Vertex-Distinguishing E-Total Coloring of Complete Bipartite Graph $K_{7,n}$ when $n \geq 978$

Xiang'en Chen

***Abstract*—Let G be a simple graph. A total coloring f of G is called an E-total coloring if no two adjacent vertices of G receive the same color, and no edge of G receives the same color as one of its endpoints. For an E-total coloring f of a graph G and any vertex x of G, let $C(x)$ denote the set of colors of vertex x and of the edges incident with x, we call $C(x)$ the color set of x. If $C(u) \neq C(v)$ for any two different vertices u and v of $V(G)$, then we say that f is a vertex-distinguishing E-total coloring of G or a VDET coloring of G for short. The minimum number of colors required for a VDET coloring of G is denoted by $\chi^e_{vt}(G)$ and is called the VDET chromatic number of G. The VDET coloring of complete bipartite graph $K_{7,n}$ $(n \geq 978)$ is discussed in this paper and the VDET chromatic number of $K_{7,n}$ $(n \geq 978)$ has been obtained.**

***Index Terms*—graph; complete bipartite graph, E-total coloring, vertex-distinguishing E-total coloring, vertex-distinguishing E-total chromatic number.**

I. Introduction and Notations

Graph theory is the historical foundation of the science of networks and the basis of information science. The problem in which we are interested is a particular case of the great variety of different ways of labeling a graph. The original motivation of studying this problem came from irregular networks. The idea was to weight the edges by positive integers such that the sum of the weights of edges incident with each vertex formed a set of distinct numbers.

For an edge coloring (proper or not) g of G and a vertex x of G, let $S(x)$ be the set (not multiset) of colors of the edges incident with x under g.

For a proper edge coloring, if $S(u) \neq S(v)$ for any two distinct vertices u and v, then the coloring is called a vertex-distinguishing proper edge coloring. The minimum number of colors required for a vertex-distinguishing proper edge coloring of G is denoted by $\chi'_s(G)$. This coloring is proposed in [5] and [4] independently. Many scholars have studied this parameter in [2], [3], [4], [5], [20], [21], [22].

For an edge coloring which is not necessary proper, if $S(u) \neq S(v)$ for any two distinct vertices u and v, then the coloring is called a point distinguishing edge coloring. The minimum number of colors required for a point distinguishing edge coloring of G is denoted by $\chi_0(G)$. This coloring is proposed in [15] by Harary et al. This parameter has been researched in many papers [6], [15], [16], [17], [18], [23], [24].

For a total coloring (proper or not) f of G and a vertex x of G, let $C(x)$ be the set (not multiset) of colors of vertex x and edges incident with x under f.

Manuscript received October 26, 2018; revised January 12, 2019. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11761064, 61163037).

Xiang'en Chen is with the College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, P R China. The corresponding author. Email: chenxe@nwnu.edu.cn; xiangenchen@163.com.

For a proper total coloring, if $C(u) \neq C(v)$, for any two distinct vertices u and v, then the coloring is called a vertex-distinguishing (proper) total coloring, or a VDT coloring of G for short. The minimum number of colors required for a VDT coloring of G is denoted by $\chi_{vt}(G)$.

The vertex distinguishing proper total colorings of graphs are introduced and studied by Zhongfu Zhang et al in [25]. After studying the vertex distinguishing proper total coloring of complete graph, star, complete bipartite graph, wheel, fan, path and cycle, a conjecture was proposed in [25]: let $\mu(G) = \min\{k :\geq n_i, \delta \leq i \leq \Delta\}$, then $\chi_{vt}(G) = \mu(G)$ or $\mu(G) + 1$. In [7], the vertex-distinguishing total coloring of n-cube were discussed, respectively. In [11], the relations of vertex distinguishing total chromatic numbers between a subgraph and its supergraph had been studied.

We will consider a kind of not necessarily proper total coloring which is vertex distinguishing. A total coloring f of G is called an E-total coloring if no two adjacent vertices of G receive the same color, and no edge of G receives the same color as one of its endpoints. If f is an E-total coloring of graph G and for any $u, v \in V(G), u \neq v$, we have $C(u) \neq C(v)$, then f is called a vertex-distinguishing E-total coloring, or a VDET coloring briefly. The minimum number of colors required for a VDET coloring of G is called the vertex-distinguishing E-total chromatic number of G and is denoted by $\chi^e_{vt}(G)$.

The VDET colorings of complete graph, complete bipartite graph $K_{2,n}$, star, wheel, fan, path and cycle were discussed in [14].

A parameter was introduced in [14]: $\eta(G) = \min\{l : \binom{l}{2} + \binom{l}{3} + \cdots + \binom{l}{i+1} \geq n_\delta + n_{\delta+1} + \cdots + n_i, 1 \leq \delta \leq i \leq \Delta\}$, n_i denote the number of vertices with degree $i, \delta \leq i \leq \Delta$. At the end of the paper [14], a Vizing-like conjecture was proposed.

Conjecture 1 ([14]) For a graph G with no isolated vertices and chromatic number at most 5, we have $\chi^e_{vt}(G) = \eta(G)$ or $\eta(G) + 1$.

We have studied the vertex-distinguishing E-total colorings of mC_3 and mC_4 in article [13] and confirmed Conjecture 1 for these two kinds of graphs.

The VDET chromatic numbers of complete bipartite graphs $K_{7,n}$ $(7 \leq n \leq 977)$ had been determined and Conjecture 1 is confirmed for $K_{7,n}$ $(7 \leq n \leq 977)$ in [8], [9], [10]. In this paper, we will consider the VDET coloring of complete bipartite graph $K_{7,n}$ $(n \geq 978)$ and confirm Conjecture 1 for $K_{7,n}$ $(n \geq 978)$.

For a vertex distinguishing E-total coloring f of a graph G and an element $z \in V(G) \cup E(G)$, we use $f(z)$ to denote the color of z under f.

Let $X = \{u_1, u_2, \cdots, u_7\}$, $Y = \{v_1, v_2, \cdots, v_n\}$, $V(K_{7,n}) = X \cup Y$ and $E(K_{7,n}) = \{u_iv_j : 1 \leq i \leq 7, 1 \leq$

$j \leq n\}$.

Given a vertex distinguishing E-total coloring f of $K_{7,n}$, let $C(X) = \{C(u_1), C(u_2), \cdots, C(u_7)\}$, $C(Y) = \{C(v_1), C(v_2), \cdots, C(v_n)\}$.

For a positive integer l, we use $[l]$ to denote the set $\{1, 2, \cdots, l\}$. If we mention an l-VDET coloring, then the colors which we have used are $1, 2, \cdots, l$. And for $c \in \{1, 2, \cdots, l\}$, we use $\overline{\{c\}}$ to denote $\{1, 2, \cdots, l\} \setminus \{c\}$, i.e., $\overline{\{c\}} = [l] \setminus \{c\}$. Generally for subset A of $[l]$, we use $\overline{A}$ to denote the complementary subset of A in $[l]$, i.e., $\overline{A} = [l] \setminus A$. A subset A of $\{1, 2, \cdots, l\}$ is called i-subset if A contain i elements, i.e., $|A| = i$.

For adjacent vertex distinguishing proper edge colorings of bicyclic graphs we may see [12]. For the hamiltonicity and hamiltonian connectivity of L-shaped supergrid graphs we may see [19]. For a particle swarm optimization (PSO) approach: shortest path planning algorithm, we may see [1].

II. Preliminaries

Lemma 1 ([8]) Let g be an l-VDET coloring of $K_{7,n}$. Suppose that there are two distinct colors among $g(u_1), g(u_2), \cdots$, and $g(u_7)$, and $g(u_i) \in \{1, 2\}, i = 1, 2, \cdots, 7$. If there exists a color $a \in \{3, 4, \cdots, l\}$ such that $\{1, 2, a\} \in C(Y)$, i.e., $\{1, 2, a\}$ is a color set of some vertex in Y, then $\{1, 2\} \subseteq C(u_i), i = 1, 2, \cdots, 7$.

Lemma 2([8]) Let g be an l-VDET coloring of $K_{7,n}$. Suppose that there are two distinct colors among $g(u_1), g(u_2), \cdots$, and $g(u_7)$, and $g(u_i) \in \{1, 2\}, i = 1, 2, \cdots, 7$. Let $a_1, a_2, \cdots, a_r$ be $r(\geq 2)$ distinct colors in $\{3, 4, \cdots, l\}$. If each 2-subset of $\{a_1, a_2, \cdots, a_r\}$ is a color set of a vertex in Y, then there exist $r - 1$ distinct colors in $\{a_1, a_2, \cdots, a_r\}$ such that these $r - 1$ distinct colors are contained in each set $C(u_i)$, $i = 1, 2, \cdots, 7$.

Lemma 3([8]) Let g be an l-VDET coloring of $K_{7,n}$. Suppose that there are two distinct colors among $g(u_1), g(u_2), \cdots$, and $g(u_7)$, and $g(u_i) \in \{1, 2\}, i = 1, 2, \cdots, 7$. Let $a_1, a_2, \cdots, a_r$ be r distinct colors in $\{3, 4, \cdots, l\}$.

(i) If $\{1, a_1, a_2\}, \{2, a_1, a_2\} \in C(Y)$, then each color set in $C(X)$ contains a_1 or a_2, i.e., $a_1 \in C(u_i)$ or $a_2 \in C(u_i)$, $i = 1, 2, \cdots, 7$;

(ii) Given $j \in \{1, 2\}$, if every 3-subset of $\{j, a_1, a_2, \cdots, a_r\}$ which contains color j belongs to $C(Y)$, then there exist $r-1$ distinct colors in $\{a_1, a_2, \cdots, a_r\}$ such that these $r - 1$ distinct colors are contained in each set $C(u_i)$ with $g(u_i) = j$;

(iii) If every 3-subset of $\{1, 2, a_1, a_2, \cdots, a_r\}$ which contains color 1 or 2 but not both belongs to $C(Y)$, then each set $C(u_i)$ contains at least $r - 1$ colors in $\{a_1, a_2, \cdots, a_r\}$, $i = 1, 2, \cdots, 7$.

Lemma 4([9]) $K_{7,472}$ has a 9-VDET coloring h_{472} such that (i) the color of u_i is 1 ($i = 1, 2, 3$) and the color of u_j is 2 ($j = 4, 5, 6, 7$); (ii) the color sets of $u_1, u_2, \cdots, u_7$ are $[9] \setminus \{3, 4\}$, $[9] \setminus \{3\}$, $[9] \setminus \{4\}$, $[9]$, $[9] \setminus \{5, 6\}$, $[9] \setminus \{5\}$ and $[9] \setminus \{6\}$ respectively; (iii) the color set of each vertex in Y is one of the following sets:

$\{3, 7\}$, $\{3, 8\}$, $\{3, 9\}$, $\{4, 7\}$, $\{4, 8\}$, $\{4, 9\}$, $\{5, 7\}$, $\{5, 8\}$, $\{5, 9\}$, $\{6, 7\}$, $\{6, 8\}$, $\{6, 9\}$, $\{7, 8\}$, $\{7, 9\}$, $\{8, 9\}$;

3-subsets of $[9]$ except for $\{1, 3, 4\}$, $\{2, 5, 6\}$;

i-subsets of $[9]$, $i = 4, 5, 6$;

7-subsets of $[9]$ except for $[9] \setminus \{3, 4\}$, $[9] \setminus \{5, 6\}$;

8-subsets of $[9]$ except for $[9] \setminus \{3\}$, $[9] \setminus \{4\}$, $[9] \setminus \{5\}$, $[9] \setminus \{6\}$.

TABLE I
The colorings of v_j and its incident edges

		u_1	u_2	u_3	u_4	u_5	u_6	u_7
v_{473}	3 10 (3)	10	10	10	10	10	10	10
v_{474}	1256789(5)	6	6	2	7	8	9	1
v_{475}	12456789(5)	6	9	2	7	8	1	4
v_{476}	12356789(5)	6	9	2	7	8	1	3
v_{477}	1234789(4)	2	9	2	7	8	1	3
v_{478}	12346789(6)	2	4	9	7	8	1	3
v_{479}	12345789(5)	2	4	9	7	8	1	3

Based on the 9-VDET coloring h_{472} stated in Lemma 4, we may give a 10-VDET coloring h_{975} of $K_{7,975}$ and a 11-VDET coloring h_{977} of $K_{7,977}$.

Let the subgraph of $K_{7,975}$ induced by $X \cup \{v_1, v_2, \cdots, v_{472}\}$ be colored using the above 9-VDET coloring h_{472} . And then color other vertices and their incident edges of $K_{7,975}$. Let v_i and its incident edges (i = 473, 474, $\cdots$, 479) be colored in the manner listed in Table I. Let vertex v_j receive color $j - 476$ and its edges receive color 10, $480 \leq j \leq 485$. Let vertices v_{486}, v_{487}, $\cdots$, v_{975} be corresponded to the following sets respectively: 3-subsets of $[10]$ which contain 10, 4-subsets of $[10]$ which contain 10, 5-subsets of $[10]$ which contain 10, 6-subsets of $[10]$ which contain 10, 7-subsets of $[10]$ which contain 10, 8-subsets of $[10]$ which contain 10 and are not $\{1, 2, 5, 6, 7, 8, 9, 10\}$, $\{1, 2, 3, 4, 7, 8, 9, 10\}$. We can color the vertex v_j ($j = 486, 487, \cdots, 975$) and its incident edges easily and omitted the process (or according to the method given in Table 5 where let $k = 10$ in [9]).

Then we determine h_{977}. Let the subgraph of $K_{7,977}$ induced by $X \cup \{v_1, v_2, \cdots, v_{975}\}$ be colored using the 10-VDET coloring h_{975}. And let v_{976} receive color 3 and all edges $u_i v_{976}$ receive color 11, let v_{977} receive color 4 and all edges $u_i v_{977}$ receive color 11. The resulting coloring h_{977} is obviously an 11-VDET coloring of $K_{7,977}$.

III. Main Result

Theorem 1 Suppose $k \geq 11, n \geq 978$. If

$$\sum_{i=2}^{8} \binom{k-1}{i} - 2k - 3 < n \leq \sum_{i=2}^{8} \binom{k}{i} - 2k - 5,$$

then $\chi_{vt}^{e}(K_{7,n}) = k$.

Proof Firstly, we prove that $K_{4,n}$ does not have a $(k-1)-$VDET coloring. Assume that $K_{4,n}$ has a $(k-1)-$VDET coloring g. There are three cases to consider.

Case 1 $u_1, u_2, \cdots, u_7$ receive the same color under g. We may suppose that $g(u_i) = 1, i = 1, 2, \cdots, 7$. So none of the $C(v_j)$ include color 1, and each $C(v_j)$ is one of the subsets of $\{2, 3, \cdots, k-1\}$. Let $\mathcal{A}$ be the set composed by the 8-, 7-, 6-, 5-, 4-, 3-, 2-subsets of $\{2, 3, \cdots, k-1\}$. Then $\mathcal{A}$ contains $\sum_{i=2}^{8} \binom{k-2}{i}$ members and $C(Y) \subseteq \mathcal{A}$. As

$$\begin{aligned} n &> \sum_{i=2}^{8} \binom{k-1}{i} - 2k - 3 \\ &= \sum_{i=2}^{8} \binom{k-2}{i} + \sum_{i=2}^{8} \binom{k-2}{i-1} - 2k - 3 \\ &\geq \sum_{i=2}^{8} \binom{k-2}{i} + 4(k-2) - 2k - 3 \\ &= \sum_{i=2}^{8} \binom{k-2}{i} + 2k - 11 > \sum_{i=2}^{8} \binom{k-2}{i} = |\mathcal{A}|. \end{aligned}$$

This is a contradiction.

Case 2 $u_1, u_2, \cdots, u_7$ receive two different colors under g. We may suppose that $g(u_i) \in \{1,2\}, i = 1,2,\cdots,7$. Each $C(v_j)$ does not include color i when $|C(v_j)| = 2, i = 1,2$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, where

$\mathcal{B}_1 = \{\{a,b\} | a,b \in \{3,4,\cdots,k-1\}, a < b\}$;

$\mathcal{B}_2 = \{\{1,2,c\} | c = 3,4,\cdots,k-1\}$;

$\mathcal{B}_3$ is the set composed by the 8-, 7-, 6-, 5-, 4-subsets of $[k-1]$ and 3-subsets of $[k-1]$ which are not in $\mathcal{B}_2$. Then $C(Y) \subseteq \mathcal{B}$ and

$$|\mathcal{B}| = \sum_{i=3}^{8} \binom{k-1}{i} + \binom{k-3}{2}.$$

By simple calculation we have that $|\mathcal{B}| - (\sum_{i=2}^{8} \binom{k-1}{i} - 2k - 3) = 8$. So we assume that $|\mathcal{B}| - 8 < n \leq |\mathcal{B}|$. At most seven subsets in $\mathcal{B}$ are not in $C(Y)$. Thus $\mathcal{B}_2 \cap C(Y) \neq \emptyset$ and, by Lemma 1, $1,2 \in C(u_i), i = 1,2,\cdots,7$.

1) If $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$ contains at most $k-8$ colors, there exist five colors $a_1, a_2, a_3, a_4, a_5 \in \{3,4,5,\cdots,k-1\}$, such that

$\{3,4,5,\cdots,k-1\} \setminus \{a_1,a_2,a_3,a_4,a_5\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$.

By Lemma 2, any 2-subsets of $\{a_1,a_2,a_3,a_4,a_5\}$ is not in $C(Y)$. Thus

$C(Y) \subseteq \mathcal{B} \setminus \{\{a_i,a_j\} | 1 \leq i < j \leq 5\}$.

So $n \leq |\mathcal{B}| - 10$, This is a contradiction.

2) If $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$ contains $k-7$ colors, there there exist four colors $a,b,c,d \in \{3,4,5,\cdots,k-1\}$, such that

$\{3,4,5,\cdots,k-1\} \setminus \{a,b,c,d\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$.

By Lemma 2, any 2-subsets of $\{a,b,c,d\}$ is not in $C(Y)$. Thus

$C(Y) \subseteq \mathcal{B} \setminus \{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$.

So $n \leq |\mathcal{B}| - 6$, and $n = |\mathcal{B}| - 6$ or $n = |\mathcal{B}| - 7$. Consider the following 12 subsets:

$\{1,a,b\},\{1,a,c\},\{1,a,d\},\{1,b,c\},\{1,b,d\},\{1,c,d\}$;

$\{2,a,b\},\{2,a,c\},\{2,a,d\},\{2,b,c\},\{2,b,d\},\{2,c,d\}$.

Therefore at least 11 subsets (possibly except for one, say $\{2,c,d\}$) are in $\in C(Y)$. From $\{1,a,b\}$, $\{1,a,c\}$, $\{1,a,d\}$, $\{1,b,c\}$, $\{1,b,d\}$, $\{1,c,d\} \in C(Y)$, by Lemma 3, we know that there exist three colors in $\{a,b,c,d\}$ which are contained in each $C(u_i)$ for color 1 vertex u_i, say $a,b,c \in C(u_i)$ for color 1 vertex u_i. So there are two choices for color 1 vertex u_i: $C(u_i) = \overline{\{d\}}$, or $[k-1]$. From $\{2,a,b\}$, $\{2,a,c\}$, $\{2,b,c\} \in C(Y)$, by Lemma 3, we know that there exist two colors in $\{a,b,c\}$ which are contained in each $C(u_i)$ for color 2 vertex u_i, say $a,b \in C(u_i)$ for color 2 vertex u_i. So there are four choices for color 2 vertex u_i, $C(u_i) \in \{[k-1] \setminus \{c,d\}, \overline{\{c\}}, \overline{\{d\}}, [k-1]\}$. This is a contradiction.

3) If $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$ contains $k-6$ colors, there there exist three colors $a,b,c \in \{3,4,5,\cdots,k-1\}$, such that

$\{3,4,5,\cdots,k-1\} \setminus \{a,b,c\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$.

Then $\{3,4,\cdots,k-1\} \setminus \{a,b,c\} \subseteq C(u_i), i = 1,2,\cdots,7$.

By Lemma 2, we have the following **Claim 1**.

Claim 1 any 2-subsets of $\{a,b,c\}$ is not in $C(Y)$.

Claim 2 $\{a,b,c\} \notin C(Y)$;

Otherwise if $\{a,b,c\} \in C(Y)$, and $C(v_{j_0}) = \{a,b,c\}, g(v_{j_0}) = a$, then each $C(u_i)$ contains b or c. So $C(X) \subseteq \{[k-1] \setminus \{a,b\}, [k-1] \setminus \{a,c\}, \overline{\{a\}}, \overline{\{b\}}, \overline{\{c\}}, [k-1]\}$. This is a contradiction.

Claim 3 One of two sets $\{1,a,b\}$ and $\{2,a,b\}$,denoted by B_1', is not in $C(Y)$; one of two sets $\{1,a,c\}$ and $\{2,a,c\}$, denoted by B_2', is not in $C(Y)$; one of two sets $\{1,b,c\}$ and $\{2,b,c\}$, denoted by B_3',is not in $C(Y)$.

Otherwise if $\{1,a,b\},\{2,a,b\} \in C(Y)$, then each $C(u_i)$ contains a or b. So $C(X) \subseteq \{[k-1] \setminus \{a,c\}, [k-1] \setminus \{b,c\}, \overline{\{c\}}, \overline{\{a\}}, \overline{\{b\}}, [k-1]\}$. This is a contradiction.

Thus exactly one subset among $\{1,a,b\}$ and $\{2,a,b\}$, denoted by B_1, is in $C(Y)$; exactly one subset among $\{1,a,c\}$ and $\{2,a,c\}$, denoted by B_2, is in $C(Y)$; exactly one subset among $\{1,b,c\}$ and $\{2,b,c\}$, denoted by B_3, is in $C(Y)$; $n = |\mathcal{B}| - 7$, and

$C(Y) = \mathcal{B} \setminus \{\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},B_1',B_2',B_3'\}$.

Claim 4 $\{1,a,b,c\},\{2,a,b,c\} \in C(Y)$, each $C(u_i)$ contains one of a,b,c, and $C(X) = \{[k-1] \setminus \{b,c\}, [k-1] \setminus \{a,c\}, \overline{\{a,b\}}, \overline{\{c\}}, \overline{\{b\}}, \overline{\{a\}}, [k-1]\}$.

By **Claim 4** we may suppose $C(u_1) = [k-1] \setminus \{b,c\}$, $C(u_2) = [k-1] \setminus \{a,c\}$, $C(u_3) = [k-1] \setminus \{a,b\}$, $C(v_1) = \{1,a,b,c\}$, $C(v_2) = \{2,a,b,c\}$. When $g(v_1) = g(v_2)$, say $g(v_1) = g(v_2) = a$, from $C(v_1) = \{1,a,b,c\}$ and $C(v_2) = \{2,a,b,c\}$ we can obtains that each $C(u_i)$ contains b or c, a contradiction to $C(u_1) = [k-1] \setminus \{b,c\}$. When $g(v_1) \neq g(v_2)$, say $g(v_1) = a, g(v_2) = b$, the color set of color 1 vertex in X contains b or c, the color set of color 2 vertex in X contains a or c. Then $g(u_1) = 2, g(u_2) = 1$. Thus $\{2,b,c\}$, $\{1,a,c\}$ are not the color sets of any vertices in Y. We may suppose that $C(v_3) = \{1,b,c\} = B_3$, $C(v_4) = \{2,a,c\} = B_2$. Thus $g(u_1v_3) = 1$, $g(u_2v_3) = b$, $g(v_3) = c$; $g(u_1v_4) = a$, $g(u_2v_4) = 2$, $g(v_4) = c$, and $g(u_3v_3) = 1$, $g(u_3v_4) = 2$. This is a contradiction to $g(u_3) \in \{1,2\}$.

4) If $C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$ contains at least $k-5$ colors, then there exist two colors $a,b \in \{3,4,5,\cdots,k-1\}$, such that

$\{3,4,5,\cdots,k-1\} \setminus \{a,b\} \subseteq C(u_1) \cap C(u_2) \cap \cdots \cap C(u_7) \cap \{3,4,5,\cdots,k-1\}$.

Then $\{3,4,\cdots,k-1\} \setminus \{a,b\} \subseteq C(u_i), i = 1,2,\cdots,7$. So $C(X) \subseteq \{[k-1] \setminus \{a,b\}, \overline{\{b\}}, \overline{\{a\}}, [k-1]\}$. This is a contradiction.

Case 3 $u_1, u_2, \cdots, u_4$ receive at least three different colors under g. We may suppose that $\{1,2,3\} \subseteq \{g(u_i) | i = 1,2,\cdots,7\}$. Each $C(v_j)$ does not include color i when $|C(v_j)| = 2, i = 1,2,3$, and each $C(v_j)$ is not $\{1,2,3\}$. Let $\mathcal{C}$ denote the set composed by the 8-, 7-, 6-, 5-, 4-subsets of $[k-1]$, 3-subsets of $[k-1]$ which is not $\{1,2,3\}$, 2-subsets of $\{4,5,\cdots,k-1\}$. Then $|\mathcal{C}| = \sum_{i=3}^{8} \binom{k-1}{i} + \binom{k-4}{2} - 1$. Thus

$$n \geq \sum_{i=2}^{8} \binom{k-1}{i} - 2k - 2 > \sum_{i=3}^{8} \binom{k-1}{i} + \binom{k-4}{2} - 1 = |\mathcal{C}|.$$

This is a contradiction to $C(Y) \subseteq \mathcal{C}$.

Hence $K_{4,n}$ does not have a $(k-1)$-VDET coloring.

Secondly, we will give a $k-$VDET coloring of $K_{4,n}$.

When $k = 11$, let the subgraph of $K_{7,1942}$ induced by $X \cup \{v_1, v_2, \cdots, v_{977}\}$ be colored using the 11-VDET coloring h_{977} given in Section 2. Let v_{978} and its incident edges u_1v_{978}, u_2v_{978}, $\cdots$, u_7v_{978} receive colors $5; 6, 10, 2, 7, 8, 9, 1$, respectively. Let v_{979} and its incident edges u_1v_{979}, u_2v_{979}, $\cdots$, u_7v_{979} receive colors $4; 2, 9, 10, 7, 8, 1, 3$, respectively. Let the vertices v_{980}, v_{981}, $\cdots$, v_{1942} be corresponded to the following subsets respectively: the 8-subsets of [11] which contains 11, the 7-subsets of [11] which contains 11, the 6-subsets of [11] which contains 11, the 5-subsets of [11] which contains 11, the

4-subsets of [11] which contains 11, the 3-subsets of [11] which contains 11, and $\{5, 11\}$, $\{6, 11\}$, $\{7, 11\}$, $\{8, 11\}$, $\{9, 11\}$, $\{10, 11\}$. And then color vertex v_j ($980 \le j \le 1942$) and its incident edges according to the manner listed in Table 2 where let $k = 11$. In such a way we have obtained an 11-VDET coloring h_{1942} of $K_{7,1942}$. The restriction of 11-VDET coloring h_{1942} of $K_{7,1942}$ on its subgraph induced by $\{u_1, u_2, \cdots, u_7, v_1, v_2, \cdots, v_i\}$ is obviously an 11-VDET coloring h_i when $978 \le i \le 1941$.

For $k = 12, 13, 14, \cdots$, we will execute the following algorithm recursively and then give a $k-$VDET coloring of $K_{4,n}$ when $k \ge 12$.

Let $s = \sum_{i=2}^{8} \binom{k-1}{i} - 2k - 3, t = \sum_{i=2}^{8} \binom{k}{i} - 2k - 5$. Note that s and t depend on k. Assume that $(k-1)$-VDET coloring h_s of $K_{4,s}$ has been constructed according to the method given in this proof. We arrange all 2-subsets, 3-subsets, 4-subsets, 5-subsets, 6-subsets, 7-subsets and 8-subsets of $\{1, 2, \cdots, k\}$ which contain k, except for $\{1, k\}$ and $\{2, k\}$, into a sequence $\mathcal{S}_k$. Then $\mathcal{S}_k$ has $\sum_{i=1}^{7} \binom{k-1}{i} - 2$ terms. Let the terms in $\mathcal{S}_k$ be corresponded to vertices $v_{s+1}, v_{s+2}, \cdots, v_t$. Then the subgraph of $K_{7,s}$ induced by $X \cup \{v_1, v_2, \cdots, v_s\}$ be colored using the $(k-1)$-VDET coloring h_s given in this proof, and then color each vertex v_j ($s+1 \le j \le t$) and its incident edges in the manner listed in Table 5 of reference [9]. The k-VDET coloring h_t of $K_{4,t}$ has been constructed. The restriction of k-VDET coloring h_t of $K_{7,t}$ on its subgraph induced by $\{u_1, u_2, \cdots, u_7, v_1, v_2, \cdots, v_j\}$ is obviously an k-VDET coloring h_j, where $s+1 \le j < t$.

The proof of Theorem 1 is completed.

IV. Conclusion

By simple computation, we may give the value of $\eta(K_{7,n})$ (see Table II)

From the results in [8], [9] and Theorem 1, we know that

1. If $n = 7, 8$, or $20 \le n \le 35$, or $51 \le n \le 95$, or $114 \le n \le 219$, or $241 \le n \le 472$, or $496 \le n \le 975$ or $\sum_{i=2}^{8} \binom{l-1}{i} + 1 \le n \le \sum_{i=2}^{8} \binom{l}{i} - 2l - 5, l \ge 11$, then $\chi_{vt}^{e}(K_{7,n}) = \eta(K_{7,n})$.

2. If $9 \le n \le 19$, or $36 \le n \le 50$, or $96 \le n \le 113$, or $220 \le n \le 240$, or $473 \le n \le 495$, or $n = 976, 977$ or $978 \le n \le 1002$, or $\sum_{i=2}^{8} \binom{l}{i} - 2l - 4 \le n \le \sum_{i=2}^{8} \binom{l}{i}, l \ge 11$, then $\chi_{vt}^{e}(K_{4,n}) = \eta(K_{4,n}) + 1$.

Thus Conjecture 1 is right for $K_{7,n}$ ($n \ge 7$).

Acknowledgment

The authors would like to thank the editor and anonymous reviewers fir their valueable suggestions.

TABLE II
The value of $\eta(K_{7,n})$

n	$\eta(K_{7,n})$
[7, 19]	5
[20, 50]	6
[51, 113]	7
[114, 240]	8
[241, 495]	9
[496, 1002]	10
$[\sum_{i=2}^{8} \binom{l-1}{i} + 1, \sum_{i=2}^{8} \binom{l}{i}]$, $l \ge 11$	l

References

[1] P. I. Adamu, J. T. Jegede, H. I. Okagbue, and P. E. Oguntunde, "Shortest Path Planning Algorithm - A Particle Swarm Optimization (PSO) Approach," *Proceedings of The World Congress on Engineering*, vol. 2235, pp19-24, 2018.

[2] P. N. Balister, A. Kostochka, H. Li, R. H. Schelp, "Balanced Edge Colorings," *J of Comb Theory, Series B*, vol. 90, pp. 3-20, 2004.

[3] P. N. Balister, O. M. Riordan, R. H. Schelp, "Vertex-Distinguishing Edge Colorings of Graphs," *J Graph Theory*, vol. 42, pp. 95-109, 2003.

[4] A. C. Burris, R. H. Schelp, " Vertex-Distinguish Proper Edge-Colorings," *J Graph Theory*, vol. 26, pp. 73-82, 1997.

[5] J. Černý, M. Horňák, R. Soták, " Observability of A Graphs," *Math Slovaca*, vol. 46, pp. 21-31, 1996.

[6] X. E. Chen, "Point-Distinguishing Chromatic Index of the Union of Paths," *Czechoslovak Mathematical Journal*,vol. 64, no. 3, pp. 629-640, 2014.

[7] X. E. Chen, "Asymptotic Behaviour of the Vertex-Distinguishing Totsl Chromatic Numbers of $n-$Cubes," *Journal of Northwest Normal University, Natural Science Edition*, vol. 41, no. 5, pp. 1-3, 2005.

[8] X. E. Chen. " Vertex-Distinguishing E-Total Coloring of Complete Bipartite Graph $K_{7,n}$ when $7 \le n \le 95$," *Communications in Matematical Research*, vol. 32, no. 4, pp. 359-374, 2016.

[9] X. E. Chen. " Vertex-Distinguishing E-Total Coloring of Complete Bipartite Graph $K_{7,n}$ when $96 \le n \le 472$," To appear *in Ars Combinatoria*, 2019.

[10] X. E. Chen. " Vertex-Distinguishing E-Total Coloring of Complete Bipartite Graph $K_{7,n}$ when $473 \le n \le 977$," To appear in *Journal of Combinatorial Mathematics and Combinatorial Computing*.

[11] X. E. Chen, Y. P. Gao, B. Yao, "Relations of Vertex Distinguishing Total Chromatic Numbers between A Subgraph and Its Supergraph," *Information Sciences*, vol, 288, pp. 246-253, 2014.

[12] X. E. Chen, and S. Y. Liu, "Adjacent Vertex Distinguishing Proper Edge Colorings of Bicyclic Graphs," *IAENG International Journal of Applied Mathematics*, vol. 48, no. 4, pp. 401-411, 2018.

[13] X. E. Chen, Y. Zu, "Vertex-Distinguishing E-Total Coloring of the Graphs mC_3 and mC_4," *Journal of Mathematical Research Exposition*, vol. 31, pp. 45-58, 2011.

[14] X. E. Chen, Y. Zu, Z. F. Zhang. Vertex-Distinguishing E-Total Colorings of Graphs," *Arab J Sci Eng*, vol. 36, pp. 1485-1500, 2011.

[15] F. Harary, M. Plantholt, "The Point-Distinguishing Chromatic Index," in *F. Harary, J. S. Maybee (Eds.), Graphs and Application, Wiley interscience, New York, 1985*, pp. 147-162.

[16] M. Horňák, R. Soták, " The Fifth Jump of the Point-Distinguishing Chromatic Index of $K_{n,n}$," *Ars Combinatoria*, vol. 42, pp. 233-242, 1996.

[17] M. Horňák, R. Soták, "Localization Jumps of the Point-Distinguishing Chromatic Index of $K_{n,n}$," *Discuss. Math. Graph Theory*, vol. 17, pp. 243-251, 1997.

[18] M. Horňák, N. Zagaglia Salvi, " On the Point-Distinguishing Chromatic Index of Complete Bipartite Graphs," *Ars Combinatoria*, vol. 80, pp. 75-85, 2006.

[19] R. W. Hung, J. L. Li, and C. -H. Lin, "The Hamiltonicity and Hamiltonian Connectivity of L-shaped Supergrid Graphs," *Proceedings of The International MultiConference of Engineers and Computer Scientists*, Vol.2233, pp 117-122, 2018.

[20] B. Liu, G. Z. Liu, " Vertex-Distinguishing Edge Colorings of Graphs with Degree Sum Conditions," *Graphs and Combinatorics* vol. 26, no.6, pp. 781-791, 2010.

[21] J. Rudašová, R. Soták, "Vertex-Distinguishing Proper Edge Colorings of Some Regular Graphs," *Discrete Math.*, vol. 308, pp. 795-802, 2008.

[22] K. Taczuk, M. Woźniak, "A note on the vertex-distinguishing index for some cubic graphs," *Opuscula Mathematica*, vol. 24, no. 2, pp. 223-229, 2004.

[23] N. Zagaglia Salvi, "On the Point-Distinguishing Chromatic Index of $K_{n,n}$," *Ars Combinatoria*, vol. 25B, pp. 93-104, 1988.

[24] N. Zagaglia Salvi. On the Value of the Point-Distinguishing Chromatic Index of $K_{n,n}$," *Ars Combinatoria*, vol. 29B, pp. 235-244, 1990.

[25] Z. F. Zhang, P. X. Qiu, J. W. Li, et al, "Vertex Distinguishing Total Colorings of Graphs," *Ars Combinatoria*, vol. 87, pp. 33-45, 2008.