, \ldots, N$ to obtain the numerical solution to (1) on the sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$.
  The procedure is the same for BNM $(5,7)$.

5.2 BVPs-Block unification algorithm

- Step 1: Choose $N, h = (x_N - x_0)/N$, on the partition Q_N.
- Step 2: Using (6), $n = 0, \mu = 1$, for BNM $(3,5)$, generate the variables
  $[y_i, y'_i, y''_i, y'''_i]^T$, i$=\frac{1}{4}(\frac{1}{4})1$
  on the interval $[x_0, x_1]$ and do not solve yet.
- Step 3: Next, for $n = 1, \mu = 2$ generate the variables
  $$[y_{\frac{5}{4}}, y_{\frac{3}{2}}, y_{\frac{7}{4}}, y_1, y'_{\frac{5}{4}}, y'_{\frac{3}{2}}, y'_{\frac{7}{4}}, y'_2, y''_{\frac{5}{4}}, y''_{\frac{3}{2}}, y''_{\frac{7}{4}}, y''_2, y'''_{\frac{5}{4}}, y'''_{\frac{3}{2}}, y'''_{\frac{7}{4}}, y'''_2]^T$$
  on the sub-interval $[x_1, x_2]$, and do not solve yet.
- Step 4: The process is continued for $n = 2, \ldots, N-1$ and $\mu = 3, \ldots, N$ until all the variables on the sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$ are obtained.
- Step 5: Create a single block matrix equation by the unification of all the blocks generated in Step 2 and Step 3 on Q_N.
- Step 6: Solve the single block matrix equation to simultaneously obtain all the solutions for (1) on the entire $[x_0, x_N]$.
  The procedure is the same for BNM $(5,7)$.

6 Numerical Examples

In this section, we give some numerical examples to illustrate the accuracy of the method. We find the absolute error of the approximate solution as $|y - y(x)|$. The rate of convergence (ROC) was calculated using the formula

$ROC = \log_2(E^{2h}/E^h)$, where E^h is the maximum absolute error $Err = Max|y(x) - y|$ using the step size h. We investigate the effectiveness and accuracy of the proposed BNM (3,5) and BNM (5,7) by solving seven test problems.
Six IVPs and three BVPs solved by different existing methods are considered. For each example considered, we find the absolute error $|y(x) - y_n(x)|$ of the approximate solution.

6.1 IVPs

Example 1a: Consider the linear fourth order problem (see Jator [21])) $y^{iv} = y''' + y'' + y' + 2y,\ y(0) = y'(0) = y''(0) = 0,$
$y'''(0) = 30,\ 0 \le t \le 2$
whose theoretical solution is y(t) = $2e^{2t}$ - $5e^{-t}$ + 3cost – 9sint.

This problem was solved by Yap and Ismail [36], Awoyemi [5], Jator [21] adopting block hybrid collocation method (BHCM4), multiderivative collocation method in (Awoyemi), finite difference method (Jator). We solve this problem using our methods, BNM (3,5) and BNM (5,7) for $0 < t < 2$ and compare the absolute error of our result at t = 2 with these existing methods and the Adams Bashforth-Adams Moulton method (Adams) as shown in Table III. BNM (3,5) and BNM (5,7) compare favourably well with these existing methods.

Table III^*: Numerical Results for Example 1a

h	Method	Absolute Error at t=2
0.1	BNM (3,5)	2.96(-8)
	BNM (5,7)	1.85(-13)
	BHCM4	1.74(-8)
	ADAMS	2.11(-3)
	JATOR	1.26(-4)
0.05	BNM (3,5)	4.62(-10)
	BNM (5,7	7.11(-14)
	BHCM4	8.45(-11)
	ADAMS	5.37(-4)
	JATOR	1.91(-6)
0.025	BNM (3,5)	6.27(-12)
	BNM (5,7)	1.14(-12)
	BHCM4	3.69(-13)
	ADAMS	5.09(-5)
	JATOR	2.96(-8)
0.02	BNM (3,5)	1.96(-12)
	BNM (5,7)	1.99(-13)
	BHCM4	7.11(-14)
	ADAMS	2.25(-5)
	JATOR	8.65(-9)

Remark
We note that the ROC of 4.28 in Tables 3-5 is due to truncation error and $***$ indicate an invalid ROC.

Table 1: Results for Example 1a

	BNM (3,5)		BNM (5,7)	
N	Err	ROC	ERr	ROC
5	1.26×10^{-4}		1.35×10^{-7}	
10	1.91×10^{-6}	6.04	5.12×10^{-10}	8.04
20	2.96×10^{-8}	6.01	1.93×10^{-12}	8.05
40	4.62×10^{-10}	6.00	9.95×10^{-14}	4.28
80	6.34×10^{-12}	6.12	7.11×10^{-13}	***

Example 1b: We consider the special fourth order problem (see Kayode [24])
$y^{iv} = x,$
$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0,\ y'''(0) = 0\ \ 0 \le x \le 1$
whose analytical solution is $y(x) = \dfrac{x^2}{120} + x.$
Example 1b was solved by Kayode et al. and Mohammed. The results are compared with the result of B(3,5) which shows its better performance.

Table IIIb: Numerical Results for Example 1b

t	MOHAMMED	KAYODE	BNM (3,5)
0.1	7.00×10^{-10}	1.832×10^{-13}	3.47×10^{-18}
0.2	8.999×10^{-10}	4.835×10^{-12}	1.39×10^{-17}
0.3	2.999×10^{-09}	7.214×10^{-12}	2.78×10^{-17}
0.4	5.100×10^{-09}	6.832×10^{-11}	5.55×10^{-16}
0.5	7.799×10^{-09}	7.416×10^{-11}	0
0.6	1.180×10^{-08}	2.714×10^{-11}	5.55×10^{-17}
0.7	1.240×10^{-08}	2.815×10^{-10}	8.33×10^{-17}
0.8	1.410×10^{-08}	3.412×10^{-10}	1.11×10^{-16}
0.9	1.880×10^{-08}	1.936×10^{-10}	1.11×10^{-16}

Example 1c: We consider homogeneous fourth order problem (see Awoyemi [7])
$y^{iv} = 4y'',$
$y(0) = 1,\ y'(0) = 3,\ y''(0) = 0,\ y'''(0) = 16\ \ 0 \le x \le 1$
whose analytical solution is $y(x) = 1 - x + e^{2x} - e^{-2x}.$
Example 1c was solved by Awoyemi et al. with no comparison of the solution with existing method. Their results are compared with the result of B(5,7).

Table IIIc: Numerical Results for Example 1c

t	AWOYEMI	BNM (5,7)
0.1	0	0
0.2	0	2.22×10^{-16}
0.3	2.22×10^{-16}	4.44×10^{-16}
0.4	2.44×10^{-15}	0
0.5	1.15×10^{-14}	4.44×10^{-16}
0.6	3.31×10^{-14}	8.88×10^{-16}
0.7	7.28×10^{-14}	8.88×10^{-16}
0.8	1.37×10^{-13}	1.78×10^{-15}
0.9	2.31×10^{-13}	3.55×10^{-15}

Example 2: Consider the nonlinear fourth order problem (see Awoyemi [5])
$y^{iv} = (y')^2 - yy''' - 4t^2 + e^t(1 + t^2 - 4t),$
$y(0) = y'(0) = 1, \ \ y''(0) = 3, \ \ y'''(0) = 1 \ \ 0 \leq t \leq 1$
whose analytical solution is $y(t) = t^2 + e^t$.

Table IV^*: Numerical Results for Example 2

h	Method	Absolute Error at t=1
0.1	BNM(3,5)	4.76(-13)
	BNM(5,7)	4.44(-16)
	BHCM4	1.95(-14)
	ADAMS	2.44(-6)
	AWOYEMI	9.26(-5)
0.2	BNM(3,5)	3.06(-11)
	BNM(5,7)	2.22(-15)
	BHCM4	2.38(-12)
	ADAMS	5.01(-7)
	AWOYEMI	5.84(-4)

Table 2: Results for Example 2

	BNM (3, 5)		BNM (5, 7)	
N	Err	ROC	ERr	ROC
5	1.26×10^{-4}		1.35×10^{-7}	
10	1.91×10^{-6}	6.04	5.12×10^{-10}	8.04
20	2.96×10^{-8}	6.01	1.93×10^{-12}	8.05
40	4.62×10^{-10}	6.00	9.95×10^{-14}	4.28
80	6.34×10^{-12}	6.12	7.11×10^{-13}	***

Example 3: This is an application problem from Ship Dynamics which was stated by Wu [35] when a sinusoidal wave of frequency Ω passes along a ship or offshore structure, the resultant fluid actions vary with time t. In a particular case study by Wu etal. [35], the fourth order problem is defined as
$y^{iv} + 3y'' + y(2 + \varepsilon \cos(\Omega t)) = 0,$
$y(0) = 1, \ \ y'(0) = y''(0) = y'''(0) = 0, \ \ t > 0$
where ε= 0 for the existence of the theoretical solution, y(t) = 2 cos t - cos(t$\sqrt{2}$). The theoretical solution is undefined when $\varepsilon \neq 0$ =0 (see Twizell [32]).

Table V^*: Performance comparison for Wu equation with $\varepsilon = 0$

h	Method	Absolute Error at t=15
0.1	BNM(3,5)	3.4(-10)
	BNM(5,7)	0
	BHCM4	2.8(-10)
	ADAMS	8.4(-5)
	CORTELL	3.7(-5)
0.25	BNM(3,5)	8.2(-8)
	BNM(5,7)	1.7(-11)
	BHCM4	5.2(-7)
	ADAMS	4.9(-3)
	TWIZELL	1.9(-4)

Table 3: Results for Example 3

	BNM (3, 5)		BNM (5, 7)	
N	Err	ROC	ERr	ROC
5	1.26×10^{-4}		1.35×10^{-7}	
10	1.91×10^{-6}	6.04	5.12×10^{-10}	8.04
20	2.96×10^{-8}	6.01	1.93×10^{-12}	8.05
40	4.62×10^{-10}	6.00	9.95×10^{-14}	4.28
80	6.34×10^{-12}	6.12	7.11×10^{-13}	***

Tables 4* and 5* show the performance comparison of results between the BNM and the existing Yap and Ismail block hybrid collocation method [36], Adams method, Jator finite difference method [21] and Awoyemi multi-derivative collocation method [5]. The superiority of BNM (5,7) which is of lower order to the order 8 of BHCM4 is established as it more accurate than the existing methods compared with.

Example 4: Consider the linear system (see Hussain

[19])

$y^{iv} = u\,e^{3x}, y(0) = 1, y'(0) = -1, y''(0) = 1, y'''(0) = -1.$
$z^{iv} = 16y\,e^{-x}, z(0) = 1, z'(0) = -2, z''(0) = 4,$
$z'''(0) = -8.$
$w^{iv} = 81z\,e^{-x}, w(0) = 1, w'(0) = -3, w''(0) = 9,$
$w'''(0) = -27.$
$u^{iv} = 256w\,e^{-x}, u(0) = 1, u'(0) = -4, u''(0) = 16,$
$u'''(0) = -64.$

The exact solution is given by

$$y(x) = e^{-x}, z(x) = e^{-2x}, w(x) = e^{-3x}, u(x) = e^{-4x}$$

The problem is integrated in the interval [0, 2].
This example was chosen to show the performance of BNM(3, 5) and BNM(5, 7) on a system. Looking at Table VI, we deduced that BNM(3, 5) and BNM(5, 7) exhibit an order 5 and 7 respectively, since on halving the step size of each method, Err is reduced by a factor 2^5 or 2^7.

Table 4: Results for Example 4

	BNM (3,5)		BNM (5,7)	
N	Err	ROC	ERr	ROC
5	1.21×10^{-3}		4.74×10^{-6}	
10	2.18×10^{-5}	5.80	2.13×10^{-8}	7.80
20	3.53×10^{-7}	5.95	8.64×10^{-11}	7.95
40	5.57×10^{-9}	5.99	3.44×10^{-13}	7.97
80	8.73×10^{-11}	6.00	1.78×10^{-15}	7.60
80	1.39×10^{-12}	5.97	6.57×10^{-15}	***

6.2 BVPs

Example 5: We consider the following nonlinear boundary value problem in [0, 1], (see [30], [31], [12]).

$$\begin{cases} y^{(iv)}(x) = (y(x))^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 \\ +120x - 48 \\ y(0) = 0, y'(0) = 0 \\ y(1) = 1, y'(1) = 1 \end{cases}$$

with exact solution y(x) $=x^5 - 2x^4 + 2x^2$.
This problem was solved by Noor and Mohyud-Din ([30] [31]) using variational method (NMD method) of approximating polynomial of degree 14 and Costabile and Napoli [12] (HBVP method) with polynomial of degree 6. We compare the result of our method, BNM (3,5) with their results as shown in Table VII.
It is obvious from the numerical results in Table VII that the method is more accurate.

Table VII: Numerical Results for Example 5

t	NMD	HBVP	BNM (3,5)
0.1	4.57×10^{-16}	7.35×10^{-16}	6.94×10^{-18}
0.2	1.59×10^{-15}	2.34×10^{-15}	1.39×10^{-17}
0.3	3.16×10^{-15}	4.11×10^{-15}	2.78×10^{-17}
0.4	4.77×10^{-15}	5.83×10^{-15}	1.11×10^{-16}
0.5	6.05×10^{-15}	5.99×10^{-15}	1.11×10^{-16}
0.6	6.66×10^{-15}	5.55×10^{-15}	1.11×10^{-16}
0.7	6.66×10^{-15}	5.21×10^{-15}	1.11×10^{-16}
0.8	5.22×10^{-15}	3.10×10^{-15}	1.11×10^{-16}
0.9	2.55×10^{-15}	5.55×10^{-16}	1.11×10^{-16}

Example 6 (see [30], [12])

$$\begin{cases} y^{(iv)}(t) = y(t) + y''(t) + e^t(t-3), t \in [0,1] \\ y(0) = 1, y'(0) = 0 \\ y(1) = 0, y'(1) = -e \end{cases}$$

Exact solution is y(t) = (1-t)e^t.
Table VIII compares the results of NMD, HBVP and BNM (3,5) methods. Its second and third columns show respectively the error in the NMD and HBVP methods each of polynomial of degree 15 while the last column contains the error in the BNM (3,5) of degree 8.

Table VIII: Numerical Results for Example 6

t	NMD	$HBVP$ (degree 15)	BNM(3,5)
0.1	2.00×10^{-10}	1.97×10^{-13}	1.23×10^{-14}
0.2	7.00×10^{-10}	1.56×10^{-13}	4.03×10^{-14}
0.3	1.35×10^{-9}	1.83×10^{-13}	7.13×10^{-14}
0.4	2.00×10^{-9}	2.06×10^{-13}	9.53×10^{-14}
0.5	2.51×10^{-9}	2.02×10^{-13}	1.06×10^{-13}
0.6	2.72×10^{-9}	2.25×10^{-13}	1.00×10^{-13}
0.7	2.21×10^{-9}	2.04×10^{-13}	7.88×10^{-14}
0.8	1.80×10^{-9}	1.98×10^{-13}	4.71×10^{-14}
0.9	7.25×10^{-10}	2.18×10^{-13}	1.56×10^{-14}

Example 7 (see [30], [12])

$$\begin{cases} y^{(iv)}(t) = \sin t + \sin^2 t - (y''(t))^2, t \in [0,1] \\ y(0) = 0, y'(0) = 1 \\ y(1) = \sin(1), y'(1) = \cos(1) \end{cases}$$

with exact solution y(t) = sin(t)
It was solved by Noor and Mohyud-Din (see [30], [31]) and Costabile and Napoli [12] taking h = 0.1 by using NMD and HBVP methods of approximating polynomials of degrees 11 and 8 respectively. We also solved for the same step size with our method, BNM (3,5) and the absolute errors at different points are shown in Table IX. The superiority of BNM (3,5) is established numerically.

Table IX: Numerical Results for Example 7

t	NMD	HBVP	BNM (3,5)
0.1	7.78×10^{-8}	4.45×10^{-10}	3.75×10^{-16}
0.2	2.72×10^{-7}	5.54×10^{-10}	1.17×10^{-15}
0.3	5.24×10^{-7}	8.95×10^{-11}	2.11×10^{-15}
0.4	7.77×10^{-7}	2.03×10^{-10}	2.78×10^{-15}
0.5	9.71×10^{-7}	3.32×10^{-11}	3.11×10^{-15}
0.6	1.05×10^{-6}	1.53×10^{-10}	3.00×10^{-15}
0.7	9.63×10^{-7}	9.48×10^{-11}	2.33×10^{-15}
0.8	6.84×10^{-7}	5.18×10^{-10}	1.33×10^{-15}
0.9	2.71×10^{-7}	4.15×10^{-10}	4.44×10^{-16}

7 Conclusion

A family of Nyström type methods BNM $(3,5)$ and BNM $(5,7)$ have been presented and implemented in a block-by-block manner to solve fourth order IVPs and BVPs. It has been shown via the numerical examples given in the Section 6 that the methods are accurate and competitive with those given in the literature. Our future research will be focused on extending these methods to solve partial differential equations via the method of lines.

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