Darboux Transforms and 2-Orthogonal Polynomials

Chadia Faghmous, Mohammed Cherif Bouras and Karima Ali Khelil

***Abstract*—The purpose of this paper is to present a new interpretation of Darboux transforms in the context of $2-$ orthogonal polynomials and find conditions in order for any Darboux transforms to yield a new set of $2-$ orthogonal polynomials. We also introduce the LU and UL factorizations of the monic Jacobi matrix J associated with a quasi-definite linear functional Γ defined on the linear space of polynomials with real coefficients, as well as the Darboux transforms without parameters.**

***Index Terms*—$2-$ orthogonal polynomials, linear functional, Jacobi matrix, Darboux transforms.**

I. Introduction

In [7], [8], [9], [11] Græunbaum and al introduced the Darboux transform to extend the classical Bochner [1], Krall [12] and Littlejohn [13] results to the bispectral problem of all *Bochner*-Krall orthogonal polynomials.

In particular, Græunbaum and Heine [4] have shown how we can obtain "Krall polynomials" by one or two Darboux transforms applied to classical orthogonal polynomials.

In 2002, Gang Joon Yoon [6] first found the conditions under which a transformation of Darboux gives a new orthogonal polynomials sequence (shortly indicated as OPS), as he also gave an interpretation of the transformation of Darboux by functional moments, which shows very clearly the role of Darboux's transformation in the context of orthogonal polynomials.

In 2004, M. I. Bueno, F. Marcellán [3], introduced the LU and UL factorizations of a tridiagonal matrix J, as well as the transformation of Darboux and the Darboux transform without parameters. They also show how to find the tridiagonal matrix $\widehat{J}_1$ associated with the linear functional $\Gamma_1 = x\ \Gamma$ in terms of the matrix J by the application of the Darboux transform without parameters.

The main purpose of this paper is to present a new interpretation of Darboux transforms in the context of $2-$ orthogonal polynomials.

The paper essentially consists of two sections. Following the introduce necessary for the sequel. In Section 2, we introduce the LU and UL factorizations of a matrix J, as well as the Darboux transforms without parameters.

Let us now recall some results which we will be used in the sequel. Let $\{B_n\}_{n\geq 0}$ be a sequence of monic polynomials and $\{\Gamma_n\}_{n\geq 1}$ be its dual sequence defined by $\langle \Gamma_n, B_m\rangle = \delta_{n,m},\ n,m \geq 0$, where $\langle\ ,\rangle$, is the duality brackets between $\mathcal{P}$ (the vector space of polynomials with coefficients in $\mathbb{C}$) and its dual $\mathcal{P}'$.

the authors are with Laboratory LANOS, Mathematics Departement, Faculty of Sciences, Badji-Mokhtar University, BP 12, Annaba, Algeria
chfaghmous21@gmail.com
bourascdz@yahoo.fr
kalikhelil@gmail.com

Definition 1: A sequence $\{B_n\}_{n\geq 0}$ is said to be $d-$orthogonal polynomial sequence (shortly indicated as $d-OPS$) $(d \geq 1)$with respect to $d-$dimensional functional $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_d)^T$ if it fulfills the following conditions [14] :

$$\begin{cases} \langle \Gamma_\alpha, x^m B_n(x)\rangle = 0, & n \geq md+\alpha, \quad m \geq 0, \\ \langle \Gamma_\alpha, x^m B_{md+\alpha-1}(x)\rangle \neq 0, & m \geq 0 \end{cases}$$

for each integer α with $1 \leq \alpha \leq d$ and $m \geq 0$. The functionals $\Gamma_1, \Gamma_2, ..., \Gamma_d$ are the d first elements of dual sequence $\{\Gamma_n\}_{n\geq 1}$ associated to the sequence of polynomials $\{B_n\}_{n\geq 0}$.

A remarkable characterization of the $d-$orthogonal polynomials is that they satisfy a standared $(d+1)-$order recurrence, $d \geq 1$, who is written in the form [14],

$$B_{m+d+1}(x) = (x-\beta_{m+d})B_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} B_{m+d-1-\nu}(x),$$
$$m \geq 0,$$

with the initial conditions

$$\begin{cases} B_0(x) = 1, \\ B_1(x) = x-\beta_0, \\ B_n(x) = (x-\beta_{n-1})B_{n-1}(x) - \displaystyle\sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} B_{n-2-\nu}(x), \\ 2 \leq n \leq d,\ d \geq 2 \end{cases}$$

where $\gamma_{m+1}^0 \neq 0, \quad m \geq 0$ (regularity conditions).

Here, we only consider the case $d = 2$, that is, $\{B_n\}_{n\geq 0}$ is $2-$monic orthogonal polynomials sequence (shortly indicated as $2-MOPS$) with respect to the linear functionals Γ_1 and Γ_2. In this case, the orthogonality relations are

$$\begin{cases} \langle \Gamma_1, x^m B_n(x)\rangle = 0, & n \geq 2m+1, \quad m \geq 0, \\ \langle \Gamma_1, x^m B_{2m}(x)\rangle \neq 0, & m \geq 0 \end{cases}$$

and

$$\begin{cases} \langle \Gamma_2, x^m B_n(x)\rangle = 0, & n \geq 2m+2, \quad m \geq 0, \\ \langle \Gamma_2, x^m B_{2m+1}(x)\rangle \neq 0, & m \geq 0 \end{cases}$$

Then $\{B_n\}_{n\geq 0}$ satisfies a third-order recurrence relation [14] which we write in the form

$$\begin{cases} B_{n+3}(x) = (x-\beta_{n+2})B_{n+2}(x) - \gamma_{n+2}B_{n+1}(x) - \rho_{n+1}B_n(x), \\ n \geq 0. \\ B_0(x) = 1, \\ B_1(x) = x-\beta_0, \\ B_2(x) = (x-\beta_1)\, B_1(x) - \gamma_1 \end{cases} \tag{1}$$

where $\rho_{n+1} \neq 0,\ n \geq 0$ (regularity conditions).

Thus, we have

$$\begin{aligned} xB_{n+2}(x) = {} & B_{n+3}(x) + \beta_{n+2}B_{n+2}(x) + \gamma_{n+2}B_{n+1}(x) \\ & + \rho_{n+1}B_n(x) \end{aligned} \tag{2}$$

Then we can express (2) as

$$JB = xB$$

where $B := (B_0(x), B_1(x), ...)^T$ and the matrix

$$J = \begin{bmatrix} \beta_0 & 1 & 0 & . & . & . & . \\ \gamma_1 & \beta_1 & 1 & 0 & . & . & . \\ \rho_1 & \gamma_2 & \beta_2 & 1 & . & . & . \\ 0 & \rho_2 & \gamma_3 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & . & . & \\ . & . & . & . & . & & . \end{bmatrix} \quad (3)$$

is the Jacobi matrix of $2-MOPS$ $\{B_n\}_{n\geq 0}$. It is also called the monic Jacobi matrix associated with the functional $\Gamma = (\Gamma_1, \Gamma_2)^T$.

Definition 2: [14] Let $\{B_n\}_{n\geq 0}$ be a $2-MOPS$ satisfying (1).

1) The $2-MOPS$ $\{B_n^{(r)}\}_{n\geq 0}$ defined by

$$\begin{cases} B_{n+3}^{(r)}(x) = (x-\beta_{n+r+2})B_{n+2}^{(r)} - \gamma_{n+r+2}B_{n+1}^{(r)}(x) \\ \qquad -\rho_{n+r+1}B_n^{(r)}(x), \quad n,\ r\geq 0, \\ B_0^{(r)}(x) = 1, \\ B_1^{(r)}(x) = x-\beta_r, \\ B_2^{(r)}(x) = (x-\beta_{r+1})B_1^{(r)}(x) - \gamma_{r+1}, \end{cases}$$

is called the associated sequence of order r of $\{B_n\}_{n\geq 0}$.

2) Given $\mu_0, \mu_1, \mu_1^1 \in \mathbb{C}$, the $2-MOPS$ $\{B_n^*(\mu_0,\mu_1,\mu_1^1;.)\}_{n\geq 0}$ (shortly indicated as $\{B_n^*(\mu;.)\}_{n\geq 0}$) defined by

$$\begin{cases} B_{n+3}^*(\mu;x) = (x-\beta_{n+2})B_{n+2}^*(\mu;x) - \gamma_{n+2}\ B_{n+1}^*(\mu;x) \\ \qquad -\rho_{n+1}B_n^*(\mu;x), \quad n\geq 0, \\ B_0^*(\mu;x) = 1, \\ B_1^*(\mu;x) = x-\beta_0-\mu_0, \\ B_2^*(\mu;x) = (x-\beta_1-\mu_1)B_0^*(\mu;x) - (\gamma_1+\mu_1^1). \end{cases}$$

is called a co-recursive sequence of $\{B_n\}_{n\geq 0}$.

Lemma 1: [14] Let $\{B_n\}_{n\geq 0}$ be a $2-MOPS$, then

$$B_n^*(x;\mu) = B_n(x) - \mu_0 B_{n-1}^{(1)}(x) - \left[\mu_1 x + \mu_1^1 - \mu_1(\beta_0+\mu_0)\right] \times B_{n-2}^{(2)}(x), n\geq 0.$$

Lemma 2: [15] Let $\{B_n\}_{n\geq 0}$, $\{\widehat{B}_n\}_{n\geq 0}$ be two $2-MOPS$ relative to $\Gamma = (\Gamma_1,\Gamma_2)^T$, $\widehat{\Gamma} = \left(\widehat{\Gamma}_1, \widehat{\Gamma}_2\right)^T$ respectively. Let ϕ be a monic polynomial with $deg\ \phi = t$.

Then, the following assertions are equivalent

1) There is a positive integer s such that

$$\phi(x)\widehat{B}_n(x) = \sum_{\nu=n-s}^{n+t} \lambda_{n,\nu} B_\nu(x),\ n\geq s,$$
$$\exists r\geq s:\quad \lambda_{r,r-s}\neq 0.$$

2) There is a polynomial matrix $\mathcal{M}_0$

$$\mathcal{M}_0 = \begin{pmatrix} P_0 & Q_0 \\ P_1 & Q_1 \end{pmatrix}$$

such that

$$\phi\Gamma = X_0\widehat{\Gamma}$$

where the polynomials P_0, Q_0 are not identically null simultaneously.

Let $\{B_n\}_{n\geq 0}$ be a $2-MOPS$ satisfying (1), and let $\left\{\widehat{B}_n\right\}_{n\geq 0}$ be a monic polynomials sequence (shortly indicated as MPS) defined by

$$\widehat{B}_n(x) = B_n(x) + \alpha_n B_{n-1}(x),\quad n\geq 1 \quad (4)$$

where $\{\alpha_n\}_{n\geq 1}$ is a complex sequence.

II. Darboux transforms

In this part, we define the Darboux transform as well as the Darboux transform without parameters, and give some results that will be very useful in the sequel.

In the same manner as in the case of a $MOPS$ [6], we introduce the following transformation on J,

$$J = LU$$

where LU denotes the LU factorization without pivoting of J. So, J has an unique LU factorization if and only if the leading principal submatrices of J are nonsingular.

Otherwise, the transformation

$$J^{(p)} = UL$$

is called Darboux transform without parameters.

Where

$$L = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ l_1 & 1 & 0 & 0 & . & . & . \\ p_2 & l_2 & 1 & 0 & . & . & . \\ 0 & p_3 & l_3 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & . & . & \\ . & . & . & . & . & & . \end{bmatrix}, \quad (5)$$

$$U = \begin{bmatrix} u_1 & 1 & 0 & . & . & . & . \\ 0 & u_2 & 1 & 0 & . & . & . \\ 0 & 0 & u_3 & 1 & . & . & . \\ 0 & 0 & 0 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & . & . & \\ . & . & . & . & . & & . \end{bmatrix} \quad (6)$$

Lemma 3: Let $\{B_n\}_{n\geq 0}$ be a $2-MOPS$ with respect to the linear form $\Gamma = (\Gamma_1,\Gamma_2)^T$ defined by the monic Jacobi matrix J given by (3). Assume that $B_n(0)\neq 0,\ n\geq 1$. If $J = LU$, then

$$u_n = -\frac{B_n(0)}{B_{n-1}(0)},\quad n\geq 1. \quad (7)$$

Moreover,

$$\begin{cases} u_1 = \beta_0 \\ u_n = \beta_{n-1} - l_{n-1},\quad n\geq 2 \end{cases} \quad (8)$$

where

$$\begin{cases} l_1 = \dfrac{\gamma_1}{\beta_0}, & l_n = \dfrac{\gamma_n - p_n}{\beta_{n-1}-l_{n-1}}, \quad n\geq 2, \\ p_2 = \dfrac{\rho_1}{\beta_0}, & p_n = \dfrac{\rho_{n-1}}{\beta_{n-2}-l_{n-2}}, \quad n\geq 3. \end{cases} \quad (9)$$

Proof: In the same manner as in the unidimensional case [6], the product of L times U gives by

$$LU = \begin{bmatrix} u_1 & 1 & 0 & . & . & . \\ l_1u_1 & l_1+u_2 & 1 & 0 & . & . \\ p_2u_1 & p_2+l_2u_2 & l_2+u_3 & 1 & . & . \\ . & . & . & . & & \\ . & . & . & & . & \\ . & . & . & & & . \end{bmatrix}$$

Comparing the J matrix elements with those of the LU matrix, we obtain the result (8) and (9).

By induction on the integer $n \geq 1$, let's show that

$$u_n = -\frac{B_n(0)}{B_{n-1}(0)}.$$

For $n = 1$ and $n = 2$, we will use the third-order recurrence relation (1), we obtain

$$B_1(0) = -\beta_0 = -u_1$$

i.e.,

$$u_1 = -\frac{B_1(0)}{B_0(0)} \tag{10}$$

further

$$B_2(0) = -\beta_1 B_1(0) - \gamma_1 B_0(0),$$

dividing the last equation by $B_1(0)$ and using (10), we obtain

$$u_2 = \beta_1 - l_1 = -\frac{B_2(0)}{B_1(0)}.$$

Suppose that the recurrence property is valid up to the order k $(k \geq 1)$, and let's show that it remains valid to the order $k+1$.

From (1), we have

$$B_{k+1}(0) = -\beta_k B_k(0) - \gamma_k B_{k-1}(0) - \rho_{k-1} B_{k-2}(0),$$

dividing the previous expression by $B_k(0)$, and by induction hypothesis, we get

$$\begin{aligned} \frac{B_{k+1}(0)}{B_k(0)} &= -\beta_k - \gamma_k \frac{B_{k-1}(0)}{B_k(0)} - \rho_{k-1}\frac{B_{k-2}(0)}{B_k(0)} \\ &= -\beta_k + \frac{\gamma_k}{u_k} - \frac{\rho_{k-1}}{u_k u_{k-1}}, \end{aligned}$$

from (9), we obtain

$$\frac{B_{k+1}(0)}{B_k(0)} = -\beta_k + l_k = -u_{k+1},$$

then

$$u_{k+1} = -\frac{B_{k+1}(0)}{B_k(0)}.$$

■

Remark 1: From Lemma 3, the LU factorization of J exists if and only if $B_n(0) \neq 0$. Moreover

$$L = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ l_1 & 1 & 0 & 0 & . & . & . \\ p_2 & l_2 & 1 & 0 & . & . & . \\ 0 & p_3 & l_3 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix},$$

$$U = \begin{bmatrix} \beta_0 & 1 & 0 & . & . & . & . \\ 0 & \beta_1 - l_1 & 1 & 0 & . & . & . \\ 0 & 0 & \beta_2 - l_2 & 1 & . & . & . \\ 0 & 0 & 0 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix}$$

with

$$\begin{cases} l_1 = \dfrac{\gamma_1}{\beta_0}, & l_n = \dfrac{\gamma_n - p_n}{\beta_{n-1} - l_{n-1}}, \quad n \geq 2, \\ p_2 = \dfrac{\rho_1}{\beta_0}, & p_n = \dfrac{\rho_{n-1}}{\beta_{n-2} - l_{n-2}}, \quad n \geq 3. \end{cases}$$

In what follows, we express the monic Jacobi matrix J as a product of an upper triangular matrix times a lower matrix, i.e., $J = UL$, where L and U are given by (5). But this factorization is not unique and depends of free parameters $S_i\,(i = 0, 1, ...)$. We will call such a kind of factorization, as in the unidimensional case [7], UL factorization.

Let $\left\{\widehat{B}_n\right\}_{n\geq 0}$ be a MPS with parameter S_0 defined by (4) and satisfying the following third-order recurrence relation:

$$\begin{cases} \widehat{B}_{n+1}(x) = (x - \widehat{\beta}_n)\widehat{B}_n(x) - \widehat{\gamma}_n \widehat{B}_{n-1}(x) - \widehat{\rho}_{n-1}\widehat{B}_{n-2}(x), \; n \geq 0, \\ \widehat{B}_{-2}(x) = 0, \; \widehat{B}_{-1}(x) = 0, \; \widehat{B}_0(x) = 1, \end{cases} \tag{11}$$

with initial condition, $\widehat{\beta}_0 = \beta_0 - S_0$.

Proposition 1: Let J be a monic Jacobi matrix associated with quasi-definite linear functional $\Gamma = (\Gamma_1, \Gamma_2)^T$ and let $\{B_n\}_{n\geq 0}$ be a $2-MOPS$ with respect to the linear form Γ. Assume that $J = UL$ and S_i denotes the entry in the position $(i+1, i+1)$ of U, i.e. $S_i := u_{i+1}$ for $i \geq 0$, where the element S_0 is a free parameter generated in the factorization (since it is not unique). Let $\left\{\widehat{B}_n\right\}_{n\geq 0}$ be a MPS with parameter S_0 defined by (11).

If $\left\{\widehat{B}_n\right\}_{n\geq 0}$ is the co-recursive sequence with parameter S_0, relative to the linear form $\widehat{\Gamma} = \left(\widehat{\Gamma}_1, \widehat{\Gamma}_2\right)^T$, and $\widehat{B}_n(0) \neq 0$ for all n, then

$$l_n = -\frac{\widehat{B}_n(0)}{\widehat{B}_{n-1}(0)}, \quad n \geq 1. \tag{12}$$

Furthermore

$$l_n = \beta_{n-1} - S_{n-1}, \quad n \geq 1$$

where

$$S_n = \frac{\gamma_n - p_{n+1}}{\beta_{n-1} - S_{n-1}}, \quad n \geq 1.$$

Proof: Using the L and U matrices defined by (5), then gives $J = UL$

$$UL = \begin{bmatrix} l_1 + u_1 & 1 & 0 & . & . & . \\ p_2 + l_1 u_2 & l_2 + u_2 & 1 & . & . & . \\ p_2 u_3 & p_3 + l_2 u_3 & l_3 + u_3 & 1 & . & . \\ 0 & p_3 u_4 & . & . & . & . \\ . & 0 & . & . & . & \\ . & . & . & & & . \end{bmatrix}$$

and by comparing the J matrix elements with those of UL matrix , we obtain

$$l_1 = \beta_0 - u_1.$$

Let us consider u_1 a free parameter, which we denote by S_0. Notice that S_0 can be take any complex value as long as $\widehat{B}_n(0) \neq 0$ for all n. Then, we have

$$l_1 = \beta_0 - S_0$$

Assume that

$$l_k = \beta_{k-1} - S_{k-1}, \quad k \leq n$$

and

$$S_{k-1} := u_k$$

then

$$\beta_k = l_{k+1} + u_{k+1} \Rightarrow l_{k+1} = \beta_k - u_{k+1} = \beta_k - S_k$$

where

$$S_k = \frac{\gamma_k - p_{k+1}}{\beta_{k-1} - S_{k-1}}$$

and

$$p_k = \frac{\rho_{k-1}}{S_k}.$$

By induction on the integer $n \geq 1$, let's show (12) .

For $n = 1$ and $n = 2$, we will use the third-order recurrence relation (11) , we obtain

$$\widehat{B}_1(0) = -\widehat{\beta}_0 \widehat{B}_0(0)$$

$$\Rightarrow \widehat{\beta}_0 = \beta_0 - S_0 = -\frac{\widehat{B}_1(0)}{\widehat{B}_0(0)} = l_1$$

and in the same manner ,

$$\widehat{B}_2(0) = -\widehat{\beta}_1 \widehat{B}_1(0) - \widehat{\gamma}_1 \widehat{B}_0(0),$$

then

$$\frac{\widehat{B}_2(0)}{\widehat{B}_1(0)} = -\widehat{\beta}_1 - \widehat{\gamma}_1 \frac{\widehat{B}_0(0)}{\widehat{B}_1(0)} = -\widehat{\beta}_1 + \frac{\gamma_1}{l_1}$$

Knowing that

$$\gamma_1 = p_2 + l_1 u_2$$

so

$$\frac{\widehat{B}_2(0)}{\widehat{B}_1(0)} = -\left(\widehat{\beta}_1 - \frac{p_2}{l_1}\right) + u_2$$

Let

$$\widehat{\beta}_1 = \beta_1 + \frac{p_2}{l_1}$$

then

$$-\frac{\widehat{B}_2(0)}{\widehat{B}_1(0)} = \beta_1 - u_2 = \beta_1 - S_1 = l_2.$$

We also have

$$\widehat{B}_3(0) = \widehat{\beta}_2 \widehat{B}_2(0) - \widehat{\gamma}_2 \widehat{B}_1(0) - \widehat{\rho}_1 \widehat{B}_0(0)$$

$$\frac{\widehat{B}_3(0)}{\widehat{B}_2(0)} = -\widehat{\beta}_2 + \frac{\widehat{\gamma}_2}{l_2} - \frac{\widehat{\rho}_1}{l_2 l_1} = -\widehat{\beta}_2 + \frac{\gamma_2}{l_2} - \frac{\rho_1}{l_2 l_1}$$

and like

$$\gamma_2 = p_3 + l_2 u_3$$

so

$$\frac{\widehat{B}_3(0)}{\widehat{B}_2(0)} = -\left(\widehat{\beta}_2 + \frac{p_3}{l_2} - \frac{\rho_1}{l_2 l_1}\right) + u_3$$

we put

$$\widehat{\beta}_2 = \beta_2 - \frac{\rho_1}{l_2 l_1} + \frac{p_3}{l_2}$$

which gives

$$-\frac{\widehat{B}_3(0)}{\widehat{B}_2(0)} = \beta_2 - u_3 = \beta_2 - S_2 = l_3.$$

Suppose that the recurrence property is valid up to the order k ($k \geq 1$), and let's show that it remains valid to the order $k + 1$.

From (11) , we have

$$l_k = -\frac{\widehat{B}_k(0)}{\widehat{B}_{k-1}(0)} = \beta_{k-1} - S_{k-1}, \quad k \leq n \tag{13}$$

and show that this relation remains true to the order $k+1$. We have

$$\widehat{B}_{k+1}(0) = -\widehat{\beta}_k \widehat{B}_k(0) - \widehat{\gamma}_k \widehat{B}_{k-1}(0) - \widehat{\rho}_{k-1} \widehat{B}_{k-2}(0)$$

dividing the previous expression by $\widehat{B}_n(0)$, and by induction hypothesis, we get

$$\frac{\widehat{B}_{n+1}(0)}{\widehat{B}_n(0)} = -\widehat{\beta}_n + \frac{\gamma_n}{l_n} - \frac{\rho_{n-1}}{l_n l_{n-1}} = -\widehat{\beta}_n + \frac{p_{n+1}}{l_n} + u_{n+1} - \frac{\rho_{n-1}}{l_n l_{n-1}} = -\left(\widehat{\beta}_n + \frac{\rho_{n-1}}{l_n l_{n-1}} - \frac{p_{n+1}}{l_n}\right) + u_{n+1}$$

if we put

$$\widehat{\beta}_n = \beta_n - \frac{\rho_{n-1}}{l_n l_{n-1}} + \frac{p_{n+1}}{l_n}$$

we have

$$-\frac{\widehat{B}_{n+1}(0)}{\widehat{B}_n(0)} = \beta_n - u_{n+1} = \beta_n - S_n = l_{n+1}$$

Hence, (12) holds. ■

Remark 2: From the last proposition, we deduce that the UL factorization of J exists if and only if the free parameters μ_0 , μ_1 and μ_1^1 take the values such that the corresponding co-recursive sequence satisfies $\widehat{B}_n(0) \neq 0$ for all n. Moreover,

$$L = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . \\ \beta_0 - S_0 & 1 & 0 & 0 & . & . & . \\ \frac{\rho_1}{S_2} & \beta_1 - S_1 & 1 & 0 & . & . & . \\ 0 & \frac{\rho_2}{S_3} & \beta_2 - S_2 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix},$$

$$U = \begin{bmatrix} S_0 & 1 & 0 & . & . & . & . \\ 0 & S_1 & 1 & 0 & . & . & . \\ 0 & 0 & S_2 & 1 & . & . & . \\ 0 & 0 & 0 & . & . & . & \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix}$$

where

$$S_n = \frac{\gamma_n - p_{n+1}}{\beta_{n-1} - S_{n-1}}, \quad n \geq 1.$$

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