Some Inequalities of Hermite–Hadamard Type for a New Kind of Convex Functions on Coordinates

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***Abstract*—In the paper, the authors introduce a new concept "(α, m_1)-(s, m_2)-convex functions on coordinates on the rectangle of the plane" and establish some inequalities of the Hermite–Hadamard type for this kind of functions.**

***Index Terms*—Coordinates, inequality of Hermite–Hadamard type, (α, m_1)-(s, m_2)-convex function.**

I. Introduction

LET us simply recall basic definitions and closely related results.

Definition I.1 ([1]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on coordinates on Δ if the partial mappings

$$f_y : u \in [a, b] \to f_y(u, y) \in \mathbb{R}$$

and

$$f_x : v \in [c, d] \to f_x(x, v) \in \mathbb{R}$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

An alternative statement of Definition I.1 may be recited as follows.

Definition I.2 ([2], [3]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on coordinates on Δ if

$$\begin{aligned} f(tx + (1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x, y) \\ +t(1-\lambda)f(x, w) + (1-t)\lambda f(z, y) \\ +(1-t)(1-\lambda)f(z, w) \end{aligned}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

In [1], S. S. Dragomir established the following theorem.

Theorem I.1 ([1, Theorem 2.2]). *Let $f : \Delta = [a, b] \times [c, d]$ be convex on coordinates on Δ. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}x \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}y\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &\le \frac{1}{4}\left\{\frac{1}{b-a}\left[\int_a^b f(x, c) \,\mathrm{d}x + \int_a^b f(x, d) \,\mathrm{d}x\right] \right. \\ &\quad \left. + \frac{1}{d-c}\left[\int_c^d f(a, y) \,\mathrm{d}y + \int_c^d f(b, y) \,\mathrm{d}y\right]\right\} \\ &\le \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \end{aligned}$$

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For more information on integral inequalities of Hermite–Hadamard type for various kinds of convex functions, please refer to the recently published papers [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], and closely related references therein.

In this paper, we introduce a new notion "(α, m_1)-(s, m_2)-convex functions on coordinates on the rectangle of the plane" and establish some new inequalities of Hermite–Hadamard type for this kind of functions.

II. A definition and lemmas

Now we introduce the new concept "(α, m_1)-(s, m_2)-convex functions on coordinates on the rectangle of the first quadrant $\mathbb{R}_0^2$".

Definition II.1. For $m_1, m_2, \alpha \in (0, 1]$ and $s \in [-1, 1]$, a function $f : [0, b] \times [0, d] \to \mathbb{R}$ is called co-ordinated (α, m_1)-(s, m_2)-convex if

$$\begin{aligned} f(tx + m_1(1-t)z, \lambda y + m_2(1-\lambda)w) \le t^\alpha \lambda^s f(x, y) \\ +m_1(1-t^\alpha)\lambda^s f(z, y) + m_2 t^\alpha (1-\lambda)^s f(x, w) \\ +m_1 m_2 (1-t^\alpha)(1-\lambda)^s f(z, w) \end{aligned}$$

holds for all $(t, \lambda) \in [0, 1] \times (0, 1)$ and $(x, y), (z, w) \in [0, b] \times [0, d]$.

In order to prove some inequalities of Hermite–Hadamard type for this class of functions, we need the following lemmas.

Lemma II.1. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ have partial derivatives of the second order. If $f_{12} \in L(\Delta)$ and $\xi, \mu \in \mathbb{R}$, then*

$$\begin{aligned} S(f, \xi, \mu) &\triangleq (1+\xi)(1+\mu)f(a, c) - (1+\xi)\mu f(a, d) \\ &- \xi(1+\mu)f(b, c) + \xi\mu f(b, d) - \frac{1}{b-a}\int_a^b [(1+\mu)f(x, c) \\ &- \mu f(x, d)] \,\mathrm{d}x - \frac{1}{d-c}\int_c^d [(1+\xi)f(a, y) - \xi f(b, y)] \,\mathrm{d}y \\ &+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &= (b-a)(d-c)\int_0^1\int_0^1 (t+\xi)(\lambda+\mu) f_{12}(ta + (1-t)b, \\ &\qquad \lambda c + (1-\lambda)d) \,\mathrm{d}t \,\mathrm{d}\lambda, \quad (1) \end{aligned}$$

where

$$f_1(x,y)\triangleq\frac{\partial f(x,y)}{\partial x},\quad f_2(x,y)\triangleq\frac{\partial f(x,y)}{\partial y},$$

$$f_{12}(x,y)\triangleq\frac{\partial^2 f(x,y)}{\partial x\partial y},\quad f_{12}(a,b)\triangleq f_{12}(x,y)|_{x=a,y=b}.$$

Proof: By integration by parts, we have

$$\begin{aligned}
&\int_0^1\int_0^1(t+\xi)(\lambda+\mu)f_{12}(ta+(1-t)b,\\
&\qquad\lambda c+(1-\lambda)d)\,\mathrm{d}t\,\mathrm{d}\lambda\\
&=\frac{1}{a-b}\int_0^1(\lambda+\mu)\bigg[(t+\xi)f_2(ta+(1-t)b,\\
&\qquad\lambda c+(1-\lambda)d)|_{t=0}^{t=1}-\int_0^1 f_2(ta+(1-t)b,\\
&\qquad\lambda c+(1-\lambda)d)\,\mathrm{d}t\bigg]\,\mathrm{d}\lambda\\
&=\frac{1}{a-b}\bigg\{\int_0^1\bigg[(1+\xi)(\lambda+\mu)f_2(a,\lambda c+(1-\lambda)d)\\
&\qquad-\xi(\lambda+\mu)f_2(b,\lambda c+(1-\lambda)d)\bigg]\,\mathrm{d}\lambda\\
&\quad-\int_0^1\int_0^1(\lambda+\mu)f_2(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}t\,\mathrm{d}\lambda\bigg\}\\
&=\frac{1}{(a-b)(c-d)}\bigg\{(1+\xi)(\lambda+\mu)f(a,\lambda c+(1-\lambda)d)\\
&\qquad-\xi(\lambda+\mu)f(b,\lambda c+(1-\lambda)d)|_{\lambda=0}^{\lambda=1}\\
&\qquad-(1+\xi)\int_0^1 f(a,\lambda c+1-\lambda)d)\,\mathrm{d}\lambda\\
&\qquad+\xi\int_0^1 f(b,\lambda c+(1-\lambda)d)\,\mathrm{d}\lambda\\
&\quad-\int_0^1(\lambda+\mu)f(ta+(1-t)b,\lambda c+(1-\lambda)d)|_{\lambda=0}^{\lambda=1}\,\mathrm{d}t\\
&\quad+\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}t\,\mathrm{d}\lambda\bigg\}\\
&=\frac{1}{(a-b)(c-d)}\bigg[(1+\xi)(1+\mu)f(a,c)-\xi(1+\mu)f(b,c)\\
&\qquad-(1+\xi)\mu f(a,d)+\xi\mu f(b,d)-(1+\xi)\int_0^1 f(a,\lambda c\\
&\qquad+(1-\lambda)d)\,\mathrm{d}\lambda+\xi\int_0^1 f(b,\lambda c+(1-\lambda)d)\,\mathrm{d}\lambda\\
&\qquad-(1+\mu)\int_0^1 f(ta+(1-t)b,c)\,\mathrm{d}t\\
&\qquad+\mu\int_0^1 f(ta+(1-t)b,d)\,\mathrm{d}t\\
&\quad+\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}t\,\mathrm{d}\lambda\bigg].
\end{aligned}$$

If further making use of the substitutions $x=ta+(1-t)b$, $y=\lambda c+(1-\lambda)d$ for $t,\lambda\in[0,1]$, we obtain (1). Lemma II.1 is proved.

Lemma II.2. *Let $-1\le\xi\le1$ and $r>-1$. Then*

$$\int_0^1|t+\xi|t^r\,\mathrm{d}t=\frac{1}{(r+1)(r+2)}\times\begin{cases}\xi(2+r)+r+1, & 0\le\xi\le1,\\ 2(-\xi)^{r+2}+\xi(2+r)+r+1, & -1\le\xi\le0,\end{cases}$$

$$\int_0^1|t+\xi|(1-t)^r\,\mathrm{d}t=\frac{1}{(r+1)(r+2)}\times\begin{cases}\xi(2+r)+1, & 0\le\xi\le1,\\ 2(1+\xi)^{r+2}-\xi(2+r)-1, & -1\le\xi\le0,\end{cases}$$

$$\int_0^1|t+\xi|^r\,\mathrm{d}t=\frac{1}{r+1}\times\begin{cases}(1+\xi)^{r+1}-\xi^{r+1}, & 0\le\xi\le1,\\ (1+\xi)^{r+1}+(-\xi)^{r+1}, & -1\le\xi\le0.\end{cases}$$

Proof: This follows from direct computation.

III. Some integral inequalities of Hermite–Hadamard type

In this section, we establish some inequalities of Hermite–Hadamard type for co-ordinated (α,m_1)-(s,m_2)-convex functions on coordinates on rectangle of the plane $\mathbb{R}_0^2$.

Theorem III.1. *Let $f:\mathbb{R}_0^2\to\mathbb{R}$ have partial derivatives of the second order and $f_{12}\in L\big([a,\frac{b}{m_1}]\times[c,\frac{d}{m_2}]\big)$, where $0\le a<b$, $0\le c<d$, and $m_1,m_2\in(0,1]$. If $|f_{12}|^q$ is co-ordinated (α,m_1)-(s,m_2)-convex on $[0,\frac{b}{m_1}]\times[0,\frac{d}{m_2}]$ for $q\ge1$, $\alpha\in(0,1]$, and $s\in(-1,1]$, then*

$$\begin{aligned}
\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)}&\le\bigg[\frac{g_1(\xi,0)g_1(\mu,0)}{4}\bigg]^{1-1/q}\\
&\times\bigg[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)(s+2)}\bigg]^{1/q}\\
&\times\bigg\{2g_1(\xi,\alpha)g_1(\mu,s)|f_{12}(a,c)|^q+2m_2g_1(\xi,\alpha)g_2(\mu,s)\\
&\times\bigg|f_{12}\bigg(a,\frac{d}{m_2}\bigg)\bigg|^q+m_1[(\alpha+1)(\alpha+2)g_1(\xi,0)\\
&-2g_1(\xi,\alpha)]g_1(\mu,s)\bigg|f_{12}\bigg(\frac{b}{m_1},c\bigg)\bigg|^q+m_1m_2[(\alpha+1)(\alpha\\
&+2)g_1(\xi,0)-2g_1(\xi,\alpha)]g_2(\mu,s)\bigg|f_{12}\bigg(\frac{b}{m_1},\frac{d}{m_2}\bigg)\bigg|^q\bigg\}^{1/q}
\end{aligned}\tag{2}$$

for $-1\le\xi,\mu\le1$, where

$$g_1(u,r)=\begin{cases}u(r+2)+r+1, & 0\le u\le1,\\ 2(-u)^{r+2}+u(r+2)+1, & -1\le u\le0\end{cases}\tag{3}$$

and

$$g_2(u,r)=\begin{cases}u(r+2)+1, & 0\le u\le1,\\ 2(1+u)^{r+2}-u(r+2)-1, & -1\le u\le0.\end{cases}$$

Proof: By Lemma II.1, Hölder's integral inequality, and the (α,m_1)-(s,m_2)-convexity of $|f_{12}|^q$, we have

$$\begin{aligned}
\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)}&\le\int_0^1\int_0^1|(t+\xi)(\lambda+\mu)||f_{12}(ta\\
&\qquad+(1-t)b,\lambda c+(1-\lambda)d)|\,\mathrm{d}t\,\mathrm{d}\lambda\\
&\le\bigg[\int_0^1\int_0^1|(t+\xi)(\lambda+\mu)|\,\mathrm{d}t\,\mathrm{d}\lambda\bigg]^{1-1/q}\\
&\times\bigg[\int_0^1\int_0^1|(t+\xi)(\lambda+\mu)||f_{12}(ta+(1-t)b,\\
&\qquad\lambda c+(1-\lambda)d)|^q\,\mathrm{d}t\,\mathrm{d}\lambda\bigg]^{1/q}
\end{aligned}$$

$$\leq \left(\int_0^1 |t+\xi|\,\mathrm{d}t \int_0^1 |\lambda+\mu|\,\mathrm{d}\lambda\right)^{1-1/q}$$
$$\times\left\{\int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)|\Big[t^\alpha\lambda^s|f_{12}(a,c)|^q + m_1(1-t^\alpha)\lambda^s\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q + m_2 t^\alpha(1-\lambda)^s\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q + m_1m_2(1-t^\alpha)(1-\lambda)^s\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Big]\,\mathrm{d}t\,\mathrm{d}\lambda\right\}^{1/q}.$$

Using Lemma II.2 leads to

$$\int_0^1 |t+\xi|\,\mathrm{d}t = \frac{1}{2}g_1(\xi,0), \quad \int_0^1 |\lambda+\mu|\,\mathrm{d}\lambda = \frac{1}{2}g_1(\mu,0),$$
$$\int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)|t^\alpha\lambda^s\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{g_1(\xi,\alpha)g_1(\mu,s)}{(\alpha+1)(\alpha+2)(s+1)(s+2)},$$
$$\int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)|(1-t^\alpha)\lambda^s\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{[(\alpha+1)(\alpha+2)g_1(\xi,0)-2g_1(\xi,\alpha)]g_1(\mu,s)}{2(\alpha+1)(\alpha+2)(s+1)(s+2)},$$
$$\int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)|t^\alpha(1-\lambda)^s\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{g_1(\xi,\alpha)g_2(\mu,s)}{(\alpha+1)(\alpha+2)(s+1)(s+2)},$$
$$\int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)|(1-t^\alpha)(1-\lambda)^s\,\mathrm{d}t\,\mathrm{d}\lambda = \frac{[(\alpha+1)(\alpha+2)g_1(\xi,0)-2g_1(\xi,\alpha)]g_2(\mu,s)}{2(\alpha+1)(\alpha+2)(s+1)(s+2)}.$$

Substituting these equalities into the above inequality and simplifying yields the inequality (2). The proof of Theorem III.1 is complete.

Corollary III.1. *Under the conditions of Theorem III.1, if $m_1=m_2=m$, then*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \leq \left[\frac{g_1(\xi,0)g_1(\mu,0)}{4}\right]^{1-1/q}\left[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)(s+2)}\right]^{1/q}$$
$$\times\Big\{2g_1(\xi,\alpha)g_1(\mu,s)|f_{12}(a,c)|^q + 2mg_1(\xi,\alpha)g_2(\mu,s)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q + m[(\alpha+1)(\alpha+2)g_1(\xi,0) - 2g_1(\xi,\alpha)]g_1(\mu,s)\left|f_{12}\left(\frac{b}{m},c\right)\right|^q + m^2[(\alpha+1)(\alpha+2)g_1(\xi,0) - 2g_1(\xi,\alpha)]g_2(\mu,s)\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Big\}^{1/q}.$$

Corollary III.2. *Under the conditions of Theorem III.1, if $q=1$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} le \frac{1}{2(\alpha+1)(\alpha+2)(s+1)(s+2)}$$
$$\times\Big\{2g_1(\xi,\alpha)g_1(\mu,s)|f_{12}(a,c)| + 2m_2g_1(\xi,\alpha)g_2(\mu,s)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right| + m_1[(\alpha+1)(\alpha+2)g_1(\xi,0) - 2g_1(\xi,\alpha)]g_1(\mu,s)\left|f_{12}\left(\frac{b}{m_1},c\right)\right| + m_1m_2[(\alpha+1)(\alpha+2)g_1(\xi,0) - 2g_1(\xi,\alpha)]g_2(\mu,s)\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|\Big\}.$$

Corollary III.3. *Under the conditions of Theorem III.1,*

1) *when $0\leq\xi,\mu\leq1$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \leq \left[\frac{(1+2\xi)(1+2\mu)}{4}\right]^{1-1/q}\left[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)(s+2)}\right]^{1/q}$$
$$\times\Big\{2[\alpha+1+\xi(\alpha+2)][s+1+\mu(s+2)]|f_{12}(a,c)|^q + 2m_2[\alpha+1+\xi(\alpha+2)][1+\mu(s+2)]\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q + m_1[\alpha(\alpha+1)+2\alpha\xi(\alpha+2)]\times[s+1+\mu(s+2)]\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q + m_1m_2[\alpha(\alpha+1)+2\alpha\xi(\alpha+2)]\times[1+\mu(s+2)]\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Big\}^{1/q};$$

2) *when $0\leq\xi,\mu\leq1$ and $m_1=m_2=\alpha=s=1$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \leq \left[\frac{(1+2\xi)(1+2\mu)}{4}\right]^{1-1/q}\left(\frac{1}{36}\right)^{1/q}\times\big[(2+3\xi)(2+3\mu)|f_{12}(a,c)|^q + (2+3\xi)(1+3\mu)\times\big|f_{12}(a,d)\big|^q + (1+3\xi)(2+3\mu)\big|f_{12}(b,c)\big|^q + (1+3\xi)(1+3\mu)\big|f_{12}(b,d)\big|^q\big]^{1/q}.$$

Theorem III.2. *Let $f:\mathbb{R}_0^2\to\mathbb{R}$ have partial derivatives of the second order and $f_{12}\in L\big(\big[a,\frac{b}{m_1}\big]\times\big[c,\frac{d}{m_2}\big]\big)$ with $0\leq a<b$, $0\leq c<d$, and $m_1,m_2\in(0,1]$. If $|f_{12}|^q$ is co-ordinated (α,m_1)-(s,m_2)-convex on $\big[0,\frac{b}{m_1}\big]\times\big[0,\frac{d}{m_2}\big]$ for $q>1$, $\alpha\in(0,1]$, and $s\in(-1,1]$, then*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \leq \left[\left(\frac{q-1}{2q-1}\right)^2 g_3(\xi,q)g_3(\mu,q)\right]^{1-1/q}\times\left[\frac{1}{(\alpha+1)(s+1)}\right]^{1/q}\left[|f_{12}(a,c)|^q + m_2\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q + \alpha m_1\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q + \alpha m_1m_2\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]^{1/q}$$

for $-1\leq\xi,\mu\leq1$, where

$$g_3(u,q) = \begin{cases}(1+u)^{(2q-1)/(q-1)} - u^{(2q-1)/(q-1)}, \\ 0\leq u\leq1, \\ (1+u)^{(2q-1)/(q-1)} + (-u)^{(2q-1)/(q-1)}, \\ -1\leq u\leq0.\end{cases} \tag{4}$$

Proof: By Lemma II.1, Hölder's integral inequality, and the (α, m_1)-(s, m_2)-convexity of $|f_{12}|^q$, we have

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \le \int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)||f_{12}(ta+(1-t)b, \lambda c+(1-\lambda)d)|\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le \left[\int_0^1\int_0^1 |(t+\xi)(s+\mu)|^{q/(q-1)}\,\mathrm{d}t\,\mathrm{d}\lambda\right]^{1-1/q} \times\left[\int_0^1\int_0^1 |f_{12}(ta+(1-t)b, \lambda c+(1-\lambda)d)|^q\,\mathrm{d}t\,\mathrm{d}\lambda\right]^{1/q}$$
$$\le \left[\int_0^1 |t+\xi|^{q/(q-1)}\,\mathrm{d}t \times\int_0^1 |\lambda+\mu|^{q/(q-1)}\,\mathrm{d}\lambda\right]^{1-1/q}\left\{\int_0^1\int_0^1\left[t^\alpha\lambda^s|f_{12}(a,c)|^q +m_1(1-t^\alpha)\lambda^s\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q +m_2t^\alpha(1-\lambda)^s\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q +m_1m_2(1-t^\alpha)(1-\lambda)^s\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\mathrm{d}t\,\mathrm{d}\lambda\right\}^{1/q}$$
$$=\left[\left(\frac{q-1}{2q-1}\right)^2 g_3(\xi,q)g_3(\mu,q)\right]^{1-1/q}\left[\frac{1}{(\alpha+1)(s+1)}\right]^{1/q} \times\left[|f_{12}(a,c)|^q+\alpha m_1\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+m_2\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q +\alpha m_1m_2\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]^{1/q}.$$

Theorem III.2 is thus proved.

Corollary III.4. *Under the conditions of Theorem III.2, when $m_1=m_2=m$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \le \left[\left(\frac{q-1}{2q-1}\right)^2 g_3(\xi,q)g_3(\mu,q)\right]^{1-1/q} \times\left[\frac{1}{(\alpha+1)(s+1)}\right]^{1/q}\left[|f_{12}(a,c)|^q+m\left|f_{12}\left(a,\frac{d}{m}\right)\right|^q +\alpha m\left|f_{12}\left(\frac{b}{m},c\right)\right|^q+\alpha m^2\left|f_{12}\left(\frac{b}{m},\frac{d}{m}\right)\right|^q\right]^{1/q},$$

where g_3 is defined by (4).

Corollary III.5. *Under the conditions of Theorem III.2, when $0\le\xi,\mu\le1$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \le \left\{\left(\frac{q-1}{2q-1}\right)^2\left[(1+\xi)^{(2q-1)/(q-1)} -\xi^{(2q-1)/(q-1)}\right]\left[(1+\mu)^{(2q-1)/(q-1)} -\mu^{(2q-1)/(q-1)}\right]\right\}^{1-1/q}\left[\frac{1}{(\alpha+1)(s+1)}\right]^{1/q}\left[|f_{12}(a,c)|^q +m_2\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q+\alpha m_1\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q +\alpha m_1m_2\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]^{1/q}.$$

Theorem III.3. *Let $f:\mathbb{R}_0^2\to\mathbb{R}$ be a partially differentiable mapping and $f_{12}\in L\big(\big[a,\frac{b}{m_1}\big]\times\big[c,\frac{d}{m_2}\big]\big)$ with $0\le a<b$, $0\le c<d$, and $m_1,m_2\in(0,1]$. If $|f_{12}|^q$ is co-ordinated (α,m_1)-(s,m_2)-convex on $\big[0,\frac{b}{m_1}\big]\times\big[0,\frac{d}{m_2}\big]$ for $q>1$, $\alpha\in(0,1]$, and $s\in(-1,1]$, then*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \le \left[\frac{q-1}{2(2q-1)}g_1(\xi,0)g_3(\mu,q)\right]^{1-1/q} \times\left[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)}\right]^{1/q} \times\left[2g_2(\xi,q)|f_{12}(a,c)|^q+2m_2g_1(\xi,\alpha)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q +m_1[(\alpha+1)(\alpha+2)g_1(\xi,0)-2g_1(\xi,\alpha)]\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q +m_1m_2[(\alpha+1)(\alpha+2)g_1(\xi,0) -2g_1(\xi,\alpha)]\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]^{1/q}$$

for $-1\le\xi,\mu\le1$, where $g_1(u,r)$ and $g_3(u,r)$ are defined by (3) *and* (4) *respectively.*

Proof: Using Lemma II.1, Hölder's integral inequality, and the (α,m_1)-(s,m_2)-convexity of $|f_{12}|^q$, we have

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \le \int_0^1\int_0^1 |(t+\xi)(\lambda+\mu)||f_{12}(ta+(1-t)b, \lambda c+(1-\lambda)d)|\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le\left[\int_0^1\int_0^1 |t+\xi||\lambda+\mu|^{q/(q-1)}\,\mathrm{d}t\,\mathrm{d}\lambda\right]^{1-1/q} \times\left\{\int_0^1\int_0^1 |t+\xi|\left[t^\alpha\lambda^s|f_{12}(a,c)|^q +m_1(1-t^\alpha)\lambda^s\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q +m_2t^\alpha(1-\lambda)^s\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q +m_1m_2(1-t^\alpha)(1-\lambda)^s\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right]\mathrm{d}t\,\mathrm{d}\lambda\right\}^{1/q}$$
$$=\left[\frac{q-1}{2(2q-1)}g_1(\xi,0)g_3(\mu,q)\right]^{1-1/q} \times\left[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)}\right]^{1/q}\left\{2g_1(\xi,q)|f_{12}(a,c)|^q +2m_2g_1(\xi,\alpha)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q+m_1[(\alpha+1)(\alpha+2)g_1(\xi,0) -2g_1(\xi,\alpha)]\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2[(\alpha+1)(\alpha+2)g_1(\xi,0) -2g_1(\xi,\alpha)]\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\right\}^{1/q}.$$

The proof of Theorem III.3 is thus completed.

Corollary III.6. *Under the conditions of Theorem III.3,*

1) *when $0\le\xi,\mu\le1$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)} \le \left\{\frac{q-1}{2(2q-1)}(1+2\xi) \times\left[(1+\mu)^{(2q-1)/(q-1)}-\mu^{(2q-1)/(q-1)}\right]\right\}^{1-1/q} \times\left[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)}\right]^{1/q}\Big\{2[\alpha+1 +\xi(\alpha+2)]|f_{12}(a,c)|^q+2m_2[\alpha+1$$

$$+\xi(\alpha+2)]\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q+m_1[\alpha(\alpha+1)$$
$$+2\alpha\xi(\alpha+2)]\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+m_1m_2[\alpha(\alpha+1)$$
$$+2\alpha\xi(\alpha+2)]\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg\}^{1/q};$$

2) *when $0\le\xi,\mu\le 1$ and $m_1=m_2=m$, we have*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)}\le\left\{\frac{q-1}{2(2q-1)}(1+2\xi)\right.$$
$$\left.\times\left[(1+\mu)^{(2q-1)/(q-1)}-\mu^{(2q-1)/(q-1)}\right]\right\}^{1-1/q}$$
$$\times\left[\frac{1}{2(\alpha+1)(\alpha+2)(s+1)}\right]^{1/q}\Big\{2[\alpha+1$$
$$+\xi(\alpha+2)]|f_{12}(a,c)|^q+2m[\alpha+1$$
$$+\xi(\alpha+2)]\left|f_{12}\left(a,\frac{d}{m}\right)\right|^q+m[\alpha(\alpha+1)$$
$$+2\alpha\xi(\alpha+2)]\left|f_{12}\left(\frac{b}{m},c\right)\right|^q+m^2[\alpha(\alpha+1)$$
$$+2\alpha\xi(\alpha+2)]\left|f_{12}\left(\frac{b}{m},\frac{d}{m}\right)\right|^q\Bigg\}^{1/q}.$$

Theorem III.4. *Let $f:\mathbb{R}_0^2\to\mathbb{R}$ have partial derivatives of the second order and $f_{12}\in L\big(\big[a,\frac{b}{m_1}\big]\times\big[c,\frac{d}{m_2}\big]\big)$ with $0\le a<b$, $0\le c<d$, and $m_1,m_2\in(0,1]$. If $|f_{12}|^q$ is co-ordinated (α,m_1)-(s,m_2)-convex on $\big[0,\frac{b}{m_1}\big]\times\big[0,\frac{d}{m_2}\big]$ for $q>1$, $\alpha\in(0,1]$, and $s\in(-1,1]$, then*

$$\frac{|S(f,\xi,\mu)|}{(b-a)(d-c)}\le\left[\frac{q-1}{2(2q-1)}g_1(\mu,0)g_3(\xi,q)\right]^{1-1/q}$$
$$\times\left[\frac{1}{(\alpha+1)(s+1)(s+2)}\right]^{1/q}\Big\{g_1(\mu,s)|f_{12}(a,c)|^q$$
$$+m_2g_2(\mu,s)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q+\alpha m_1g_1(\mu,s)$$
$$\times\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+\alpha m_1m_2g_2(\mu,s)\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg\}^{1/q}$$

for $-1\le\xi,\mu\le 1$, where $g_1(u,r)$ and $g_3(u,r)$ are defined by (3) *and* (4) *respectively.*

Proof: The proof is similar to that of Theorem III.3.

Theorem III.5. *Let $f:\mathbb{R}_0^2\to\mathbb{R}$ have partial derivatives of the second order and $f_{12}\in L\big(\big[a,\frac{b}{m_1}\big]\times\big[c,\frac{d}{m_2}\big]\big)$ with $0\le a<b$, $0\le c<d$, and $m_1,m_2\in(0,1]$. If $|f_{12}|^q$ is co-ordinated (α,m_1)-(s,m_2)-convex on $\big[0,\frac{b}{m_1}\big]\times\big[0,\frac{d}{m_2}\big]$ for $q>1$, $q\ge p$, $\ell\ge 0$, $\alpha\in(0,1]$, and $s\in(-1,1]$, then*

$$\left|f(b,d)-\frac{1}{b-a}\int_a^b f(x,d)\,\mathrm{d}x-\frac{1}{d-c}\int_c^d f(b,y)\,\mathrm{d}y\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right|$$
$$\le\frac{1}{(b-a)(d-c)}\left[\frac{(q-1)^2}{(2q-p-1)(2q-\ell-1)}\right]^{1-1/q}$$
$$\times\left[\frac{1}{(p+1)(s+\ell+1)}\right]^{1/q}\Big\{(p+1)(s+\ell+1)$$
$$\times B(p+1,\alpha+1)B(\ell+1,s+1)|f_{12}(a,c)|^q$$
$$+m_2(p+1)B(p+1,\alpha+1)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q$$
$$+m_1(s+\ell+1)[1-(p+1)B(p+1,\alpha+1)]$$
$$\times B(\ell+1,s+1)\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+\alpha m_1m_2[1-(p+1)$$
$$\times B(p+1,\alpha+1)]\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg\}^{1/q},$$

where $B(\alpha,\beta)$ is the beta function

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t,\quad\alpha,\beta>0.$$

Proof: Using Lemma II.1, Hölder's integral inequality, and the (α,m_1)-(s,m_2)-convexity of $|f_{12}|^q$, we have

$$\frac{|S(f,-1,-1)|}{(b-a)(d-c)}\le\int_0^1\int_0^1(1-t)(1-\lambda)\big|f_{12}(ta+(1-t)b,$$
$$\lambda c+(1-\lambda)d)\big|\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le\left[\int_0^1\int_0^1(1-t)^{\frac{q-p}{q-1}}(1-\lambda)^{(q-\ell)/(q-1)}\,\mathrm{d}t\,\mathrm{d}\lambda\right]^{1-1/q}$$
$$\times\Bigg\{\int_0^1\int_0^1(1-t)^p(1-\lambda)^\ell\Big[t^\alpha\lambda^s|f_{12}(a,c)|^q$$
$$+m_1(1-t^\alpha)\lambda^s\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q$$
$$+m_2t^\alpha(1-\lambda)^s\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q$$
$$+m_1m_2(1-t^\alpha)(1-\lambda)^s\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Big]\,\mathrm{d}t\,\mathrm{d}\lambda\Bigg\}^{1/q}$$
$$=\frac{1}{(b-a)(d-c)}\left[\frac{(q-1)^2}{(2q-p-1)(2q-\ell-1)}\right]^{1-1/q}$$
$$\times\left[\frac{1}{(p+1)(s+\ell+1)}\right]^{1/q}\Big\{(p+1)(s+\ell+1)B(p+1,$$
$$\alpha+1)B(\ell+1,s+1)|f_{12}(a,c)|^q$$
$$+m_1(s+\ell+1)[1-(p+1)B(p+1,\alpha+1)]$$
$$\times B(\ell+1,s+1)\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q$$
$$+m_2(p+1)B(p+1,\alpha+1)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q$$
$$+\alpha m_1m_2[1-(p+1)B(p+1,\alpha+1)]$$
$$\times\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\Bigg\}^{1/q}.$$

The proof of Theorem III.5 is thus completed.

Theorem III.6. *Let $f:\mathbb{R}_0^2\to\mathbb{R}$ have partial derivatives of the second order and $f_{12}\in L\big(\big[a,\frac{b}{m_1}\big]\times\big[c,\frac{d}{m_2}\big]\big)$ with $0\le a<b$, $0\le c<d$, and $m_1,m_2\in(0,1]$. If $|f_{12}|^q$ is co-ordinated (α,m_1)-(s,m_2)-convex on $\big[0,\frac{b}{m_1}\big]\times\big[0,\frac{d}{m_2}\big]$ for $q>1$, $q\ge p$, $\ell\ge 0$, $\alpha\in(0,1]$, and $s\in(-1,1]$, then*

$$\left|f(a,c)-\frac{1}{b-a}\int_a^b f(x,c)\,\mathrm{d}x-\frac{1}{d-c}\int_c^d f(a,y)\,\mathrm{d}y\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right|$$
$$\le\frac{1}{(b-a)(d-c)}\left[\frac{(q-1)^2}{(2q-p-1)(2q-l-1)}\right]^{1-1/q}$$

$$\times\left[\frac{1}{(p+1)(\alpha+p+1)(s+\ell+1)}\right]^{1/q}$$
$$\times\bigg[(p+1)|f_{12}(a,c)|^q+\alpha m_2(p+1)(\ell+s+1)$$
$$\times B(\ell+1,s+1)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q$$
$$+m_1\alpha\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+\alpha m_1m_2\alpha(\ell+a+1)$$
$$\times B(\ell+1,s+1)\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\bigg]^{1/q}.$$

Proof: Using Lemma II.1, Hölder's integral inequality, and the (α,m_1)-(s,m_2)-convexity of $|f_{12}|^q$, we have

$$\frac{|S(f,0,0)|}{(b-a)(d-c)}\le\int_0^1\int_0^1 t\lambda\big|f_{12}(ta+(1-t)b,$$
$$\lambda c+(1-\lambda)d)\big|\,\mathrm{d}t\,\mathrm{d}\lambda$$
$$\le\left[\int_0^1\int_0^1 t^{(q-p)/(q-1)}\lambda^{(q-\ell)/(q-1)}\,\mathrm{d}t\,\mathrm{d}\lambda\right]^{1-1/q}$$
$$\times\bigg\{\int_0^1\int_0^1 t^p\lambda^\ell\bigg[t^\alpha\lambda^s|f_{12}(a,c)|^q$$
$$+m_1(1-t^\alpha)\lambda^s\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q$$
$$+m_2t^\alpha(1-\lambda)^s\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q$$
$$+m_1m_2(1-t^\alpha)(1-\lambda)^s\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\bigg]\,\mathrm{d}t\,\mathrm{d}\lambda\bigg\}^{1/q}$$
$$=\frac{1}{(b-a)(d-c)}\left[\frac{(q-1)^2}{(2q-p-1)(2q-l-1)}\right]^{1-1/q}$$
$$\times\left[\frac{1}{(p+1)(\alpha+p+1)(s+\ell+1)}\right]^{1/q}$$
$$\times\bigg[(p+1)|f_{12}(a,c)|^q+\alpha m_2(p+1)(\ell+s+1)$$
$$\times B(\ell+1,s+1)\left|f_{12}\left(a,\frac{d}{m_2}\right)\right|^q$$
$$+m_1\alpha\left|f_{12}\left(\frac{b}{m_1},c\right)\right|^q+\alpha m_1m_2\alpha(\ell+a+1)$$
$$\times B(\ell+1,s+1)\left|f_{12}\left(\frac{b}{m_1},\frac{d}{m_2}\right)\right|^q\bigg]^{1/q}.$$

The proof of Theorem III.6 is thus completed.

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