Estimation for Constantinides-Ingersol Model with Small Lévy Noises from Discrete Observations

Chao Wei, Xinru Lian and Feimeng Yuan

Abstract—**This paper is concerned with the parameter estimation problem for Constantinides-Ingersol model with small Lévy noises from discrete observations. The least squares method is used to obtain the parameter estimators and the explicit formula of the estimation error is given. The consistency of the estimators are derived when a small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. The simulation is made to verify the effectiveness of the least squares estimators.**

Index Terms—**Least squares estimator, Lévy noises, discrete observations, consistency.**

I. Introduction

Itô stochastic differential equations are important tools for studying random phenomena and are widely used in the modeling of stochastic phenomena in the fields of physics, chemistry, medicine and finance [2], [3], [5], [6], [13], [18]. However, part or all of the parameters in stochastic model are always unknown. In the past few decades, some popular methods have been put forward to estimate the parameters in Itô stochastic differential equations, such as maximum likelihood estimation [1], [20], [21], least squares estimation [4], [17], [19] and Bayes estimation [8]–[10], [12]. But, in fact, non-Gaussian noise can more accurately reflect the practical random perturbation. Lévy noise, as a kind of important non-Gaussian noise, has attracted wide attention in the research and practice in the fields of engineering, economy and society. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for stochastic differential equations with small Lévy noises. Recently, a number of literatures have been devoted to the parameter estimation for the models driven by small Lévy noises. When the coefficient of the Lévy jump term is constant, drift parameter estimation has been investigated by some authors [14], [15].

The Constantinides-Ingersol model([7]), which was introduced in 1992, is a nonlinear economic model introduced to exam the value of the timing option regarding the realization of capital gains and losses on bondsand analyze the effect of capital gains tax on their pricing. It is known that parameter estimation for Constantinides-Ingersol model driven by Brownian motion has been well developed based on discrete observations([22]). However, some features of the financial processes cannot be captured by Constantinides-Ingersol model, for example, discontinuous sample paths and heavy tailed properties. Therefore, it is natural to replace the Brownian motion by the Lévy process. However, there has few literatures about the parameter estimation for Constantinides-Ingersol model driven by Lévy process.

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Chao Wei, Xinru Lian and Feimeng Yuan are with the School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China.(Email: chaowei0806@aliyun.com.)

In this paper, we consider the parameter estimation problem for Constantinides-Ingersol model with small Lévy noises from discrete observations. The decomposition of the Lévy process is different from that in ([11], [16]), so the methods used to prove the asymptotic property of the estimators are different. The process is discreted based on Euler-Maruyama scheme, the least squares method is used to obtain the explicit formula of the estimator and the estimation error is given as well. when the small dispersion coefficient $\varepsilon \to 0$ and $n \to \infty$ simultaneously, the consistency of the least squares estimator is proved by applying the Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence. Finally, the simulation result is provided to verify the effectiveness of the obtained estimator.

This paper is organized as follows. In Section 2, the Constantinides-Ingersol model driven by small Lévy noises is introduced, the contrast function is given and the explicit formula of the least squares estimator is obtained. In Section 3, the estimation error is derived and the consistency of the estimator is proved. In Section 4, the results are extended to semi-martingale noises. In Section 5, some simulation results are made. The conclusion is given in Section 6.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $(\{\mathscr{F}_t\}_{t\geq 0})$. Let $(L_t, t \geq 0)$ be an $(\{\mathscr{F}_t\})$-adapted Lévy noises with decomposition

$$L_t = B_t + \int_0^t \int_{|z|>1} zN(ds, dz) + \int_0^t \int_{|z|\leq 1} z\widetilde{N}(ds, dz), \tag{1}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion, $N(ds, dz)$ is a Poisson random measure independent of $(B_t, t \geq 0)$ with characteristic measure $dt\nu(dz)$, and $\widetilde{N}(ds, dz) = N(ds, dz) - \nu(dz)$ is a martingale measure. We assume that $\nu(dz)$ is a Lévy measure on $\mathbb{R}\backslash 0$ satisfying $\int(|z|^2 \wedge 1)\nu(dz) < \infty$.

In this paper, we study the parameter estimation for Constantinides-Ingersol model with small Lévy noises described by the following stochastic differential equation:

$$\begin{cases} dX_t = \alpha X_t^2 dt + \varepsilon X_t^{\frac{3}{2}} dL_t, & t \in [0, 1] \\ X_0 = x_0, \end{cases} \tag{2}$$

where α is an unknown parameter. Without loss of generality, it is assumed that $\varepsilon \in (0, 1]$.

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - \alpha X_{t_{i-1}}^2 \Delta t_{i-1}|^2}{\varepsilon^2 X_{t_{i-1}}^3 \Delta t_{i-1}}, \tag{3}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the estimator

$$\widehat{\alpha}_{n,\varepsilon} = \frac{\sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}}{\Delta t_{i-1} \sum_{i=1}^{n} X_{t_{i-1}}}. \quad (4)$$

Before giving the main results, we introduce some assumptions below.

Let $X^0 = (X_t^0, t \geq 0)$ be the solution to the underlying ordinary differential equation under the true value of the parameter:

$$dX_t^0 = \alpha_0 (X_t^0)^2 dt, \quad X_0^0 = x_0.$$

Assumption 1: α_0 is positive true valve of the parameter α.

Assumption 2: $\inf_{0 \leq t \leq 1} \{X_t\} > 0$, $\sup_{0 \leq t \leq 1} \{X_t\} \leq K < \infty$.

In the next sections, the consistency of the least squares estimators are derived and the simulation is made to verify the effectiveness of the estimators.

III. Main Result and Proofs

In the following theorem, the consistency in probability of the least squares estimators is proved by using Cauchy-Schwarz inequality, Gronwall's inequality, Markov inequality and dominated convergence.

Theorem 1: The least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ is consistent in probability, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0.$$

Proof: By using the Euler-Maruyama scheme, from (2), we have

$$X_{t_i} - X_{t_{i-1}} = \alpha (X_{t_{i-1}})^2 \Delta t_{i-1} + \varepsilon (X_{t_{i-1}})^{\frac{3}{2}} (L_{t_i} - L_{t_{i-1}}). \quad (5)$$

Then, it is easy to see that

$$\begin{aligned} & \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} \quad (6) \\ = \; & \alpha \frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}} + \varepsilon \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}). \end{aligned}$$

Substituting (6) into the expression of $\widehat{\alpha}_{n,\varepsilon}$, it follows that

$$\widehat{\alpha}_{n,\varepsilon} - \alpha = \frac{\varepsilon \sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}})}{\frac{1}{n} \sum_{i=1}^{n} X_{t_{i-1}}}. \quad (7)$$

Let $M_t^{n,\varepsilon} = X_{[nt]/n}$, in which $[nt]$ denotes the integer part of nt. We will prove that the sequence $\{M_t^{n,\varepsilon}\}$ converges to the deterministic process $\{X_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Observe that

$$X_t - X_t^0 = \alpha_0 \int_0^t ((X_s)^2 - (X_s^0)^2) ds + \varepsilon \int_0^t (X_s)^{\frac{3}{2}} dL_s. \quad (8)$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & |X_t - X_t^0|^2 \\ \leq \; & 2(\alpha_0)^2 |\int_0^t ((X_s)^2 - (X_s^0)^2) ds|^2 \\ + \; & 2\varepsilon^2 |\int_0^t (X_s)^{\frac{3}{2}} dL_s|^2 \\ \leq \; & 2t(\alpha_0)^2 \int_0^t |(X_s)^2 - (X_s^0)^2|^2 ds \\ + \; & 2\varepsilon^2 |\int_0^t (X_s)^{\frac{3}{2}} dL_s|^2 \\ = \; & 2t(\alpha_0)^2 \int_0^t |X_s - X_s^0|^2 |X_s + X_s^0|^2 ds \\ + \; & 2\varepsilon^2 |\int_0^t (X_s)^{\frac{3}{2}} dL_s|^2 \\ \leq \; & 8K^4 t(\alpha_0)^2 \int_0^t |X_s - X_s^0|^2 ds \\ + \; & 2\varepsilon^2 |\int_0^t (X_s)^{\frac{3}{2}} dL_s|^2 \end{aligned}$$

.

According to the Gronwall's inequality, we obtain

$$|X_t - X_t^0|^2 \leq 2\varepsilon^2 e^{8K^4 t^2 (\alpha_0)^2} |\int_0^t (X_s)^{\frac{3}{2}} dL_s|^2. \quad (9)$$

Then, it follows that

$$\begin{aligned} & \sup_{0 \leq t \leq T} |X_t - X_t^0| \quad (10) \\ \leq \; & \sqrt{2} \varepsilon e^{4K^4 T^2 (\alpha_0)^2} \sup_{0 \leq t \leq T} |\int_0^t (X_s)^{\frac{3}{2}} dL_s|. \end{aligned}$$

Therefore, for each $T > 0$, it is easy to check that

$$\sup_{0 \leq t \leq T} |X_t - X_t^0| \xrightarrow{P} 0. \quad (11)$$

As $[nt]/n \to t$ when $n \to \infty$, we get that the sequence $\{M_t^{n,\varepsilon}\}$ converges to the deterministic process $\{X_t^0\}$ uniformly in probability as $\varepsilon \to 0$ and $n \to \infty$.

Next we will prove that

$$\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \sqrt{X_s^0} dL_s.$$

Note that

$$\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}) = \int_0^1 \sqrt{M_s^{n,\varepsilon}} dL_s. \quad (12)$$

Then, it is elementary to see that

$$\begin{aligned} & |\int_0^1 \sqrt{M_s^{n,\varepsilon}} dL_s - \int_0^1 \sqrt{X_s^0} dL_s| \\ = & |\int_0^1 (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) dB_s \\ + & \int_0^1 \int_{|z|>1} (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) z N(ds,dz) \\ + & \int_0^1 \int_{|z|\le 1} (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) z \widetilde{N}(ds,dz)| \\ \le & |\int_0^1 (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) dB_s| \\ + & |\int_0^1 \int_{|z|>1} (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) z N(ds,dz)| \\ + & |\int_0^1 \int_{|z|\le 1} (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) z \widetilde{N}(ds,dz)|. \end{aligned}$$

It is easy to check that

$$\begin{aligned} & |\int_0^1 \int_{|z|>1} (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) z N(ds,dz)| \\ \le & \int_0^1 \int_{|z|>1} |\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}| |z| N(ds,dz) \\ \le & \sup_{0\le s\le 1} |\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}| \int_0^1 \int_{|z|>1} |z| N(ds,dz) \\ \xrightarrow{P} & \ 0, \end{aligned}$$

as $\varepsilon \to 0$ and $n \to \infty$.

By using the Markov inequality and dominated convergence, we have

$$|\int_0^1 (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) dB_s| \xrightarrow{P} 0, \tag{13}$$

and

$$|\int_0^1 \int_{|z|\le 1} (\sqrt{M_s^{n,\varepsilon}} - \sqrt{X_s^0}) z \widetilde{N}(ds,dz)| \xrightarrow{P} 0. \tag{14}$$

Thus, combining the previous results, it follows that

$$\sum_{i=1}^n \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \sqrt{X_s^0} dL_s. \tag{15}$$

Let

$$X_N = \inf_{0\le t_{i-1}\le 1} \{X_{t_{i-1}}\}. \tag{16}$$

From (16), we obtain

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \ge X_N.$$

Then, we get

$$\frac{1}{\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}} \le \frac{1}{X_N}.$$

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, we have

$$\varepsilon\sigma \sum_{i=1}^n \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} 0, \tag{17}$$

and

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0. \tag{18}$$

The proof is complete. ■

Theorem 2: When $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{P} \frac{\int_0^1 \sqrt{X_s^0} dL_s}{\int_0^1 X_s^0 ds}.$$

Proof: Since

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) = \frac{\sum_{i=1}^n \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}.$$

According to the results in Section 3, it is easy to check that

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_s^0 ds.$$

Since

$$\sum_{i=1}^n \sqrt{X_{t_{i-1}}} (L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \sqrt{X_s^0} dL_s.$$

We obtain that

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{P} \frac{\int_0^1 \sqrt{X_s^0} dL_s}{\int_0^1 X_s^0 ds}. \tag{19}$$

The proof is complete. ■

IV. Generalization to semi-martingale noises

In this section, we discuss the extension of our main results in Section 3 to the general case when the driving noise is a semi-martingale. Let $Q_t = Q_0 + M_t + A_t$ be a semi-martingale, where M_t is a local martingale and A_t is a finite variation process. Then, we can replace the driving Lévy process L_t by the semi-martingale Q_t to get

$$\begin{cases} dX_t = \alpha X_t^2 dt + \varepsilon X_t^{\frac{3}{2}} dQ_t, & t \in [0,1] \\ X_0 = x_0, \end{cases}$$

where α is an unknown parameter. Without loss of generality, it is assumed that $\varepsilon \in (0,1]$.

All the related information about the least squares estimator of α discussed in this section is same to Section 2. We are interested in the consistency and asymptotic behavior of the least squares estimator of α.

Now we state the new results as follows.

Theorem 3: Under Assumptions $1-2$, $\varepsilon \to 0$ and $n \to \infty$, the least squares estimators $\widehat{\alpha}_{n,\varepsilon}$ is consistent, namely

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0.$$

Proof: According to the results in Section 3, it is easy to get the error of estimation

$$\widehat{\alpha}_{n,\varepsilon} - \alpha_0 = \frac{\varepsilon \sum_{i=1}^n \sqrt{X_{t_{i-1}}} (Q_{t_i} - Q_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}}.$$

By applying the same methods, it can be checked that

$$\sum_{i=1}^n \sqrt{X_{t_{i-1}}} (Q_{t_i} - Q_{t_{i-1}}) \xrightarrow{P} \int_0^1 \sqrt{X_s^0} dQ_s.$$

Since

$$\frac{1}{n} \sum_{i=1}^n X_{t_{i-1}} \ge X_N.$$

Then, we get

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}} \leq \frac{1}{X_N} < \infty.$$

Therefore, when $\varepsilon \to 0$ and $n \to \infty$, we obtain

$$\widehat{\alpha}_{n,\varepsilon} \xrightarrow{P} \alpha_0.$$

The proof is complete. ■

Theorem 4: When $\varepsilon \to 0$ and $n \to \infty$,

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{P} \frac{\int_0^1 \sqrt{X_s^0} dQ_s}{\int_0^1 X_s^0 ds}.$$

Proof: Since

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) = \frac{\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(Q_{t_i} - Q_{t_{i-1}})}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}.$$

According to the results in Section 3, it is easy to check that

$$\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_s^0 ds.$$

Since

$$\sum_{i=1}^{n} \sqrt{X_{t_{i-1}}}(L_{t_i} - L_{t_{i-1}}) \xrightarrow{P} \int_0^1 \sqrt{X_s^0} dQ_s.$$

We obtain that

$$\varepsilon^{-1}(\widehat{\alpha}_{n,\varepsilon} - \alpha_0) \xrightarrow{P} \frac{\int_0^1 \sqrt{X_s^0} dQ_s}{\int_0^1 X_s^0 ds}. \tag{20}$$

The proof is complete. ■

Remark 1: If Constantinides-Ingersol model is driven by small α-stable noises as follows

$$\begin{cases} dX_t = \alpha X_t^2 dt + \varepsilon X_t^{\frac{3}{2}} dZ_t, & t \in [0,1] \\ X_0 = x_0, \end{cases}$$

where α is an unknown parameter, $\varepsilon \in (0,1]$ and $Z = \{Z_t, t \geq 0\}$ is a strictly symmetric α-stable Lévy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0,2]$, scale parameter $\sigma \in (0,\infty)$, skewness parameter $\beta \in [-1,1]$ and location parameter $\mu \in (-\infty,\infty)$ if it has the following characteristic function:

$$\phi_\eta(u) = \begin{cases} \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta sgn(u) \tan\frac{\alpha\pi}{2}) + i\mu u\} \\ \qquad if \alpha \neq 1, \\ \exp\{-\sigma|u|(1 + i\beta\frac{2}{\pi} sgn(u) \log|u|) + i\mu u\} \\ \qquad if \alpha = 1. \end{cases}$$

Consider the following contrast function

$$\rho_{n,\varepsilon}(\alpha) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - \alpha X_{t_{i-1}}^2 \Delta t_{i-1}|^2}{X_{t_{i-1}}^3},$$

where $\Delta t_{i-1} = t_i - t_{i-1} = \frac{1}{n}$.

It is easy to obtain the estimator

$$\widehat{\alpha}_{n,\varepsilon} = \frac{\sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}.$$

Note that

$$X_{t_i} - X_{t_{i-1}} = \alpha \int_{t_{i-1}}^{t_i} X_s^2 ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s^{\frac{3}{2}} dZ_s.$$

Then, we can give a more explicit decomposition for $\widehat{\alpha}_{n,\varepsilon}$ as follows

$$\widehat{\alpha}_{n,\varepsilon} = \frac{\alpha \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{X_s^2}{X_{t_{i-1}}} ds}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}} + \frac{\varepsilon \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{X_s^{\frac{3}{2}}}{X_{t_{i-1}}} dZ_s}{\frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}}$$

This equation is different from Equation (7). Therefore, the methods to prove the consistency of $\widehat{\alpha}_{n,\varepsilon}$ is different as well and it is more difficult.

V. Simulation

In this experiment, we generate a discrete sample $(X_{t_i})_{i=0,1,\ldots,n}$ and compute $\widehat{\alpha}_{n,\varepsilon}$ from the sample. We let $x_0 = 0.1$. For every given true value of the parameters-α_0, the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, $\varepsilon = 0.05$, the size is increasing from 500 to 3000. In Table 2, $\varepsilon = 0.001$, the size is increasing from 5000 to 30000. The tables list the value of "$\alpha_0 - LSE$" and the absolute errors (AE) of LSE, LSE means least squares estimator.

Two tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. Therefore, the methods used in this paper are effective and the obtained estimators are good.

TABLE I
LSE simulation results of α_0

True	Aver		AE
(α_0)	Size n	$\alpha_0 - LSE$	α_0
1	500	0.9652	0.0348
	1000	0.9736	0.0264
	3000	0.9814	0.0186
2.5	500	2.4663	0.0337
	1000	2.4782	0.0218
	3000	2.4875	0.0125
3.5	500	3.4653	0.0347
	1000	3.4791	0.0209
	3000	3.4856	0.0144

Next we give some simulation results of the confidence interval of α_0 under 0.95 confidence level. In Table 3, We

TABLE II
LSE SIMULATION RESULTS OF α_0

True	Aver		AE
(α_0)	Size n	$\alpha_0 - LSE$	α_0
1	5000	0.9752	0.0248
	10000	0.9814	0.0186
	30000	0.9935	0.0065
2.5	5000	2.4768	0.0232
	10000	2.4821	0.0179
	30000	2.4963	0.0037
3.5	5000	3.4763	0.0237
	10000	3.4882	0.0118
	30000	3.4971	0.0029

let $\sigma = 0.5$, $x_0 = 0.1$. In Table 4, We let $\sigma = 0.1$, $x_0 = 0.5$. For every given true value of α_0, let $\varepsilon = 0.01$, the size of the sample is increasing from 2000 to 10000. Table 3 and Table 4 list the value of $\alpha_0 - LSE$ and in the last column of the table list the confidence interval of α_0. Table 3 and Table 4 illustrate that the length of the confidence interval is becoming small when the size of the sample is increasing.

VI. Conclusion

In this paper, the parameter estimation problem for Constantinides-Ingersol model with small Lévy noises has been studied from discrete observations. The least squares method has been used to obtain the estimator. The explicit formula of the estimation error has been given and the consistency of the least squares estimators has been proved. The results have been extended to the semi-martingale noises as well. However, due to the complexity of Lévy process, it is difficult to obtain the explicit expression of estimator and estimation error for diffusion parameter in Constantinides-Ingersol model. Therefore, further research topics will include the diffusion parameter estimation for Constantinides-Ingersol model with small Lévy noises and general nonlinear stochastic differential equations driven by lévy noises.

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TABLE III
SIMULATION RESULTS OF CONFIDENCE INTERVAL OF α_0

True	Aver		0.95
α_0	Size n	$\alpha_0 - LSE$	confidence interval
1	2000	0.9963	[0.9825,1.2246]
	5000	1.0018	[0.9917,1.1148]
	10000	1.0007	[0.9956,1.1032]
2	2000	1.9951	[1.9792,2.3423]
	5000	2.0027	[1.9845,2.2247]
	10000	2.0011	[1.9983,2.1162]
3	2000	2.9954	[2.9763,3.3592]
	5000	3.0028	[2.9894,3.2334]
	10000	3.0015	[2.9932,3.1221]

TABLE IV
SIMULATION RESULTS OF CONFIDENCE INTERVAL OF α_0

True	Aver		0.95
α_0	Size n	$\alpha_0 - LSE$	confidence interval
2	2000	2.0032	[1.9875,2.2354]
	5000	2.0019	[1.9986,2.1241]
	10000	2.0008	[2.0001,2.1032]
3	2000	3.0043	[2.9721,3.3246]
	5000	3.0025	[2.9875,3.2314]
	10000	3.0009	[2.9986,3.1105]
4	2000	4.0035	[3.9749,4.3368]
	5000	4.0021	[3.9875,4.2389]
	10000	4.0010	[3.9983,4.1568]

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