Mixed (p, q)-Affine Surface Areas and Related Inequalities

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***Abstract*—In this paper, the mixed (p, q)-affine surface area is introduced. Its related inequalities, such as affine isoperimetric inequality, Blaschke-Santal*ó* inequality, monotonous inequality, cyclic inequality, and Brunn-Minkowski inequality, are established.**

***Index Terms*—mixed (p, q)-affine surface area, L_p affine surface area, affine surface area, Lutwak's inequalities.**

I. Introduction

Throughout let $\mathbf{R}^n$ denote n-dimensional Euclidean space. A convex body is a compact convex subset of $\mathbf{R}^n$ with nonempty interior. We denote by $\mathcal{K}^n$ the set of convex bodies, by $\mathcal{K}_c^n$ the set of convex bodies whose centroids lie at the origin, and by $\mathcal{K}_o^n$ the set of convex bodies containing the origin in their interiors. Denote by $V(K)$ the n-dimensional volume of a body K. For the standard unit ball B in $\mathbf{R}^n$, we write $\omega_n = V(B)$ to denote its volume. The unit sphere in $\mathbf{R}^n$ will be denoted by S^{n-1}.

For all $x \in \mathbf{R}^n \setminus \{0\}$, the support function of $K \in \mathcal{K}^n$ is defined by

$$h(K, x) = h_K(x) = \max\{x \cdot y : y \in K\},$$

where $x \cdot y$ denotes the standard inner product of x and y.

The radial function, $\rho_K = \rho(K, \cdot) : \mathbf{R}^n \setminus \{0\} \to [0, \infty)$, of a compact star-shaped (about the origin) set $K \subset \mathbf{R}^n$ is defined by

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then K is called a star body (about the origin). The set of all star bodies about the origin in $\mathbf{R}^n$ is denoted by $\mathcal{S}_o^n$. Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is an arbitrary nonempty subset of $\mathbf{R}^n$, then the set

$$E^* = \{x \in \mathbf{R}^n : x \cdot y \leq 1 \text{ for all } y \in E\}$$

is called the polar set of E. The polar set is always closed and convex and contains the origin.

The mixed volume $V_1(K, L)$ of convex bodies K, L is defined by

$$\begin{aligned} V_1(K, L) &:= \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(K, u), \end{aligned}$$

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where $S(K, \cdot)$ is the surface area measure of K.

The classical affine surface area, first introduced by Blaschke [3], has received a lot of attention in the last forty years (see e.g. [15], [16], [17], [18], [21], [24], [25], [26], [27], [35], [37], [41], [31], [32], [6]). In particular, based on the mixed volume Leichtweiß [16] defined the affine surface area, $\Omega(K)$, of a convex body $K \in \mathcal{K}^n$ by

$$n^{-\frac{1}{n}} \Omega(K)^{\frac{n+1}{n}} = \inf\{nV_1(K, Q^*)V(Q)^{\frac{1}{n}} : Q \in \mathcal{S}_o^n\}. \quad (1)$$

For $p \geq 1$, the L_p mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_o^n$ was defined, in [28], by

$$V_p(K, L) := \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon},$$

where $K +_p \varepsilon \cdot L$ is the L_p Minkowski-Firey combination, see [8], defined by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p.$$

It was shown in [28] that the L_p mixed volume has the following integral representation:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u), \quad (2)$$

where $S_p(K, \cdot)$ is the L_p surface area measure of K.

In 1996, Lutwak [29] extended the classical affine surface area to L_p affine surface area according to the L_p mixed volume. For $p \geq 1$ and $K \in \mathcal{K}_o^n$, the L_p affine surface area, denoted by $\Omega_p(K)$, is defined by

$$n^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K, Q^*)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}. \quad (3)$$

When $p = 1$, $\Omega_1(K)$ is just the classical affine surface area $\Omega(K)$.

From the definition of the L_p affine surface area, Lutwak [29] established the following well-known inequalities.

Theorem A (affine isoperimetric inequality). *If $K \in \mathcal{K}_c^n$ and $p \geq 1$, then*

$$\Omega_p(K)^{n+p} \leq n^{n+p} \omega_n^{2p} V(K)^{n-p}, \quad (4)$$

with equality if and only if K is an ellipsoid.

Theorem B (Blaschke-Santal*ó* inequality). *If $K \in \mathcal{K}_c^n$ and $p \geq 1$, then*

$$\Omega_p(K)\Omega_p(K^*) \leq (n\omega_n)^2, \quad (5)$$

with equality if and only if K is an ellipsoid.

Theorem C (monotonous inequality). *If $K \in \mathcal{F}_o^n$ and $1 \leq p \leq q$, then*

$$\left(\frac{\Omega_p(K)^{n+p}}{n^{n+p}V(K)^{n-p}}\right)^{\frac{1}{p}} \leq \left(\frac{\Omega_q(K)^{n+q}}{n^{n+q}V(K)^{n-q}}\right)^{\frac{1}{q}}, \quad (6)$$

with equality if and only if $K \in \{K \in \mathcal{F}_o^n : K^*$ *and* ΛK *are dilates*$\}$.

Here, $\mathcal{F}_o^n$ denotes the set of all bodies in $\mathcal{K}_o^n$ that has a positive continuous curvature function and ΛK denotes the curvature image of K.

Theorem D (cyclic inequality). *If* $K \in \mathcal{K}_o^n$ *and* $1 \le p < q < r$, *then*

$$\Omega_q(K)^{(n+q)(r-p)} \le \Omega_p(K)^{(n+p)(r-q)}\Omega_r(K)^{(n+r)(q-p)}. \quad (7)$$

The L_p affine surface area is closely related to the theory of valuations (see e.g. [1], [2], [14], [19], [21], [22], [48]) and the theory of information (see e.g. [13], [34], [43]). Recently, it was further extended to all $p \in \mathbf{R}$ via geometric interpretations (see [33], [38], [39], [42]), to the mixed L_p affine surface area (see [40], [44]), to the general affine surface area (see [22], [20]), to the general mixed affine surface area (see [45]) as well as to the Orlicz affine surface area (see [47], [46]). Important applications of the L_p affine surface area can be found in [11], [23].

Suppose $p, q \in \mathbf{R}$, $K \in \mathcal{K}_o^n$ and $L \in \mathcal{S}_o^n$. Lutwak, Yang and Zhang [30] defined the L_p dual curvature measures, $\widetilde{C}_{p,q}(K, L, \cdot)$, by

$$\begin{aligned} &\int_{S^{n-1}} g(v)d\widetilde{C}_{p,q}(K,L,v) \\ = \ &\frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_L(u)^{n-q}du \end{aligned}$$

for each continuous $g : S^{n-1} \to \mathbf{R}$, where α_K is the radial Gauss map (see Section 3 in [30]). Since they introduced the new concept of the L_p dual curvature measures, it quickly became the center of attention, see e.g., [7], [4], [5], [10], [12].

Using the L_p dual curvature measures, the (p, q)-mixed volume was given by Lutwak, Yang and Zhang in [30]. Suppose $p, q \in \mathbf{R}$. If $K, Q \in \mathcal{K}_o^n$, and $L \in \mathcal{S}_o^n$, then the (p, q)-mixed volume $\widetilde{V}_{p,q}(K, Q, L)$ is defined by

$$\widetilde{V}_{p,q}(K,Q,L) = \int_{S^{n-1}} h_Q^p(v)d\widetilde{C}_{p,q}(K,L,v). \quad (8)$$

The (p, q)-mixed volume can also be written as the following formula:

$$\begin{aligned} &\widetilde{V}_{p,q}(K,Q,L) \\ = \ &\frac{1}{n}\int_{S^{n-1}} \left(\frac{h_Q}{h_K}\right)^p(\alpha_K(u))\left(\frac{\rho_K}{\rho_L}\right)^q(u)\rho_L^n(u)du. \quad (9) \end{aligned}$$

In [30], Lutwak, Yang and Zhang extended formula (2) from $p \ge 1$ to all $p \in \mathbf{R}$. For $p \in \mathbf{R}$, the L_p mixed volume, $V_p(K, Q)$, of $K, Q \in \mathcal{K}_o^n$ is defined by

$$V_p(K,Q) = \frac{1}{n}\int_{S^{n-1}} h_Q^p(v)dS_p(K,v)$$

for $v \in S^{n-1}$. Moreover, for $q \in \mathbf{R}$ and $K, Q \in \mathcal{S}_o^n$ they also, in [30], defined the q-th dual mixed volume, $\widetilde{V}_q(K, Q)$, by

$$\widetilde{V}_q(K,Q) = \frac{1}{n}\int_{S^{n-1}} \rho_K^q(u)\rho_Q^{n-q}(u)du \quad (10)$$

for $u \in S^{n-1}$, where the integration is with respect to spherical Lebesgue measure.

Suppose $p, q \in \mathbf{R}$. In [30], the L_p mixed volume, the q-th dual mixed volume and the volume of a convex body were shown to be special cases of the (p, q)-mixed volumes:

$$\widetilde{V}_{p,q}(K,Q,K) = V_p(K,Q); \quad (11)$$

$$\widetilde{V}_{p,n}(K,Q,L) = V_p(K,Q); \quad (12)$$

$$\widetilde{V}_{p,q}(K,K,L) = \widetilde{V}_q(K,L); \quad (13)$$

$$\widetilde{V}_{p,q}(K,K,K) = V(K). \quad (14)$$

Motivated by the work of Lutwak, Yang and Zhang [30], in this paper we introduce the mixed (p, q)-affine surface area based on the (p, q)-mixed volume. The detailed descriptions are provided below.

Definition 1.1. *For* $p, q \in \mathbf{R}$, $K \in \mathcal{K}_o^n$ *and* $L \in \mathcal{S}_o^n$, *the mixed* (p, q)*-affine surface area,* $\widetilde{\Omega}_{p,q}(K, L)$, *is defined by*

$$\begin{aligned} &n^{-\frac{p}{n}}\widetilde{\Omega}_{p,q}(K,L)^{\frac{n+p}{n}} \\ = \ &\inf\{n\widetilde{V}_{p,q}(K,Q^*,L)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_o^n\}. \quad (15) \end{aligned}$$

If we take $L = K$ or $q = n$ in (15), then from (11) or (12) we see that the definition is just Lutwak's L_p affine surface area for $p > 1$ and Leichtweiß's affine surface area for $p = 1$.

For the mixed (p, q)-affine surface area, our first results are to establish some analogous inequalities of Theorems A-D. Then we prove a Brunn-Minkowski inequality.

Theorem 1.1. *Let* $p, q \in \mathbf{R}$ *be such that* $p > 0$ *and* $0 < q \le n$. *If* $K \in \mathcal{K}_c^n$ *and* $L \in \mathcal{S}_o^n$, *then*

$$\widetilde{\Omega}_{p,q}(K,L)^{n+p} \le n^{n+p}\omega_n^{2p}V(K)^{q-p}V(L)^{n-q}, \quad (16)$$

with equality if and only if K *and* L *are dilates, and* K *is an ellipsoid.*

If $L = K$ or $q = n$, then for $p \ge 1$, inequality (16) becomes inequality (4).

Theorem 1.2. *Let* $p, q \in \mathbf{R}$ *be such that* $0 < p < q \le n$. *If* $K, L \in \mathcal{K}_c^n$, *then*

$$\widetilde{\Omega}_{p,q}(K,L)\widetilde{\Omega}_{p,q}(K^*,L^*) \le (n\omega_n)^2, \quad (17)$$

with equality if and only if K *and* L *are dilates, and an ellipsoid, respectively.*

If $L = K$ or $q = n$, then for $p \ge 1$, inequality (17) becomes inequality (5).

Theorem 1.3. *Let* $q \in \mathbf{R}$. *If* $K \in \mathcal{K}_o^n$ *and* $L \in \mathcal{S}_o^n$, *then for* $0 < r < s$,

$$\left(\frac{\widetilde{\Omega}_{r,q}(K,L)^{n+r}}{n^{n+r}\widetilde{V}_q(K,L)^{n-r}}\right)^{\frac{1}{r}} \le \left(\frac{\widetilde{\Omega}_{s,q}(K,L)^{n+s}}{n^{n+s}\widetilde{V}_q(K,L)^{n-s}}\right)^{\frac{1}{s}}; \quad (18)$$

for $0 < s < r$, *the reverse inequality holds. In every inequality, equality holds if and only if* K *and* L *are dilates.*

If $L = K$ or $q = n$, then for $1 \le r < s$, inequality (18) becomes inequality (6).

Theorem 1.4. *Let* $q \in \mathbf{R}$. *If* $K \in \mathcal{K}_o^n$ *and* $L \in \mathcal{S}_o^n$, *then for* $0 < r < s < t$,

$$\begin{aligned} &\widetilde{\Omega}_{s,q}(K,L)^{(n+s)(t-r)} \\ \le \ &\widetilde{\Omega}_{r,q}(K,L)^{(n+r)(t-s)}\widetilde{\Omega}_{t,q}(K,L)^{(n+t)(s-r)}, \quad (19) \end{aligned}$$

with equality if and only if K *and* L *are dilates.*

If $L = K$ or $q = n$, then for $1 \le r < s < t$, inequality (19) becomes inequality (7).

Next, we are to establish the Brunn-Minkowski inequality for the mixed (p,q)-affine surface area.

Theorem 1.5. *Suppose $p,q \in \boldsymbol{R}$ are such that $0<\frac{n-q}{q}<1$ and $q \neq n$, and let $\lambda,\mu \in \boldsymbol{R}$. If $K \in \mathcal{K}_o^n$ and $L_1, L_2 \in \mathcal{S}_o^n$, then*

$$\begin{aligned} &\widetilde{\Omega}_{p,q}(K, \lambda \cdot L_1 \tilde{+}_q \mu \cdot L_2)^{\frac{q(n+p)}{n(n-q)}} \\ \geq\ & \lambda \widetilde{\Omega}_{p,q}(K, L_1)^{\frac{q(n+p)}{n(n-q)}} + \mu \widetilde{\Omega}_{p,q}(K, L_2)^{\frac{q(n+p)}{n(n-q)}}, \end{aligned} \tag{20}$$

with equality if and only if L_1 and L_2 are dilates.

Here, $\lambda \cdot L_1 \tilde{+}_q \mu \cdot L_2$ is the radial q-combination, see [30], defined by

$$\rho(\lambda \cdot L_1 \tilde{+}_q \mu \cdot L_2, \cdot)^q = \lambda \rho(L_1, \cdot)^q + \mu \rho(L_2, \cdot)^q, \tag{21}$$

for $q \neq 0$.

II. Preliminaries

In the following we collect some basic facts about convex bodies. Good references are the books of Gardner [9] and Schneider [36].

The support and radial functions of a convex body $K \in \mathcal{K}_o^n$ and its polar set are related by

$$\rho_K = 1/h_{K^*} \quad \text{and} \quad h_K = 1/\rho_{K^*}. \tag{22}$$

From the definition of polar set, it is easily verified that for all $K \in \mathcal{K}_o^n$,

$$K^{**} = K. \tag{23}$$

The well-known Blaschke-Santal*ó* inequality states that if $K \in \mathcal{K}_c^n$, then

$$V(K)V(K^*) \leq \omega_n^2, \tag{24}$$

with equality if and only if K is an ellipsoid.

The polar coordinate formula for the volume of a compact set K is

$$V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(u)du. \tag{25}$$

From formula (10) and the H*ö*lder inequality, it easily follows that if $K, L \in \mathcal{S}_o^n$, then for $0<q\leq n$,

$$\widetilde{V}_q(K,L) \leq V(K)^{\frac{q}{n}} V(L)^{\frac{n-q}{n}}, \tag{26}$$

with equality if and only if K and L are dilates.

III. Proofs of Theorems 1.1-1.5

In this section, we will give the proofs of Theorems 1.1-1.5.

Proof of Theorem 1.1. It follows from (15) that for all $Q \in \mathcal{S}_o^n$,

$$n^{-\frac{p}{n}} \widetilde{\Omega}_{p,q}(K,L)^{\frac{n+p}{n}} \leq n\widetilde{V}_{p,q}(K, Q^*, L) V(Q)^{\frac{p}{n}}. \tag{27}$$

For $K \in \mathcal{K}_c^n$, we let $Q = K^*$ in (27). Then from (23) and (13) we see

$$n^{-\frac{p}{n}} \widetilde{\Omega}_{p,q}(K,L)^{\frac{n+p}{n}} \leq n\widetilde{V}_q(K,L)V(K^*)^{\frac{p}{n}}.$$

Together with inequality (26), we get that for $0<q\leq n$,

$$n^{-\frac{p}{n}} \widetilde{\Omega}_{p,q}(K,L)^{\frac{n+p}{n}} \leq nV(K)^{\frac{q}{n}} V(L)^{\frac{n-q}{n}} V(K^*)^{\frac{p}{n}},$$

i.e.,

$$\widetilde{\Omega}_{p,q}(K,L)^{n+p} \leq n^{n+p} V(K)^q V(L)^{n-q} V(K^*)^p. \tag{28}$$

From (24), we obtain that for $p>0$

$$V(K^*)^p \leq \omega_n^{2p} V(K)^{-p}. \tag{29}$$

Combining (28) and (29), it follows that

$$\widetilde{\Omega}_{p,q}(K,L)^{n+p} \leq n^{n+p}\omega_n^{2p} V(K)^{q-p} V(L)^{n-q}.$$

From the equality conditions of inequalities (26) and (24), we see that equality holds in inequality (16) if and only if K and L are dilates, and K is an ellipsoid.

Proof of Theorem 1.2. For $p>0$ and $0<q\leq n$, we know that for $K, L \in \mathcal{K}_c^n$,

$$\widetilde{\Omega}_{p,q}(K,L)^{n+p} \leq n^{n+p}\omega_n^{2p} V(K)^{q-p} V(L)^{n-q} \tag{30}$$

and

$$\widetilde{\Omega}_{p,q}(K^*,L^*)^{n+p} \leq n^{n+p}\omega_n^{2p} V(K^*)^{q-p} V(L^*)^{n-q}. \tag{31}$$

Together with (30) and (31), it follows from (24) that for $q>p$,

$$\widetilde{\Omega}_{p,q}(K,L)\widetilde{\Omega}_{p,q}(K^*,L^*) \leq (n\omega_n)^2.$$

From the equality condition of inequality (16), we know that equality holds in inequality (17) if and only if K and L are dilates, and an ellipsoid, respectively.

Proof of Theorem 1.3. By (9), the H*ö*lder inequality, (9) again, and definition (10), it follows that for $\frac{s}{r}>1$,

$$\begin{aligned} &\widetilde{V}_{s,q}(K,Q^*,L) \\ =\ & \frac{1}{n}\int_{S^{n-1}} \left(\frac{h_{Q^*}}{h_K}\right)^s (\alpha_K(u)) \left(\frac{\rho_K}{\rho_L}\right)^q (u)\rho_L^n(u)du \\ =\ & \frac{1}{n}\int_{S^{n-1}} \left[\left(\frac{h_{Q^*}}{h_K}\right)^r (\alpha_K(u)) \left(\frac{\rho_K}{\rho_L}\right)^q (u)\rho_L^n(u)\right]^{\frac{s}{r}} \\ & \times \left[\left(\frac{\rho_K}{\rho_L}\right)^q (u)\rho_L^n(u)\right]^{1-\frac{s}{r}} du \\ \geq\ & \left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{h_{Q^*}}{h_K}\right)^r (\alpha_K(u)) \left(\frac{\rho_K}{\rho_L}\right)^q (u)\rho_L^n(u)du\right]^{\frac{s}{r}} \\ & \times \left[\frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L}\right)^q (u)\rho_L^n(u)du\right]^{1-\frac{s}{r}} \\ =\ & \widetilde{V}_{r,q}(K,Q^*,L)^{\frac{s}{r}} \widetilde{V}_q(K,L)^{1-\frac{s}{r}}. \end{aligned}$$

For $s>0$, this has

$$\left(\frac{\widetilde{V}_{s,q}(K,Q^*,L)}{\widetilde{V}_q(K,L)}\right)^{\frac{1}{s}} \geq \left(\frac{\widetilde{V}_{r,q}(K,Q^*,L)}{\widetilde{V}_q(K,L)}\right)^{\frac{1}{r}}. \tag{32}$$

It follows from the equality condition of the H*ö*lder inequality that equality in (32) holds if and only if K, Q^*, L are dilates.

From definition (15) and inequality (32) we see that for $r>0$,

$$\begin{aligned} &\left(\frac{\widetilde{\Omega}_{r,q}(K,L)^{n+r}}{n^{n+r}\widetilde{V}_q(K,L)^{n-r}}\right)^{\frac{1}{r}} \\ =\ & \inf\left\{\left(\frac{\widetilde{V}_{r,q}(K,Q^*,L)}{\widetilde{V}_q(K,L)}\right)^{\frac{n}{r}} \widetilde{V}_q(K,L)V(Q) : Q \in \mathcal{S}_o^n\right\} \\ \leq\ & \inf\left\{\left(\frac{\widetilde{V}_{s,q}(K,Q^*,L)}{\widetilde{V}_q(K,L)}\right)^{\frac{n}{s}} \widetilde{V}_q(K,L)V(Q) : Q \in \mathcal{K}_o^n\right\} \\ =\ & \left(\frac{\widetilde{\Omega}_{s,q}(K,L)^{n+s}}{n^{n+s}\widetilde{V}_q(K,L)^{n-s}}\right)^{\frac{1}{s}}. \end{aligned}$$

This is, for $0 < r < s$

$$\left(\frac{\widetilde{\Omega}_{r,q}(K,L)^{n+r}}{n^{n+r}\widetilde{V}_q(K,L)^{n-r}}\right)^{\frac{1}{r}} \leq \left(\frac{\widetilde{\Omega}_{s,q}(K,L)^{n+s}}{n^{n+s}\widetilde{V}_q(K,L)^{n-s}}\right)^{\frac{1}{s}}.$$

The equality condition of inequality (32) implies that the equality of the inequality holds if and only if K, L are dilates.

Similarly, for $0 < s < r$ the reverse inequality can be obtained.

Proof of Theorem 1.4. For any $Q_1, Q_2 \in \mathcal{S}_o^n$, we define $Q_3 \in \mathcal{S}_o^n$ by

$$\rho_{Q_3}^{s(t-r)} = \rho_{Q_1}^{r(t-s)}\rho_{Q_2}^{t(s-r)}. \tag{33}$$

This has

$$\rho_{Q_3}^n = \left(\rho_{Q_1}^n\right)^{\frac{r(t-s)}{s(t-r)}}\left(\rho_{Q_2}^n\right)^{\frac{t(s-r)}{s(t-r)}}. \tag{34}$$

It follows from (33) and (22) that

$$h_{Q_3^*}^{s(t-r)} = h_{Q_1^*}^{r(t-s)}h_{Q_2^*}^{t(s-r)}. \tag{35}$$

From (25), (34), the Hölder inequality, and (25) again, we have that for $0 < r < s < t$ or $0 < t < s < r$ or $t < s < r < 0$ or $r < s < t < 0$,

$$\begin{aligned} V(Q_3) &= \frac{1}{n}\int_{S^{n-1}}\rho_{Q_3}^n(u)du \\ &= \frac{1}{n}\int_{S^{n-1}}\left(\rho_{Q_1}^n(u)\right)^{\frac{r(t-s)}{s(t-r)}}\left(\rho_{Q_2}^n(u)\right)^{\frac{t(s-r)}{s(t-r)}}du \\ &\leq \left(\frac{1}{n}\int_{S^{n-1}}\rho_{Q_1}^n(u)du\right)^{\frac{r(t-s)}{s(t-r)}} \\ &\quad\times\left(\frac{1}{n}\int_{S^{n-1}}\rho_{Q_2}^n(u)du\right)^{\frac{t(s-r)}{s(t-r)}} \\ &= V(Q_1)^{\frac{r(t-s)}{s(t-r)}}V(Q_2)^{\frac{t(s-r)}{s(t-r)}}. \end{aligned}$$

For $s(t-r) > 0$, this implies

$$V(Q_3)^{s(t-r)} \leq V(Q_1)^{r(t-s)}V(Q_2)^{t(s-r)}. \tag{36}$$

In addition, by (9), the Hölder inequality and (9) again, we get that for $r < s < t$ or $t < s < r$,

$$\begin{aligned} &\widetilde{V}_{s,q}(K,Q_3^*,L) \\ =& \frac{1}{n}\int_{S^{n-1}}\left(\frac{h_{Q_3^*}}{h_K}\right)^s(\alpha_K(u))\left(\frac{\rho_K}{\rho_L}\right)^q(u)\rho_L^n(u)du \\ =& \frac{1}{n}\int_{S^{n-1}}\left[\left(\frac{h_{Q_1^*}}{h_K}\right)^r(\alpha_K(u))\left(\frac{\rho_K}{\rho_L}\right)^q(u)\rho_L^n(u)\right]^{\frac{t-s}{t-r}} \\ &\times\left[\left(\frac{h_{Q_2^*}}{h_K}\right)^t(\alpha_K(u))\left(\frac{\rho_K}{\rho_L}\right)^q(u)\rho_L^n(u)\right]^{\frac{s-r}{t-r}}du \\ \leq& \left[\frac{1}{n}\int_{S^{n-1}}\left(\frac{h_{Q_1^*}}{h_K}\right)^r(\alpha_K(u))\left(\frac{\rho_K}{\rho_L}\right)^q(u)\right. \\ &\left.\times\,\rho_L^n(u)du\right]^{\frac{t-s}{t-r}} \\ &\times\left[\frac{1}{n}\int_{S^{n-1}}\left(\frac{h_{Q_2^*}}{h_K}\right)^t(\alpha_K(u))\left(\frac{\rho_K}{\rho_L}\right)^q(u)\right. \\ &\left.\times\,\rho_L^n(u)du\right]^{\frac{s-r}{t-r}} \\ =& \widetilde{V}_{r,q}(K,Q_1^*,L)^{\frac{t-s}{t-r}}\widetilde{V}_{t,q}(K,Q_2^*,L)^{\frac{s-r}{t-r}}. \end{aligned}$$

This implies that for $t > r$,

$$\begin{aligned} &\widetilde{V}_{s,q}(K,Q_3^*,L)^{t-r} \\ \leq& \widetilde{V}_{r,q}(K,Q_1^*,L)^{t-s}\widetilde{V}_{t,q}(K,Q_2^*,L)^{s-r}. \end{aligned} \tag{37}$$

From (36) and (37), it follows that for $0 < r < s < t$,

$$\begin{aligned} &\left[\widetilde{V}_{s,q}(K,Q_3^*,L)V(Q_3)^{\frac{s}{n}}\right]^{t-r} \\ \leq& \left[\widetilde{V}_{r,q}(K,Q_1^*,L)V(Q_1)^{\frac{r}{n}}\right]^{t-s} \\ &\times\left[\widetilde{V}_{t,q}(K,Q_2^*,L)V(Q_2)^{\frac{t}{n}}\right]^{s-r}. \end{aligned} \tag{38}$$

From (15) and (38), we have that for $0 < r < s < t$,

$$\begin{aligned} &\left[n^{-\frac{s}{n}}\widetilde{\Omega}_{s,q}(K,L)^{\frac{n+s}{n}}\right]^{t-r} \\ =& \inf\left\{\left[n\widetilde{V}_{s,q}(K,Q_3^*,L)V(Q_3)^{\frac{s}{n}}\right]^{t-r} : Q_3 \in \mathcal{S}_o^n\right\} \\ \leq& \inf\left\{\left[n\widetilde{V}_{r,q}(K,Q_1^*,L)V(Q_1)^{\frac{r}{n}}\right]^{t-s} : Q_1 \in \mathcal{S}_o^n\right\} \\ \times& \inf\left\{\left[n\widetilde{V}_{t,q}(K,Q_2^*,L)V(Q_2)^{\frac{t}{n}}\right]^{s-r} : Q_2 \in \mathcal{S}_o^n\right\} \\ =& \left[n^{-\frac{r}{n}}\widetilde{\Omega}_{r,q}(K,L)^{\frac{n+r}{n}}\right]^{t-s} \\ &\times\left[n^{-\frac{t}{n}}\widetilde{\Omega}_{t,q}(K,L)^{\frac{n+t}{n}}\right]^{s-r}, \end{aligned}$$

i.e.,

$$\begin{aligned} &\widetilde{\Omega}_{s,q}(K,L)^{(n+s)(t-r)} \\ \leq& \widetilde{\Omega}_{r,q}(K,L)^{(n+r)(t-s)}\widetilde{\Omega}_{t,q}(K,L)^{(n+t)(s-r)}. \end{aligned}$$

The equality condition of (19) directly follows from the equality condition of the Hölder inequality.

Proof of Theorem 1.5. By (9), (21), and the Minkowski's integral inequality, we get that for $0 < \frac{n-q}{q} < 1$, $q \neq n$ and any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned} &\widetilde{V}_{p,q}(K,Q^*,\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2)^{\frac{q}{n-q}} \\ =& \left[\frac{1}{n}\int_{S^{n-1}}\left(\frac{h_{Q^*}}{h_K}\right)^p(\alpha_K(u))\rho_K^q(u)\right. \\ &\left.\times\,\rho_{\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2}^{n-q}(u)du\right]^{\frac{q}{n-q}} \\ =& \left[\frac{1}{n}\int_{S^{n-1}}\left[\left(\frac{h_{Q^*}}{h_K}\right)^{\frac{pq}{n-q}}(\alpha_K(u))\rho_K^{\frac{q^2}{n-q}}(u)\right.\right. \\ &\left.\left.\times\,\rho_{\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2}^q(u)\right]^{\frac{n-q}{q}}du\right]^{\frac{q}{n-q}} \\ =& \left[\frac{1}{n}\int_{S^{n-1}}\left[\left(\frac{h_{Q^*}}{h_K}\right)^{\frac{pq}{n-q}}(\alpha_K(u))\rho_K^{\frac{q^2}{n-q}}(u)\right.\right. \\ &\left.\left.\times\,\left(\lambda\rho_{L_1}^q(u)+\mu\rho_{L_2}^q(u)\right)\right]^{\frac{n-q}{q}}du\right]^{\frac{q}{n-q}} \\ \geq& \lambda\widetilde{V}_{p,q}(K,Q^*,L_1)^{\frac{q}{n-q}}+\mu\widetilde{V}_{p,q}(K,Q^*,L_2)^{\frac{q}{n-q}}, \end{aligned}$$

i.e.,

$$\begin{aligned} &\widetilde{V}_{p,q}(K,Q^*,\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2)^{\frac{q}{n-q}} \\ \geq& \lambda\widetilde{V}_{p,q}(K,Q^*,L_1)^{\frac{q}{n-q}}+\mu\widetilde{V}_{p,q}(K,Q^*,L_2)^{\frac{q}{n-q}}. \end{aligned} \tag{39}$$

From the equality condition of the Minkowski's integral inequality, we see that equality holds in (39) if and only if L_1 and L_2 are dilates.

Moreover, based on (15) and (39) it follows that

$$\begin{aligned}
&\left[n^{-\frac{p}{n}}\widetilde{\Omega}_{p,q}(K,\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2)^{\frac{n+p}{n}}\right]^{\frac{q}{n-q}}\\
=\;&\inf\left\{\left[n\widetilde{V}_{p,q}(K,Q^*,\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2)\right]^{\frac{q}{n-q}}\right.\\
&\left.\times\left[V(Q)^{\frac{p}{n}}\right]^{\frac{q}{n-q}}:Q\in\mathcal{S}_o^n\right\}\\
\geq\;&\lambda\inf\left\{\left[n\widetilde{V}_{p,q}(K,Q^*,L_1)\right]^{\frac{q}{n-q}}\left[V(Q)^{\frac{p}{n}}\right]^{\frac{q}{n-q}}:\right.\\
&\left.Q\in\mathcal{S}_o^n\right\}\\
&+\mu\inf\left\{\left[n\widetilde{V}_{p,q}(K,Q^*,L_2)\right]^{\frac{q}{n-q}}\left[V(Q)^{\frac{p}{n}}\right]^{\frac{q}{n-q}}:\right.\\
&\left.Q\in\mathcal{S}_o^n\right\}\\
=\;&\lambda\left[n^{-\frac{p}{n}}\widetilde{\Omega}_{p,q}(K,L_1)^{\frac{n+p}{n}}\right]^{\frac{q}{n-q}}\\
&+\mu\left[n^{-\frac{p}{n}}\widetilde{\Omega}_{p,q}(K,L_2)^{\frac{n+p}{n}}\right]^{\frac{q}{n-q}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\widetilde{\Omega}_{p,q}(K,\lambda\cdot L_1\tilde{+}_q\mu\cdot L_2)^{\frac{q(n+p)}{n(n-q)}}\\
\geq\;&\lambda\widetilde{\Omega}_{p,q}(K,L_1)^{\frac{q(n+p)}{n(n-q)}}+\mu\widetilde{\Omega}_{p,q}(K,L_2)^{\frac{q(n+p)}{n(n-q)}}.
\end{aligned}$$

The equality condition of the above inequality follows from the equality condition of inequality (39).

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