Dynamic Behaviors of an N-species Lotka-Volterra Model with Nonlinear Impulses

Mengxin He*, Zhong Li, Fengde Chen, Zhenliang Zhu

***Abstract*—For a logistic model with nonlinear impulses, we investigate the permanence, extinction and existence of positive equilibrium of the model. All our results show that the nonlinear impulse plays an important role in the dynamic behaviors of the model. Further we utilize the obtained results to study the dynamic behaviors of an N-species Lotka-Volterra model with nonlinear impulses.**

***Index Terms*—Logistic model, Lotka-Volterra model, Nonlinear Impulse, Permanence, Extinction.**

I. Introduction

THE logistic equation is one of the most important models in mathematical ecology, which admits a unique globally asymptotically stable positive equilibrium. In recent decades, many perfect results of its modified models have been obtained (for the logistic system with discrete time delay, see [1], [2]; for the logistic system with Allee effect and feedback control, see [3]).

However, external effects on the development of the species can cause jumps in the quantity of biomass, such as with a single removal of part of the biomass or with the introduction of a supplementary quantity of biomass into the bioreactor [4]. Such processes can also be seen in control theory, optimization theory, population dynamics, biology, and some physics or mechanics problems. So to naturally describe such observed evolution processes, many scholars have considered impulsive differential equations, which are regarded as an important mathematical tool for a better understanding of several real world problems in applied sciences. [5] offered a systematic treatment of the theory of impulsive differential equations. [6], [7], [8] investigated some periodic logistic model with impulses. [9] considered an almost periodic logistic model with impulses. [10] studied a logistic model with linear impulse and discussed the permanence and global attractivity of the model. For more results on impulsive differential equations, please see [11], [12], [13], [14], [15], [16], [17], [18], [19], [20] and references therein.

But all the above impulsive perturbations are linear. Note that the ecological system is often inevitably perturbed by human activities such as planting and harvesting, which can not also be linear. Thus, it is important to consider systems with nonlinear impulses, which cover those with linear impulses as species cases. Clearly, the results obtained as well as the techniques used in the above papers cannot be applied to study systems with nonlinear impulses. Motivated by this, we propose the following logistic model with nonlinear impulse

$$\begin{aligned} &\dot{x}(t) = x(t)(a - bx(t)), \quad t \neq t_k, \\ &x(t_k^+) = \frac{x(t_k)}{h + dx(t_k)}, \qquad k = 1, 2, \cdots, \end{aligned} \tag{1}$$

where $x(t)$ is the density of the species at time t; $a > 0$, $b > 0$, $h > 0$ and $d \geq 0$ are constants; $t_{k+1} = t_k + \theta$ are impulse points with $\theta > 0$ being a constant and $\lim_{k\to+\infty} t_k = +\infty$. It is obvious that the system considered in [10] is a special case of (1).

On the other hand, two or more species also interact with each other in order to compete for the limit resource (see [21], [22], [23], [24], [25], [26]), which can be explained by competitive Lotka-Volterra systems. The dynamics of N-species competitive Lotka-Volterra systems such as the permanence, stability and extinction have been extensively investigated. [27], [28], [29], [30], [31] considered the continuous Lotka-Voleterra systems. [11], [32], [33], [34], [35], [36], [37] investigated the Lotka-Voleterra systems with linear impulses. [38] studied a predator-prey Lotka-Voleterra system with nonlinear impulse on the prey. [39] discussed the existence of periodic solutions of a Lotka-Voleterra system with linear pulses. In this paper, we propose the following N-species Lotka-Volterra competitive system with nonlinear impulses

$$\begin{aligned} &\dot{x}_i(t) = x_i(t)\Big(a_i(t) - \sum_{j=1}^{N} b_{ij}(t)x_j(t)\Big), \quad t \neq t_k, \\ &x_i(t_k^+) = \frac{x_i(t_k)}{h_{ik} + d_{ik}x_i(t_k)}, \qquad k = 1, 2, \cdots, \end{aligned} \tag{2}$$

where $x_i(t)$ $(i = 1, \cdots, N)$ is the density of species x_i at time t; $a_i(t)$ and $b_{ij}(t)$ are continuous functions, bounded above and below by positive constants; $h_{ik} > 0$ and $d_{ik} \geq 0$ are constants; $t_{k+1} = t_k + \theta$ with $\theta > 0$ being a constant and $\lim_{k\to+\infty} t_k = +\infty$; $\{h_{ik} : k = 1, 2, \cdots\}$ is a positive sequence bounded above and below by positive constants; $\{d_{ik} : k = 1, 2, \cdots\}$ is a bounded nonnegative sequence. When $d_{ik} \equiv 0$, system (2) is reduced to system (1.3) in [36].

This paper is organized as follows. In Section 2, we investigate the permanence and global attractivity of system (1), while in Section 3, we utilize the obtained results to study the dynamic behaviors of system (2). In Section 4, we give some examples with their numerical simulations.

II. Logistic model

In this section, we first present the following definition and lemmas which are useful in proving our main results.

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Mengxin He (Corresponding author) is with the College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian, 350116 China (e-mail: hemx@fzu.edu.cn).

Zhong Li is with the College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian, 350116 China (e-mail: liz@fzu.edu.cn).

Fengde Chen is with the College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian, 350116 China (e-mail: fdchen@fzu.edu.cn).

Zhenliang Zhu is with the College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian, 350116 China.

Let $PC([0,+\infty),R^N)$=$\{\phi\ :\ [0,+\infty)\ \to\ R^N,\ \phi$ is continuous for $t\neq t_k$. Also $\phi(t_k^-)$ and $\phi(t_k^+)$ exist with $\phi(t_k^-)=\phi(t_k),\ k=1,\ 2,\cdots\}$. By the basic theories of impulsive differential equations in [1-2], system (1) has a unique solution $x(t)=x(t,x_0)\in PC([0,+\infty),R^+)$.

Define $G_k=(t_{k-1},t_k)\times R^N$, $k=1,2,\cdots$; $G=\bigcup_{k=1}^{+\infty}G_k$; $V_0=\{V\in C[G,R^+]$, there exist the limits $V(t_k^-,X_0)$ and $V(t_k^+,X_0)$ with $V(t_k^-,X_0)=V(t_k,X_0)$, and V is locally Lipschitz continuous$\}$.

Definition 2.1 Let $V\in V_0$. For any $(t,x(t))\in[t_{k-1},t_k)\times R^N$, the right-hand derivative $D^+V(t,x(t))$ along the solution $x(t,x_0)$ of system (1) is defined by

$$D^+V(t,x(t))=\liminf_{h\to0^+}\tfrac{1}{h}[V(t+h,x(t+h))-V(t,x(t))].$$

Lemma 2.1[5] *Assume that $m\in PC[R^+,R]$ with points of discontinuity at $t=t_k$ is left continuous at $t=t_k$, $k=1,2,\cdots$, and that*

$$\begin{array}{lll} D^+m(t)\le g(t,m(t)), & t\neq t_k, & k=1,2,\cdots,\\ m(t_k^+)\le\phi_k(m(t_k)), & t=t_k, & k=1,2,\cdots,\end{array}\tag{3}$$

where $g\in C[R_+\times R_+,R]$, $\phi_k\in C[R,R]$ and $\phi_k(u)$ is nondecreasing in u for each $k=1,2,\cdots$. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{array}{ll} \dot u=g(t,u), & t\neq t_k,\quad k=1,2,\cdots,\\ u(t_k^+)=\phi_k(u(t_k))\ge0, & t=t_k,\ \ t_k>t_0,\ \ k=1,2,\cdots,\\ u(t_0^+)=u_0, & \end{array}\tag{4}$$

existing on $[t_0,+\infty)$, then $m(t_0^+)\le u_0$ implies $m(t)\le r(t),\ t\ge t_0$.

Lemma 2.2 *Let $x(t)$ be any solution of model* (1) *with $x(0^+)>0$, then $x(t)>0$, for all $t\ge0$.*

Proof. We prove this lemma by induction. For $t\in[0,t_1)$, from the first equation of (1) and $x(0^+)>0$, we can obtain

$$x(t)=x(0^+)\exp\Big(\int_0^t(a-bx(s))\mathrm{d}s\Big)>0,$$

which implies $x(t_1^+)=x(t_1)/(h+dx(t_1))>0$. Then for $t\in[t_1,t_2)$, it is obvious that

$$x(t)=x(t_1^+)\exp\Big(\int_{t_1}^t(a-bx(s))\mathrm{d}s\Big)>0.$$

Assume that for $t\in[t_{k-1},t_k)$, there is

$$x(t)=x(t_{k-1}^+)\exp\Big(\int_{t_{k-1}}^t(a-bx(s))\mathrm{d}s\Big)>0.$$

Obviously, $x(t_k^+)=x(t_k)/(h+dx(t_k))>0$. Therefore for $t\in[t_k,t_{k+1})$,

$$x(t)=x(t_k^+)\exp\Big(\int_{t_k}^t(a-bx(s))\mathrm{d}s\Big)>0.$$

Thus $x(t)>0$, for all $t\ge0$, which completes the proof of Lemma 2.2.

Theorem 2.1 *Let $x(t)$ be any positive solution of system* (1) *with $x(0^+)>0$.*

(I) *If $0<h\le1$, there is*

$$m_1\le\liminf_{t\to+\infty}x(t)\le\limsup_{t\to+\infty}x(t)\le M_1,$$

where $m_1=\Big(\dfrac{b}{a}+\dfrac{de^{a\theta}}{h(e^{a\theta}-1)}\Big)^{-1}$ and $M_1=\Big(\dfrac{bh\theta}{a\theta-\ln h}+\dfrac{dh}{e^{a\theta}-h}\Big)^{-1}$.

(II) *If $h>1$ and $a\theta-\ln h>0$, then*

$$m_2\le\liminf_{t\to\infty}x(t)\le\limsup_{t\to\infty}x(t)\le M_2,$$

where $m_2=\Big(\dfrac{b\theta h}{a\theta-\ln h}+\dfrac{dhe^{a\theta}}{e^{a\theta}-h}\Big)^{-1}$ and $M_2=\Big(\dfrac{b}{a}+\dfrac{d}{h(e^{a\theta}-1)}\Big)^{-1}$.

(III) *For any positive solutions $x_1(t)$ and $x_2(t)$ of system* (1) *with $x_1(0^+)>0$ and $x_2(0^+)>0$ respectively, if* (I) *or* (II) *holds, we have*

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0.$$

Proof. Let $x(t)=1/y(t)$, then system (1) is transformed into

$$\begin{array}{ll} \dot y(t)=-ay(t)+b, & t\neq t_k,\\ y(t_k^+)=hy(t_k)+d, & k=1,2,\cdots.\end{array}\tag{5}$$

Let $y(t)$ be any solution of system (5), according to [4], we can obtain

$$y(t)=w(t,0)y(0)+b\int_0^t w(t,s)\mathrm{d}s+\sum_{0\le t_k<t}w(t,t_k^+)d,$$

where $w(t,s)=\Big(\prod_{s\le t_k<t}h\Big)e^{-a(t-s)}$. Note that $w(t,t_k^+)=\dfrac{1}{h}w(t,t_k)$, then

$$\begin{aligned} y(t)\ =\ &\Big(\prod_{0\le t_k<t}h\Big)e^{-at}y(0)+b\int_0^t\Big(\prod_{s\le t_k<t}h\Big)e^{-a(t-s)}\mathrm{d}s\\ &+\frac{d}{h}\sum_{0\le t_k<t}\Big(\prod_{t_k\le t_j<t}h\Big)e^{-a(t-t_k)}.\end{aligned}\tag{6}$$

(I) When $0<h\le1$, we have

$$\begin{aligned} y(t)\ &\le\ e^{-at}y(0)+b\int_0^t e^{-a(t-s)}\mathrm{d}s+\frac{d}{h}\sum_{0\le t_k<t}e^{-a(t-t_k)}\\ &=\ e^{-at}y(0)+\frac{b}{a}(1-e^{-at})+\frac{d(1-e^{-a\theta n})}{h(1-e^{-a\theta})},\end{aligned}$$

where n is the number of the impulse points in the interval $[0,t)$.

Since $e^{-a}<1$, by letting $t\to+\infty$ and obviously $n\to+\infty$, it follows that

$$\limsup_{t\to+\infty}y(t)\le\frac{b}{a}+\frac{de^{a\theta}}{h(e^{a\theta}-1)}=\frac{1}{m_1}.$$

Next we prove that $\liminf_{t\to+\infty}y(t)\ge1/M_1$. From (6), we obtain

$$\begin{aligned} y(t)\ &\ge\ h^{\frac{t}{\theta}+1}e^{-at}y(0)+b\int_0^t h^{\frac{t-s}{\theta}+1}e^{-a(t-s)}\mathrm{d}s\\ &\quad+\frac{d}{h}\sum_{0\le t_k<t}h^{\frac{t-t_k}{\theta}+1}e^{-a(t-t_k)}\\ &\ge\ h\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t y(0)+\frac{bh\theta}{\ln h-a\theta}\Big[\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t-1\Big]\\ &\quad+\frac{dh\Big(1-\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{n\theta}\Big)}{e^{a\theta}-h},\end{aligned}$$

where n is the number of the impulse points in the interval $[0,t)$. Similarly, we have

$$\liminf_{t\to+\infty} y(t) \geq \frac{b\theta h}{a\theta - \ln h} + \frac{dh}{e^{a\theta} - h} = \frac{1}{M_1}.$$

The above analysis shows that $m_1 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_1$.

(II) When $h > 1$, we have

$$\begin{aligned} y(t) &\leq h^{\frac{t}{\theta}+1}e^{-at}y(0) + b\int_0^t h^{\frac{t-s}{\theta}+1}e^{-a(t-s)}\mathrm{d}s \\ &\quad +\frac{d}{h}\sum_{0\leq t_k<t} h^{\frac{t-t_k}{\theta}+1}e^{-a(t-t_k)} \\ &= h\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t y(0) + bh\int_0^t \Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{t-s}\mathrm{d}s \\ &\quad +d\sum_{0\leq t_k<t}\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{(t-t_k)} \\ &\leq h\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t y(0) + \frac{b\theta h}{\ln h - a\theta}\Big[\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t - 1\Big] \\ &\quad +\frac{d\Big(1-\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{n\theta}\Big)}{1-\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{\theta}}, \end{aligned}$$

where n is the number of the impulse points in the interval $[0,t)$.

Since $a\theta - \ln h > 0$ that is $\frac{h^{\frac{1}{\theta}}}{e^a} < 1$, by letting $t \to +\infty$ and $n \to +\infty$, it follows that

$$\limsup_{t\to+\infty} y(t) \leq \frac{b\theta h}{a\theta - \ln h} + \frac{dhe^{a\theta}}{e^{a\theta} - h} = \frac{1}{m_2}.$$

From (6) and $h > 1$, we obtain

$$\begin{aligned} y(t) &\geq e^{-at}y(0) + b\int_0^t e^{-a(t-s)}\mathrm{d}s + \frac{d}{h}\sum_{0\leq t_k<t} e^{-a(t-t_k)} \\ &= e^{-at}y(0) + \frac{b}{a}\Big(1-e^{-at}\Big) + \frac{d\Big(1-e^{-an\theta}\Big)}{h(e^{a\theta}-1)}, \end{aligned} \tag{7}$$

where n is the number of the impulse points in the interval $[0,t)$. Similarly, we have

$$\liminf_{t\to+\infty} y(t) \geq \frac{b}{a} + \frac{d}{h(e^{a\theta}-1)} = \frac{1}{M_2}.$$

Therefore,

$$m_2 \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M_2.$$

(III) Denote $x_1(t) = 1/y_1(t)$ and $x_2(t) = 1/y_2(t)$. Then $y_1(t)$ and $y_2(t)$ are any solutions of (5). It follows from (6) that

$$|y_1(t) - y_2(t)| = \Big(\prod_{0\leq t_k<t} h\Big)e^{-at}|y_1(0) - y_2(0)|.$$

Since $0 < h \leq 1$, it is obvious that $\lim_{t\to+\infty}|y_1(t) - y_2(t)| = 0$.

Note that

$$|x_1(t) - x_2(t)| = \frac{|y_1(t) - y_2(t)|}{y_1(t)y_2(t)}.$$

This together with the boundedness of $y_1(t)$ and $y_2(t)$ implies that $\lim_{t\to+\infty}|x_1(t) - x_2(t)| = 0$. This completes the proof of Theorem 2.1.

Remark 2.1. When $h = 1$ and $d = 0$, system (1) is reduced to the continuous logistic model. It follows from Theorem 2.1 that $a/b \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq a/b$, that is $\lim_{t\to+\infty} x(t) = a/b$, which is consistent with the previous result of the traditional logistic equation.

Theorem 2.2 *If $\frac{1-h}{d} = \frac{a}{b}$ with $0 < h < 1$ and $d \neq 0$, then* (1) *admits a positive equilibrium $x^* = \frac{a}{b}$, which is globally asymptotically stable, that is for any solution $x(t)$ of system* (1)*, there is $\lim_{t\to+\infty} x(t) = a/b$.*

Proof. Obviously x^* satisfies the following equations

$$\begin{aligned} &a - bx^* = 0, \\ &x^* = \frac{x^*}{h + dx^*}. \end{aligned}$$

The assumption of the theorem deduces $x^* = \frac{1-h}{d} = \frac{a}{b}$. Next we prove the global asymptotic stability of x^*. Let $x(t)$ be any solution of system (1). According to Theorem 3.1, for any $\varepsilon > 0$, there exists a $T > 0$ such that for $t > T$,

$$x(t) \leq M_1 + \varepsilon.$$

Define a Lyapunov function as follows

$$V(t) = \Big|\frac{1}{x(t)} - \frac{1}{x^*}\Big|.$$

For $t > T$ and $t \neq t_k$, $k = 1,\ 2, \cdots$, calculating the upper right derivatives of $V(t)$, we have

$$\begin{aligned} D^+(V(t)) &= \mathrm{sgn}(x^* - x(t))\frac{bx(t) - a}{x(t)} \\ &= \mathrm{sgn}(x^* - x(t))\frac{b(x(t) - x^*)}{x(t)} \\ &= -\frac{b|x(t) - x^*|}{x(t)} \\ &\leq -\frac{b}{M_1 + \varepsilon}|x(t) - x^*|. \end{aligned} \tag{8}$$

For $t = t_k$, $k = 1,\ 2, \cdots$, we obtain

$$\begin{aligned} V(t_k^+) &= \Big|\frac{1}{x(t_k^+)} - \frac{1}{x^*}\Big| = \Big|\frac{dx(t_k) + h}{x(t_k)} - \frac{dx^* + h}{x^*}\Big| \\ &= \Big|\frac{h}{x(t_k)} - \frac{h}{x^*}\Big| \leq \Big|\frac{1}{x(t_k)} - \frac{1}{x^*}\Big| = V(t_k). \end{aligned}$$

Consider the following inequalities

$$\begin{aligned} &D^+(V(t)) \leq -\frac{b}{M_1 + \varepsilon}|x(t) - x^*|, \qquad t \neq t_k, \\ &V(t_k^+) \leq V(t_k), \qquad k = 1, 2, \cdots. \end{aligned} \tag{9}$$

According to Lemma 2.1, we can easily verify that

$$V(t) + \frac{b}{M_1 + \varepsilon}\int_T^t |x(t) - x^*|\mathrm{d}s \leq V(T) < +\infty.$$

Therefore, $V(t)$ is bounded on $[T, +\infty)$ and $\int_T^{+\infty} |x(t) - x^*|\mathrm{d}s < +\infty$. Then we claim that $\lim_{t\to+\infty}|x(t) - x^*| = 0$. Otherwise, for any given $\varepsilon_1 > 0$ there are two cases:

Case A: For any $T > 0$, when $t > T$, $|x(t) - x^*| \geq \varepsilon_1$.

Case B: For any $T > 0$, when $t > T$, $|x(t) - x^*|$ is oscillatory about ε_1.

We first consider Case A. It is obvious that

$$\int_T^{+\infty} |x(t) - x^*|\mathrm{d}s \geq \int_T^{+\infty} \varepsilon_1\mathrm{d}s \to +\infty,$$

which is a contradiction.

For Case B, we can choose two sequences $\{\rho_n\}$ and $\{\rho_n^*\}$ satisfying $T < \rho_1 < \rho_1^* < \cdots < \rho_n < \rho_n^* < \cdots$ and $\lim_{t\to+\infty}\rho_n = \lim_{t\to+\infty}\rho_n^* = +\infty$ such that

$$\begin{aligned}
&|x(\rho_n)-x^*| \ge \varepsilon_1; \quad |x(\rho_n^+)-x^*| \le \varepsilon_1;\\
&|x(\rho_n^*)-x^*| \le \varepsilon_1; \quad |x(\rho_n^{*+})-x^*| \ge \varepsilon_1,\\
&|x(t)-x^*| \le \varepsilon_1, \quad \text{for all} \quad t \in (\rho_n, \rho_n^*);\\
&|x(t)-x^*)| \ge \varepsilon_1, \quad \text{for all} \quad t \in (\rho_n^*, \rho_{n+1}).
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_T^{+\infty} |x(t)-x^*|\mathrm{d}s\\
= &\Big(\int_T^{\rho_1} + \sum_{n=1}^{+\infty}\int_{\rho_n}^{\rho_n^*} + \sum_{n=1}^{+\infty}\int_{\rho_n^*}^{\rho_{n+1}}\Big)|x(t)-x^*|\mathrm{d}s\\
\ge &\Big(\int_T^{\rho_1} + \sum_{n=1}^{+\infty}\int_{\rho_n}^{\rho_n^*}\Big)|x(t)-x^*|\mathrm{d}s + \sum_{n=1}^{+\infty}\int_{\rho_n^*}^{\rho_{n+1}} \varepsilon_1 \mathrm{d}s\\
\to &+\infty,
\end{aligned}$$

which is also a contradiction. This completes the proof of Theorem 2.2.

Theorem 2.3 *Assume that $a\theta - \ln h \le 0$. Let $x(t)$ be any positive solution of system* (1). *Then* $\lim\limits_{t\to+\infty} x(t) = 0$.

Proof. First we consider $a\theta - \ln h = 0$, that is $\frac{h^{\frac{1}{\theta}}}{e^a} = 1$. Let $y(t)$ be any solution of (5). Then from (6), we obtain by letting $t \to +\infty$ and $n \to +\infty$ that

$$\begin{aligned}
y(t) \ \ge\ & h^{\frac{t}{\theta}-1}e^{-at}y(0) + b\int_0^t h^{\frac{t-s}{\theta}-1}e^{-a(t-s)}\mathrm{d}s\\
&+\frac{d}{h}\sum_{0\le t_k<t} h^{\frac{t-t_k}{\theta}-1}e^{-a(t-t_k)}\\
\ge\ & \frac{1}{h}\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t y(0) + \frac{b}{h}\int_0^t \Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{t-s}\mathrm{d}s\\
&+\frac{d}{h^2}\sum_{0\le t_k<t}\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{t-t_k}\\
=\ & \frac{1}{h}y(0) + \frac{b}{h}t + \frac{d}{h^2}n \to +\infty,
\end{aligned}$$

where n is the number of the impulse points in the interval $[0,t)$.

Next we consider $a\theta - \ln h < 0$, which implies $\frac{h^{\frac{1}{\theta}}}{e^a} > 1$ and $h > 1$. Similarly we have

$$\begin{aligned}
y(t) \ \ge\ & \frac{1}{h}\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t y(0) + \frac{b\theta}{h(\ln h - a\theta)}\Big[\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^t - 1\Big]\\
&+\frac{d\Big(\Big(\frac{h^{\frac{1}{\theta}}}{e^a}\Big)^{n\theta} - 1\Big)}{h(h-e^{a\theta})} \to +\infty,
\end{aligned}$$

by letting $t \to +\infty$ and $n \to +\infty$. By the transformation $x(t) = 1/y(t)$, the above analysis shows that when $a\theta - \ln h \le 0$, $\lim\limits_{t\to+\infty} x(t) = 0$. This complete the proof of Theorem 2.3.

III. Lotka-Volterra model

In this section, we study the dynamics of model (2). Similar to the proof of Lemma 2.2, we can obtain the following lemma.

Lemma 3.1 *Let $(x_1(t), x_2(t), \cdots, x_N(t))^T$ be any solution of model* (2) *with $x_i(0^+) > 0$, then $x(t) > 0$, for all $t \ge 0$.*

Theorem 3.1 *Let $(x_1(t), x_2(t), \cdots, x_N(t))^T$ be any solution of system* (2) *with $x_i(0^+) > 0$ $(i = 1, 2, \cdots, N)$. Assume that*

$$\begin{aligned}
&a_{iL} - \sum_{j=1,j\ne i}^{N} b_{ijM}M_j > 0,\\
&\Big(a_{iL} - \sum_{j=1,j\ne i}^{N} b_{ijM}M_j\Big)\xi - \ln h_{iM} > 0,
\end{aligned} \tag{10}$$

then we can obtain

$$m_i \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le M_i,$$

where

$$\begin{aligned}
&m_i = \min\{m_i^{(1)},\ m_i^{(2)}\}, \quad M_i = \max\{M_i^{(1)},\ M_i^{(2)}\},\\
&M_i^{(1)} = \Big(\frac{b_{iiL}h_{iL}\xi}{a_{iM}\xi - \ln h_{iL}} + \frac{d_{iL}h_{iL}}{e^{a_{iM}\xi} - h_{iL}}\Big)^{-1};\\
&M_i^{(2)} = \Big(\frac{b_{iiL}\xi}{h_{iL}(a_{iM}\xi - \ln h_{iL})} + \frac{d_{iL}}{h_{iL}(e^{a_{iM}\xi} - h_{iL})}\Big)^{-1};\\
&m_i^{(1)} = \Big(\frac{b_{iiM}\xi h_{iM}}{A_i\xi - \ln h_{iM}} + \frac{d_{iM}e^{A_i\xi}}{e^{A_i\xi} - h_{iM}}\Big)^{-1};\\
&m_i^{(2)} = \Big(\frac{b_{iiM}\xi}{h_{iM}(A_i\xi - \ln h_{iM})} + \frac{d_{iM}e^{A_i\xi}}{h_{iM}^2(e^{A_i\xi} - h_{iM})}\Big)^{-1},
\end{aligned}$$

with $A_i = a_{iL} - \sum\limits_{j=1,i\ne j}^{N} b_{ijM}M_j$.

Proof. Let $(x_1(t), x_2(t), \cdots, x_N(t))^T$ be any solution of system (2) with $x_i(0^+) > 0$ $(i = 1, 2, \cdots, N)$.

(i) We first prove $\limsup\limits_{t\to+\infty} x_i(t) \le M_i$. From the ith $(i = 1, 2, \cdots, N)$ equation of system (2) we can obtain

$$\begin{aligned}
&\dot{x_i}(t) \le x_i(t)(a_{iM} - b_{iiL}x_i(t)), \quad t \ne t_k,\\
&x_i(t_k^+) \le \frac{x_i(t_k)}{h_{iL} + d_{iL}x_i(t_k)}, \qquad k = 1, 2, \cdots.
\end{aligned}$$

According to Lemma 2.1 and Theorem 2.1, if $h_{iL} \le 1$, that is $a_{iM}\xi - \ln h_{iL} > 0$, then

$$\limsup_{t\to+\infty} x_i(t) \le \Big(\frac{b_{iiL}h_{iL}\xi}{a_{iM}\xi - \ln h_{iL}} + \frac{d_{iL}h_{iL}}{e^{a_{iM}\xi} - h_{iL}}\Big)^{-1} \triangleq M_i^{(1)}.$$

If $h_{iL} > 1$, noting that $\Big(a_{iL} - \sum\limits_{j=1,j\ne i}^{N} b_{ijM}M_j\Big)\xi - \ln h_{iM} > 0$ implies $a_{iM}\xi - \ln h_{iL} > 0$, then

$$\begin{aligned}
&\limsup_{t\to+\infty} x_i(t)\\
\le\ & \Big(\frac{b_{iiL}\xi}{h_{iL}(a_{iM}\xi - \ln h_{iL})} + \frac{d_{iL}}{h_{iL}(e^{a_{iM}\xi} - h_{iL})}\Big)^{-1}\\
\triangleq\ & M_i^{(2)}.
\end{aligned}$$

Let $M_i = \max\{M_i^{(1)},\ M_i^{(2)}\}$, therefore,

$$\limsup_{t\to+\infty} x_i(t) \le M_i, \quad i = 1,\ \cdots,\ N.$$

(ii) We prove $\liminf\limits_{t\to+\infty} x_i(t) \ge m_i$. For any $\varepsilon_1 > 0$ small enough satisfying

$$\begin{aligned}&\Big(a_{iL}-\sum_{j=1,j\neq i}^{N}b_{ijM}(M_j+\varepsilon_1)\Big)\xi-\ln h_{iM}>0,\\&a_{iL}-\sum_{j=1,j\neq i}^{N}b_{ijM}(M_j+\varepsilon_1)>0,\end{aligned}\tag{11}$$

there exists a $T_1>0$, such that for $t>T_1$ there is

$$x_i(t)\leq M_i+\varepsilon_1.\tag{12}$$

Substituting (12) into system (2) deduces

$$\begin{aligned}\dot{x_i}(t)&\geq x_i(t)\Big(a_{iL}-\sum_{j=1,i\neq j}^{N}b_{ijM}(M_j+\varepsilon_1)-b_{iiM}x_i(t)\Big),\\&\qquad\qquad t\neq t_k,\\x_i(t_k^+)&\geq\frac{x_i(t_k)}{h_{iM}+d_{iM}x_i(t_k)},\qquad k=1,2,\cdots.\end{aligned}$$

If $h_{iM}>1$, from (11), according to Lemma 2.1 and Theorem 2.1, letting $\varepsilon_1\to0$, then we have

$$\liminf_{t\to+\infty}x_i(t)\geq\Big(\frac{b_{iiM}\xi h_{iM}}{A_i\xi-\ln h_{iM}}+\frac{d_{iM}e^{A_i\xi}}{e^{A_i\xi}-h_{iM}}\Big)^{-1}\triangleq m_i^{(1)},$$

where $A_i=a_{iL}-\sum_{j=1,i\neq j}^{N}b_{ijM}M_j>0$.

If $h_{iM}\leq1$, from (11), similarly we can obtain

$$\begin{aligned}&\liminf_{t\to+\infty}x_i(t)\\\geq&\Big(\frac{b_{iiM}\xi}{h_{iM}(A_i\xi-\ln h_{iM})}+\frac{d_{iM}e^{A_i\xi}}{h_{iM}^2(e^{A_i\xi}-h_{iM})}\Big)^{-1}\\\triangleq&\ m_i^{(2)}.\end{aligned}$$

Let $m_i=\min\{m_i^{(1)},\ m_i^{(2)}\}$, then we derive

$$m_i\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq M_i,\quad i=1,2,\cdots,N.$$

This completes the proof of Theorem 3.1.

According to the proof of Theorem 3.1, we can obviously obtain the following corollary.

Corollary 3.1 *Let* $(x_1(t),x_2(t),\cdots,x_N(t))^T$ *be any solution of* (2) *with* $x_i(0^+)>0$ $(i=1,2,\cdots,N)$.

(1) *When* $h_{iL}\leq1$, $h_{iM}>1$ *and* $\Big(a_{iL}-\sum_{j=1,j\neq i}^{N}b_{ijM}M_j^{(1)}\Big)\xi-\ln h_{iM}>0$, *there is*

$$m_i^{(1)}\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq M_i^{(1)}.$$

(2) *When* $h_{iL}\leq1,\ h_{iM}\leq1$ *and* $a_{iL}-\sum_{j=1,j\neq i}^{N}b_{ijM}M_j^{(1)}>0$, *there is*

$$m_i^{(2)}\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq M_i^{(1)}.$$

(3) *When* $h_{iL}>1$ *and* $\Big(a_{iL}-\sum_{j=1,j\neq i}^{N}b_{ijM}M_j^{(2)}\Big)\xi-\ln h_{iM}>0$, *there is*

$$m_i^{(1)}\leq\liminf_{t\to+\infty}x_i(t)\leq\limsup_{t\to+\infty}x_i(t)\leq M_i^{(2)},$$

where $M_i^{(1)},M_i^{(2)},m_i^{(1)},m_i^{(2)}$ *are defined in* Theorem 3.1.

Theorem 3.2 *Assume that all the conditions of* Theorem 3.1 *are satisfied, and there exist* ρ_i *and* $\delta_i>0$ $(i=1,\ 2,\cdots,N)$ *such that*

$$b_{iiL}\rho_i-\sum_{j=1,j\neq i}^{N}b_{jiM}\rho_j>\delta_i,\tag{13}$$

$$\alpha^{\frac{1}{\xi}}e^{-\delta}<1,\tag{14}$$

where

$$\alpha=\max\Big\{1,\ \frac{d_{1M}M_1}{h_{1L}+d_{1L}m_1},\cdots,\frac{d_{NM}M_N}{h_{NL}+d_{NL}m_N}\Big\},$$

$$\delta=\min\Big\{\frac{\delta_im_i}{\rho_i}\Big|i=1,\ 2,\cdots,N\Big\},$$

with $m_i=\min\{m_i^{(1)},m_i^{(2)}\}$ *and* $M_i=\max\{M_i^{(1)},M_i^{(2)}\}$ *being defined in* Theorem 3.1. *For any solutions* $(x_1(t),x_2(t),\cdots,x_N(t))^T$ *and* $(y_1(t),y_2(t),\cdots,y_N(t))^T$ *of system* (2), *there are*

$$\lim_{t\to+\infty}|x_i(t)-y_i(t)|=0,\qquad i=1,\ 2,\cdots,N.$$

Proof. Let $(x_1(t),x_2(t),\cdots,x_N(t))^T$ and $(y_1(t),y_2(t),\cdots,y_N(t))^T$ be any positive solutions of system (2). According to Theorem 3.1, for any $\varepsilon_2>0$ small enough satisfying

$$m_i-\varepsilon_2>0\quad\text{and}\quad\alpha_{\varepsilon_2}^{\frac{1}{\xi}}e^{-\delta_{\varepsilon_2}}<1,\tag{15}$$

where

$$\begin{aligned}\alpha_{\varepsilon_2}&=\max\Big\{1,\ \frac{d_{1M}(M_1+\varepsilon_2)}{h_{1L}+d_{1L}(m_1-\varepsilon_2)},\\&\qquad\cdots,\frac{d_{NM}(M_N+\varepsilon_2)}{h_{NL}+d_{NL}(m_N-\varepsilon_2)}\Big\},\end{aligned}$$

$$\delta_{\varepsilon_2}=\min\Big\{\frac{\delta_i(m_i-\varepsilon_2)}{\rho_i}\Big|i=1,\ 2,\cdots,N\Big\},$$

there is a $T_2>0$, such that for $t>T_2$,

$$m_i-\varepsilon_2\leq x_i(t)\leq M_i+\varepsilon_2,\quad i=1,\ 2,\cdots,N.\tag{16}$$

Using the mean value theorem, it follows that

$$\begin{aligned}\frac{1}{M_i+\varepsilon_2}|x_i(t)-y_i(t)|&\leq|\ln x_i(t)-\ln y_i(t)|\\&\leq\frac{1}{m_i-\varepsilon_2}|x_i(t)-y_i(t)|.\end{aligned}$$

Define a Lyapunov function

$$V(t)=\sum_{i=1}^{N}\rho_i|\ln x_i(t)-\ln y_i(t)|.$$

For $t>T_2$ and $t\neq t_k$, calculating the upper right derivatives of $V(t)$, we have

$$\begin{aligned}D^+V(t)&=\sum_{i=1}^{N}\rho_i\mathrm{sgn}(x_i(t)-y_i(t))\Big(\sum_{j=1}^{N}b_{ij}(t)y_j(t)\\&\qquad-\sum_{j=1}^{N}b_{ij}(t)x_j(t)\Big)\\&=\sum_{i=1}^{N}\rho_i\mathrm{sgn}(x_i(t)-y_i(t))\Big[b_{ii}(t)(y_i(t)-x_i(t))\\&\qquad+\sum_{j=1,i\neq j}^{N}b_{ij}(t)(y_j(t)-x_j(t))\Big]\\&\leq\sum_{i=1}^{N}\Big(-b_{iiL}\rho_i+\sum_{j=1,i\neq j}^{N}b_{jiM}\rho_j\Big)|x_i(t)-y_i(t)|\\&\leq\sum_{i=1}^{N}-\delta_i|x_i(t)-y_i(t)|\\&\leq\sum_{i=1}^{N}-\delta_i(m_i-\varepsilon_2)|\ln x_i(t)-\ln y_i(t)|\\&\leq-\delta_{\varepsilon_2}V(t),\end{aligned}$$

where $\delta_{\varepsilon_2} = \min\Big\{\dfrac{\delta_i(m_i-\varepsilon_2)}{\rho_i}\Big| i=1,\ 2,\cdots,N\Big\}$. For $t = t_k$,

$$\begin{aligned} & V(t_k^+) \\ = & \sum_{i=1}^N \rho_i |\ln x_i(t_k^+) - \ln y_i(t_k^+)| \\ = & \sum_{i=1}^N \rho_i \Big| \ln \frac{x_i(t_k)}{h_{ik}+d_{ik}x_i(t_k)} - \ln \frac{y_i(t_k)}{h_{ik}+d_{ik}y_i(t_k)}\Big| \\ = & \sum_{i=1}^N \rho_i |(\ln x_i(t_k) - \ln y_i(t_k)) \\ & -[\ln(h_{ik}+d_{ik}x_i(t_k)) - \ln(h_{ik}+d_{ik}y_i(t_k))]|. \end{aligned}$$

Using the mean value theorem, there is

$$\begin{aligned} & |\ln(h_{ik}+d_{ik}x_i(t_k)) - \ln(h_{ik}+d_{ik}y_i(t_k))| \\ = & \frac{d_{ik}}{h_{ik}+d_{ik}\xi_i(t_k)}|x_i(t_k)-y_i(t_k)| \\ = & \frac{d_{ik}\varsigma_i(t_k)}{h_{ik}+d_{ik}\xi_i(t_k)}|\ln x_i(t_k) - \ln y_i(t_k)| \\ \le & \frac{d_{iM}(M_i+\varepsilon_2)}{h_{iL}+d_{iL}(m_i-\varepsilon_2)}|\ln x_i(t_k) - \ln y_i(t_k)|, \end{aligned}$$

where both $\xi_i(t_k)$ and $\varsigma_i(t_k)$ lie between $x_i(t_k)$ and $y_i(t_k)$.

Noticing that $\ln x_i(t_k) - \ln y_i(t_k)$ and $\ln(h_{ik}+d_{ik}x_i(t_k)) - \ln(h_{ik}+d_{ik}y_i(t_k))$ have the same sign, we can easily obtain

$$\begin{aligned} V(t_k^+) \le & \max\Big\{1,\ \frac{d_{1M}(M_1+\varepsilon_2)}{h_{1L}+d_{1L}(m_1-\varepsilon_2)}, \\ & \cdots, \frac{d_{NM}(M_N+\varepsilon_2)}{h_{NL}+d_{NL}(m_N-\varepsilon_2)}\Big\} V(t_k) \\ \triangleq & \alpha_{\varepsilon_2} V(t_k). \end{aligned}$$

From $\alpha_{\varepsilon_2} \ge 1$ and (15), for $t \to +\infty$, there is

$$\begin{aligned} V(t) \le & V(T_2^+)\Big(\prod_{T_2 \le t_k < t} \alpha_{\varepsilon_2}\Big) e^{-\delta_{\varepsilon_2}(t-T_2)} \\ = & \alpha_{\varepsilon_2}^{\frac{t-T_2}{\xi}+1} e^{-\delta_{\varepsilon_2}(t-T_2)} \\ = & \alpha_{\varepsilon_2}\Big(\alpha_{\varepsilon_2}^{\frac{1}{\xi}} e^{-\delta_{\varepsilon_2}}\Big)^{t-T_2} \\ \to & 0. \end{aligned}$$

This implies

$$\lim_{t\to+\infty} |\ln x_i(t) - \ln y_i(t)| = 0, \quad i = 1,\ 2,\cdots,N,$$

that is

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1,\ 2,\cdots,N.$$

The proof of Theorem 3.2 is complete.

Theorem 3.3 *Let* $(x_1(t),x_2(t),\cdots,x_N(t))^T$ *be any solution of system* (2) *with* $x_i(0^+) > 0$ $(i=1,2,\cdots,N)$. *For* $1 < r < N$, *assume the following conditions*

$$a_{iL} - \sum_{j=1,j\ne i}^r b_{ijM}M_j > 0, \tag{17}$$

$$\Big(a_{iL} - \sum_{j=1,j\ne i}^r b_{ijM}M_j\Big)\xi - \ln h_{iM} > 0, \tag{18}$$

$$a_{iM}\xi - \ln h_{iL} \le 0 \tag{19}$$

hold, then we have

$$\bar m_i \le \liminf_{t\to+\infty} x_i(t) \le \limsup_{t\to+\infty} x_i(t) \le M_i, \quad 1 \le i \le r,$$
$$\lim_{t\to+\infty} x_i(t) = 0, \quad r < i \le N,$$

where M_i $(1 \le i \le r)$ *are defined in* Theorem 3.1 *and*

$$\bar m_i = \min\{\bar m_i^{(1)},\ \bar m_i^{(2)}\},$$
$$\bar m_i^{(1)} = \Big(\frac{b_{iiM}\xi h_{iM}}{\bar A_i\xi - \ln h_{iM}} + \frac{d_{iM}e^{\bar A_i\xi}}{e^{\bar A_i\xi}-h_{iM}}\Big)^{-1},$$
$$\bar m_i^{(2)} = \Big(\frac{b_{iiM}\xi}{h_{iM}(\bar A_i\xi - \ln h_{iM})} + \frac{d_{iM}e^{\bar A_i\xi}}{h_{iM}^2(e^{\bar A_i\xi}-h_{iM})}\Big)^{-1}$$

with $\bar A_i = a_{iL} - \sum_{j=1,i\ne j}^r b_{ijM}M_j$.

Proof. Let $(x_1(t),x_2(t),\cdots,x_N(t))^T$ and $(y_1(t),y_2(t),\cdots,y_N(t))^T$ be any solutions of system (2) with $x_i(0^+) > 0$ and $y_i(0^+) > 0$, respectively, $i = 1,2,\cdots,N$. From system (2), there are

$$\dot x_i(t) \le x_i(t)(a_{iM} - b_{iiL}x_i(t)),\ \ t \ne t_k,$$
$$x_i(t_k^+) \le \frac{x_i(t_k)}{h_{iL}+d_{iL}x_i(t_k)}, \quad k = 1,2,\cdots.$$

According to conditions (17) and (18), similar to the proof of Theorem 3.1, we can easily obtain

$$\limsup_{t\to+\infty} x_i(t) \le M_i, \quad i \le r.$$

From condition (19), according to Theorem 2.3, it deduces

$$\lim_{t\to+\infty} x_i(t) = 0, \quad r < i \le N.$$

For any $\varepsilon_3 > 0$ small enough satisfying

$$a_{iL} - \sum_{j=1,i\ne j}^r b_{ijM}(M_j+\varepsilon_3) - \sum_{j=r+1}^N b_{ijM}\varepsilon_3 > 0,$$
$$\Big(a_{iL} - \sum_{j=1,i\ne j}^r b_{ijM}(M_j+\varepsilon_3) - \sum_{j=r+1}^N b_{ijM}\varepsilon_3\Big)\xi - \ln h_{iM} > 0,$$

there exists a $T_3 > 0$, such that for $t > T_3$ we have

$$\begin{aligned} & x_i(t) \le M_i+\varepsilon_3, \quad i \le r, \\ & x_i(t) \le \varepsilon_3, \quad r < i \le N. \end{aligned} \tag{20}$$

Substituting (20) into system (2) leads to

$$\begin{aligned} \dot x_i(t) \ge & x_i(t)\Big(a_{iL} - \sum_{j=1,i\ne j}^r b_{ijM}(M_j+\varepsilon_3) \\ & - \sum_{j=r+1}^N b_{ijM}\varepsilon_3 - b_{iiM}x_i(t)\Big),\ \ t \ne t_k, \\ x_i(t_k^+) \ge & \frac{x_i(t_k)}{h_{iM}+d_{iM}x_i(t_k)}, \quad k = 1,2,\cdots. \end{aligned}$$

From Theorem 2.1, letting $\varepsilon_3 \to 0$, one has

$$\liminf_{t\to+\infty} x_i(t) \ge \bar m_i, \quad i = 1,2,\cdots,r,$$

where $\bar m_i = \min\{\bar m_i^{(1)},\ \bar m_i^{(2)}\}$,

$$\bar m_i^{(1)} = \Big(\frac{b_{iiM}\xi h_{iM}}{\bar A_i\xi - \ln h_{iM}} + \frac{d_{iM}e^{\bar A_i\xi}}{e^{\bar A_i\xi}-h_{iM}}\Big)^{-1},$$
$$\bar m_i^{(2)} = \Big(\frac{b_{iiM}\xi}{h_{iM}(\bar A_i\xi - \ln h_{iM})} + \frac{d_{iM}e^{\bar A_i\xi}}{h_{iM}^2(e^{\bar A_i\xi}-h_{iM})}\Big)^{-1},$$

with $\bar A_i = a_{iL} - \sum_{j=1,i\ne j}^r b_{ijM}M_j$. The proof is complete.

Consider the following system

$$\begin{aligned} & \dot x_i(t) = x_i(t)\Big(a_i(t) - \sum_{j=1}^r b_{ij}(t)x_j(t)\Big),\ \ t \ne t_k, \\ & \qquad\qquad i = 1,\ 2,\cdots,\ r, \\ & x_i(t_k^+) = \frac{x_i(t_k)}{h_{ik}+d_{ik}x_i(t_k)}, \quad k = 1,2,\cdots. \end{aligned} \tag{21}$$

Theorem 3.4 *Assume that all the conditions of* Theorem 3.3 *hold, and there exist $\bar{\rho}_i$ and $\bar{\delta}_i > 0$ $(i = 1, 2, \cdots, r)$ such that*

$$b_{iiL}\bar{\rho}_i - \sum_{j=1,j\neq i}^{N} b_{jiM}\bar{\rho}_j > \bar{\delta}_i, \tag{22}$$

$$\bar{\alpha}^{\frac{1}{\xi}} e^{-\bar{\delta}} < 1, \tag{23}$$

where

$$\bar{\alpha} = \max\Big\{1,\ \frac{d_{1M}M_1}{h_{1L} + d_{1L}\bar{m}_1}, \cdots, \frac{d_{rM}M_r}{h_{rL} + d_{rL}\bar{m}_r}\Big\},$$
$$\bar{\delta} = \min\Big\{\frac{\bar{\delta}_i\bar{m}_i}{\bar{\rho}_i}\Big| i = 1,\ 2, \cdots, r\Big\},$$

here $\bar{m}_i = \min\{\bar{m}_i^{(1)}, \bar{m}_i^{(2)}\}$ *and* $M_i = \max\{M_i^{(1)}, M_i^{(2)}\}$ *are defined in* Theorem 3.3.

Let $(x_1(t), x_2(t), \cdots, x_N(t))^T$ *be any positive solution of system* (2)*, and* $(y_1(t), y_2(t), \cdots, y_r(t))^T$ *be any positive solution of system* (21)*, then*

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1,\ 2, \cdots, r.$$

Proof. Let $(x_1(t), x_2(t), \cdots, x_N(t))^T$ be any positive solution of (2), and $(y_1(t), y_2(t), \cdots, y_r(t))^T$ be any positive solution of system (21). According to Theorem 3.3, for any $\varepsilon_4 > 0$ small enough satisfying

$$\bar{m}_i - \varepsilon_4 > 0 \quad \text{and} \quad \bar{\alpha}_{\varepsilon_4}^{\frac{1}{\xi}} e^{-\bar{\delta}_{\varepsilon_4}} < 1, \tag{24}$$

where

$$\begin{aligned}\bar{\alpha}_{\varepsilon_4} &= \max\Big\{1,\ \frac{d_{1M}(M_1+\varepsilon_4)}{h_{1L} + d_{1L}(\bar{m}_1 - \varepsilon_4)},\\ &\qquad \cdots,\ \frac{d_{NM}(M_N+\varepsilon_4)}{h_{NL} + d_{NL}(\bar{m}_N - \varepsilon_4)}\Big\},\\ \bar{\delta}_{\varepsilon_4} &= \min\Big\{\frac{\delta_i(\bar{m}_i - \varepsilon_4)}{\rho_i}\Big| i = 1,\ 2, \cdots, r\Big\},\end{aligned}$$

there exists a $T_4 > 0$, such that for $t > T_4$,

$$\begin{aligned}&\bar{m}_i - \varepsilon_4 \le x_i(t) \le M_i + \varepsilon_4, \quad i = 1,\ 2, \cdots, r,\\ &x_i(t) \le \varepsilon_4, \quad i > r,\\ &\sum_{i=1}^{r}\sum_{j=r+1}^{N} \bar{\rho}_i b_{ijM} x_j(t) \le \varepsilon_4.\end{aligned} \tag{25}$$

Using the mean value theorem, it follows that

$$\begin{aligned}\frac{1}{M_i + \varepsilon_4}|x_i(t) - y_i(t)| &\le |\ln x_i(t) - \ln y_i(t)|\\ &\le \tfrac{1}{\bar{m}_i - \varepsilon_4}|x_i(t) - y_i(t)|.\end{aligned}$$

Define a Lyapunov as follows

$$\bar{V}(t) = \sum_{i=1}^{r} \bar{\rho}_i |\ln x_i(t) - \ln y_i(t)|.$$

For $t > T_4$ and $t \neq t_k$, calculating the upper right derivatives of $\bar{V}(t)$, we have

$$\begin{aligned}&D^+\bar{V}(t)\\ =& \sum_{i=1}^{r} \bar{\rho}_i \text{sgn}(x_i(t) - y_i(t))\Big(\sum_{i=1}^{r} b_{ij}(t)y_j(t) - \sum_{i=1}^{N} b_{ij}(t)x_j(t)\Big)\\ =& \sum_{i=1}^{r} \bar{\rho}_i \text{sgn}(x_i(t) - y_i(t))\Big[b_{ii}(t)(y_i(t) - x_i(t))\\ &+ \sum_{j=1,i\neq j}^{r} b_{ij}(t)(y_j(t) - x_j(t)) - \sum_{j=r+1}^{N} b_{ij}(t)x_j(t)\Big]\\ \le& \sum_{i=1}^{r} -\bar{\delta}_i(\bar{m}_i - \varepsilon_4)|\ln x_i(t) - \ln y_i(t)| + \varepsilon_4\\ \le& -\bar{\delta}_{\varepsilon_4}\bar{V}(t) + \varepsilon_4,\end{aligned}$$

where $\delta_{\varepsilon_4} = \min\Big\{\frac{\bar{\delta}_i(\bar{m}_i - \varepsilon_4)}{\bar{\rho}_i}\Big| i = 1,\ 2, \cdots, N\Big\}$.

For $t = t_k$, similar to the analysis of Theorem 3.2, we can easily obtain

$$\begin{aligned}\bar{V}(t_k^+) &\le \max\Big\{1,\ \frac{d_{1M}(M_1+\varepsilon_4)}{h_{1L} + d_{1L}(\bar{m}_1 - \varepsilon_4)},\\ &\qquad \cdots,\ \frac{d_{rM}(M_r+\varepsilon_4)}{h_{rL} + d_{rL}(\bar{m}_r - \varepsilon_4)}\Big\}\bar{V}(t_k)\\ &\triangleq \bar{\alpha}_{\varepsilon_4}\bar{V}(t_k).\end{aligned}$$

From $\bar{\alpha}_{\varepsilon_4} \ge 1$ and (24), setting $t \to +\infty$ and $\varepsilon_4 \to 0$, there is

$$\begin{aligned}\bar{V}(t) &\le \bar{V}(T_4^+)\Big(\prod_{T_4 \le t_k < t} \bar{\alpha}_{\varepsilon_4}\Big) e^{-\bar{\delta}_{\varepsilon_4}(t - T_4)}\\ &\quad + \varepsilon_4 \int_{T_4}^{t} \Big(\bar{\alpha}_{\varepsilon_4}\Big)^{\frac{t-s}{\xi}+1} e^{-\bar{\delta}_{\varepsilon_4}(t-s)} ds\\ &= \bar{\alpha}_{\varepsilon_4}^{\frac{t-T_4}{\xi}+1} e^{-\bar{\delta}(t-T_4)}\bar{V}(T_4^+)\\ &\quad + \frac{\varepsilon_4\xi\bar{\alpha}_{\varepsilon_4}}{\bar{\delta}_{\varepsilon_4}\xi - \ln\bar{\alpha}_{\varepsilon_4}}\Big[1 - \Big(\bar{\alpha}_{\varepsilon_4}\Big)^{\frac{t-T_4}{\xi}} e^{-\bar{\delta}_{\varepsilon_4}(t-T_4)}\Big]\\ &\to 0,\end{aligned}$$

hence $\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0$, $i = 1,\ 2, \cdots, r$. The proof of the theorem is complete.

Similar to the proof of Theorem 2.3, we can easily verify the following result.

Theorem 3.5 *Let* $(x_1(t), x_2(t), \cdots, x_N(t))^T$ *be any solution of system* (2) *with* $x_i(0^+) > 0$ $(i = 1, 2, \cdots, N)$*. Assume that* $a_{iM}\xi - \ln h_{iL} \le 0$, $i = 1, 2, \cdots, N$, *then system* (2) *is extinct, that is*

$$\lim_{t\to+\infty} x_i(t) = 0, \quad i = 1, 2, \cdots, N.$$

IV. Numerical simulation

In this section, we first show the influence of the nonlinear impulse on dynamic behaviors of the logistic model.

TABLE I
DYNAMICS OF SYSTEM (1)

Case	a	b	h	d	θ	Figure
1	2	3	0.2	2.1	2	Fig. 1(a)
2	2	3	exp(2)	2.1	2	Fig. 1(b)
3	2	3	0.2	1.2	2	Fig. 1(c)
4	2	3	exp(5)	2.1	2	Fig. 1(d)

When $h < 1$, Fig. 1(a) shows that the species is permanent and globally attractive; especially when the relation $\frac{1-h}{d} = \frac{a}{b}$ holds, the species is globally attractive to a positive equilibrium (shown in Fig. 1(c)), which are in accordance with the results of Theorem 2.1 (I) and Theorem 2.2 respectively. But when $h > 1$, the dynamic behavior is more complex. Fig. 1(b) shows the global attractivity of the species, while Fig. 1(d) shows the extinction of the species; these behaviors can be verified by the results of Theorem 2.1 (II) and Theorem 2.3 respectively.

Next, we show the influence of the nonlinear impulses on dynamic behaviors of a three-species Lotka-Volterra model. Let its coefficients be $a_1(t) = 2.15 - 0.15\cos(\sqrt{2}t)$; $b_{11} = 9.2$; $b_{12} = 1.1 + 0.1\sin(\sqrt{3}t)$; $b_{13} = 0.95 + 0.05\sin(\sqrt{3}t)$; $a_2(t) = 1.35 + 0.05\sin(\sqrt{5}t)$; $b_{21}(t) = 0.9 - 0.1\sin(\sqrt{3}t)$; $b_{22} = 9.25 + 0.05\sin(\sqrt{2}t)$; $b_{23} = 0.4 - 0.1\cos(\sqrt{3}t)$; $a_3 = 3.55 + 0.05\cos(\sqrt{3}t)$; $b_{31} = 2.2$; $b_{32} = 5.1 + 0.1\cos(\sqrt{3}t)$; $b_{33} = 9.9$; $d_{1k} = 2.55 + 0.05\cos(\pi k)$; $d_{2k} = 2.3 + 0.2\cos(\pi k)$; $d_{3k} = 2.35 + 0.15\sin(\pi k)$; $\xi = 0.78$.

TABLE II
DYNAMICS OF SYSTEM (2)

Case	1	2
h_{1k}	$0.95 + 0.15\cos(\pi k)$	$0.95 + 0.15\cos(\pi k)$
h_{2k}	$0.55 + 0.05\sin(\pi k)$	$0.55 + 0.05\sin(\pi k)$
h_{3k}	$1.2 + 0.1\cos(\pi k)$	$17.1 + 0.1\cos(\pi k)$
Figure	Fig. 2	Fig. 3

Choose $\rho_1 = 1.9$, $\rho_2 = 1.2$, $\rho_3 = 1.5$, $\delta_1 = 12.9$, $\delta_2 = 0.95$ and $\delta_3 = 12.35$. Considering Case 1 in TABLE II, we can easily verify that all the conditions of Theorem 3.2 hold, then the species x_1, x_2 and x_3 are globally attractive, which are respectively shown in Fig. 2 (a)-(c). When only changing the values of h_{ik} presented by Case 2 in TABLE II, according to Theorem 3.4, the species x_1 and x_2 are also globally attractivity, but x_3 is extinct, which are respectively shown in Fig. 3 (a)-(c). In TABLES I and II, we keep the intrinsic growth rate and the inter-species competition rate unchanged but only adjust the values of the impulsive perturbation parameters, then simulations show that the permanence and extinction of the species are significantly changed.

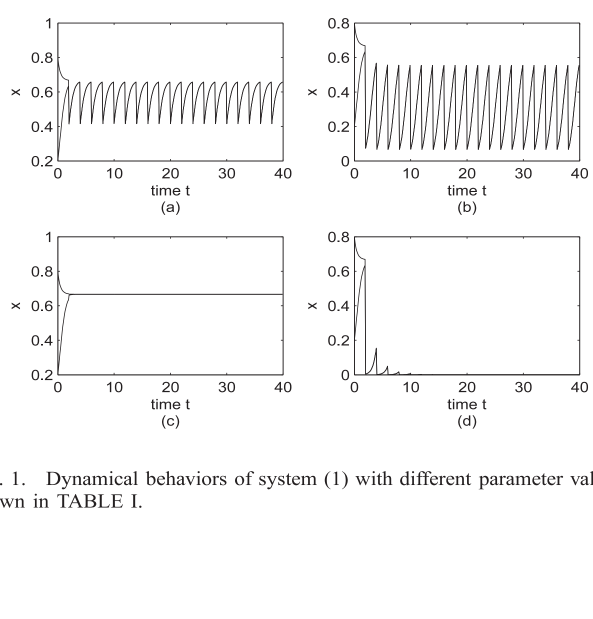

Fig. 1. Dynamical behaviors of system (1) with different parameter values shown in TABLE I.

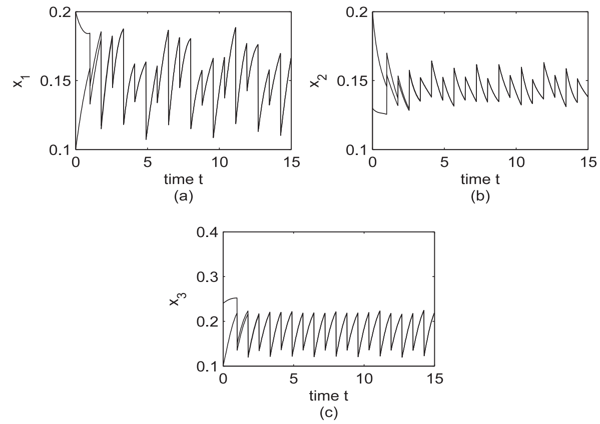

Fig. 2. Dynamics of the Lotka-Volterra model with Case 1 in TABLE II.

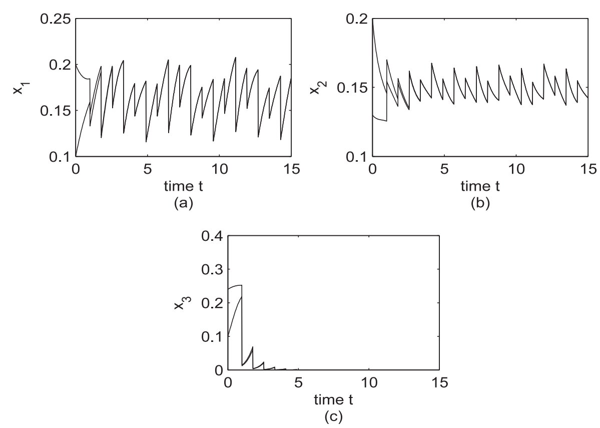

Fig. 3. Dynamics of the Lotka-Volterra model with Case 2 in TABLE II.

V. CONCLUSION

In this paper, we first consider a logistic model with nonlinear impulse. It is interesting that with this type of nonlinear impulse, the system admits a globally stable equilibrium, which is absolutely impossible for the model with linear impulse. When $h_k < 1$, the species is always permanent and globally attractive. However when $h_k > 1$, the behavior of the species is more complicated, which changes between the permanence and extinction according to the relationship between the impulsive perturbation parameter h_k and the intrinsic growth rate a. For a given intrinsic growth rate a, the species is extinct when the value of the impulsive perturbations h_k is large enough, but is globally attractive when h_k is small enough. However for a given $h_k > 1$, the opposite is the case.

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