A New Halley's Family of Third-Order Methods For Solving Nonlinear Equations

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Abstract—**In this work, we present an important scheme for constructing one-point third order iterative formulas for finding simple roots of nonlinear equations. The scheme is powerful since it regenerates new, simple and fast methods. The convergence analysis shows that all the methods of the proposed family are cubically convergent for a simple root. The originality of this family lies in the fact that these methods are generated in a recurring way depending on an natural integer parameter p. Furthermore, under certain hypothesis, its methods become faster by increasing p. Comparison theoretical and/or numerical with several other existing third order and higher order methods shows that the proposed methods are robust.**

Keywords: Nonlinear equations, One-parameter family, Iterative methods, Order of convergence, third order method, Newton's method, Halley's method.

1 Introduction

Solving non-linear equations is one of the most important problems in mathematics, engineering and economy [1]-[2]. In this research, we consider iterative methods to find a simple root of a non-linear equation:

$$f(x) = 0 \tag{1}$$

where f is a real analytic function. Since it is often impossible to obtain its exact solution by analytic methods, the numerical iterative methods are generally used to obtain an approximate solution of such problems.

The root α of f, supposed simple, can be find as a fixed point of some iteration function (I.F.) by means of the one-point iteration method [3]-[6], [36]-[40]:

$$x_{n+1} = F(x_n) \qquad for \quad n = 0, 1, 2, \cdots \tag{2}$$

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where x_0 is the initial value. A point α is called a fixed point of F if $F(\alpha) = \alpha$. By making a good choice of iterative function F and respecting certain hypothesis, we can ensure the convergence of the sequence (x_n) towards α.

Newton's method for a single non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots \tag{3}$$

This is the best known iterative method [7], which converges quadratically for simple roots.

In order to increase the rate of convergence of Newton's method, the many new techniques with third order have been proposed. Among the most used methods of order 3, we cite in particular Halley's method [1], [4], [7]-[11], given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} W_0(L_n) \quad n = 0, 1, 2, 3, \ldots \tag{4}$$

where $W_0(L_n) = \dfrac{2}{2 - L_n}$

and $L_n = L_f(x_n) = \dfrac{f(x_n) f''(x_n)}{f'(x_n)^2}$

a special case of (2) with the following (I.F) :

$$F_0(x) = x - \frac{f(x)}{f'(x)} \left(\frac{2}{2 - L_f(x)} \right) \tag{5}$$

We also quote methods of: Chebyshev [1], [7]-[8], [12]-[13], [1], [7]-[8], [11], [13], Ostrowski [13], Hansen-Patrick [18], Laguerre [13], Super-Halley [8], [14]-[17], Chun [15], Jiang-Han [17], Sharma [16], [19], Amat [8], Traub [7], Weerakon and Fernando [20], Nedzhibov and al. [21], Kou and al. [22], Osban [23], Frontini and Sormany [24], Chun and Neta [25], which are interesting and well-known third-order methods.

In recent years, several researches have been conducted with the aim to create multi-step iterative methods with the improved convergence order, see [12], [26]-[34]. Fang L and al. have developed some higher-order convergent iterative methods [29], [30]. Chun and Ham have suggested a family of sixth-order methods by weight function methods in [28]. Thukral have proposed a new family of two-step iterative methods of sixth-order in [35]. Fernandez-Torres and al. in [12] have constructed a method with sixth-order convergence. Noor and al. in [33] have proposed a new predictor–corrector method whit fifth-order convergence. Kou and al. in [32] have constructed a family of variants of Ostrowski's method with seventh-order convergence. Wang and al. have proposed two families of sixth-order methods in [34].

In this paper, based on Taylor polynomial and Halley's method, we will develop a new family for finding simple roots of nonlinear equations with cubical convergence. The first important advantage of this family is that its methods are generated in a recurring fashion via a natural integer parameter p and the second is that, under certain conditions, its sequences converge more rapidly when the value of parameter p increases. To illustrate the power of new techniques, several numerical examples will presented. A comparison with many third, five and sixth order methods will be realized.

2 Development of a new family

The geometric construction of Newton's method consists in considering the tangent line

$$y(x) = f(x_n) + f^{'}(x_n)(x - x_n)$$

To the graph of f at $(x_n, f(x_n))$. The point of intersection $(x_{n+1}, 0)$ of this tangent line with x-axis, gives the celebrated sequence (2).

The linear approximation in Newton method is none other than first-order Taylor polynomial of f at x_n. By using a second-order Taylor polynomial, we obtain

$$y(x) = f(x_n) + f^{'}(x_n)(x - x_n) + \frac{f^{''}(x_n)}{2}(x - x_n)^2,$$

where x_n is again an approximate value of the zero α of the equation (1). The goal is to obtain a point $(x_{n+1}, 0)$, where the curve of y will intersect the x-axis, which is the solution of the following equation for x_{n+1}:

$$0 = f(x_n)+f^{'}(x_n)(x_{n+1}-x_n)+\frac{f^{''}(x_n)}{2}(x_{n+1}-x_n)^2 \quad (6)$$

Replacing the quantity $x_{n+1} - x_n$ remaining in the last term of the right-hand side of (6) by Halley's approximation given in (4), we obtain

$$0 = f(x_n)+f^{'}(x_n)(x_{n+1}-x_n)+\frac{f^{''}(x_n)}{2}\cdot\left(\frac{f(x_n)}{f'(x_n)}(\frac{2}{2-L_n})\right)^2 \quad (7)$$

From which it follows that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}W_1(L_n) \quad (8)$$

where $\qquad W_1(L_n) = 1 + \frac{L_n}{2}W_0^2(L_n) = \dfrac{L_n^2 - 2L_n + 4}{L_n^2 - 4L_n + 4}$

By repeating several times the same scenario as before, we derive the following iterative scheme which depends on a natural integer parameter p:

$$x_{n+1}^p = x_n^p - W_p(L_n)\frac{f(x_n^p)}{f'(x_n^p)}$$

To make writing easier, we note:

$$x_{n+1} = x_n - W_p(L_n)\frac{f(x_n)}{f'(x_n)} \quad (9)$$

where $W_0(x) = \dfrac{2}{2-x}$

and $W_{p+1}(x) = 1 + \frac{x}{2}W_p^2(x) \qquad p = 0, 1, 2, \ldots$

(9) is a special case of fixed point method with the following (I.F.) :

$$F_p(x) = x - \frac{f(x)}{f'(x)}.W_p(L_f(x)) \qquad p = 0, 1, 2, \ldots \quad (10)$$

The iterative formulas (9) represents a general Halley's family (Hp), to solve nonlinear equations that have simple roots.

3 Analysis of convergence

3.1 Order of convergence

We will prove that the proposed Halley's family (H_p) given by (9) is cubically convergent for any natural integer p.

Theorem 3.1. *Let $\alpha \in D$ be a simple root of sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. If x_0 is sufficiently close to α , then the methods*

(H_p) defined by (9) converges cubically to α, for any natural integer parameter p and it then satisfies the error equation

$$\begin{cases} e_{n+1} = & (c_2^2 - c_3)e^3 + \mathcal{O}(e_n^4) \quad \text{for} \quad p = 0 \\ e_{n+1} = & -c_3 e^3 + \mathcal{O}(e_n^4) \quad \text{for} \quad p \neq 0 \end{cases} \tag{11}$$

where $e_n = x_n - \alpha$ is the error at n^{th} iteration and $c_i = \frac{f^{(i)}(\alpha)}{i! f'(\alpha)}, \quad i = 2, 3, \ldots$

Proof. Let α be a simple root of f (i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$), and $e_n = x_n - \alpha$ be the error in approximating α by the n^{th} iterate x_n. We use the following Taylor expansions about α :

$$\begin{aligned} f(x_n) &= f'(a)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \mathcal{O}(e_n^5)], \\ f'(x_n) &= f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + \\ &\quad 4c_4 e_n^3 + \mathcal{O}(e_n^4)], \\ f''(x_n) &= f'(\alpha)[2c_2 + 6c_3 e_n + 12c_4 e_n^2 + \mathcal{O}(e_n^3)], \end{aligned} \tag{12}$$

Using (12) we get

$$[f'(x_n)]^2 = [f'(\alpha)]^2[1 + 4c_2 e_n + 2(2c_2^2 + 3c_3)e_n^2 + 4(3c_2c_3 + 2c_4)e_n^3 + \mathcal{O}(e_n^4)],$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + \mathcal{O}(e_n^4) \tag{13}$$

and

$$L_n = \frac{f(x_n) f''(x_n)}{[f'(x_n)]^2} = 2c_2 e_n - 6(c_2^2 - c_3)e_n^2 + 4(4c_2^3 - 7c_2c_3 + 3c_4)e_n^3 + \mathcal{O}(e_n^4) \tag{14}$$

Using the Taylor's series expansion [19] of $W_p(L_n)$ about $L(\alpha)$ leads to

$W_p(L_n) = W_p(L(\alpha)) + (L_n - L(\alpha))W_p'(L(\alpha)) + \frac{1}{2}(L_n - L(\alpha))^2 W_p''(L(\alpha)) + \mathcal{O}\left((L_n - L(\alpha))^3\right),$

Where p is a natural integer parameter.

Since $L(\alpha) = 0$, we obtain:

$$W_p(L_n) = W_p(0) + L_n W_p'(0) + \frac{1}{2} L_n^2 W_p''(0) + \mathcal{O}\left(L_n^3\right) \tag{15}$$

where $W_0(t) = \frac{2}{2-t}$ and $W_{p+1} = 1 + \frac{t}{2} W_p^2(t)$

Taking into account that:

$$W_0'(t) = \frac{2}{(2-t)^2}, W_0''(t) = \frac{4}{(2-t)^3},$$

$$W_{p+1}'(t) = \frac{1}{2} W_p^2(t) + t.W_p(t)W_p'(t)$$

$$W_{p+1}''(t) = 2W_p(t)W_p'(t) + t\left(W_p'^2(t) + W_p(t)W_p''(t)\right) \tag{16}$$

It is easy to prove that, for all $p \in \mathbb{N}^*$, the function W_p checks the following conditions:

$$\begin{cases} \text{For all p} \in \mathbb{N} & W_p(0) = 1 \quad \text{and} \quad W_p'(0) = \frac{1}{2} \\ \text{For all p} \in \mathbb{N}^* & W_p''(0) = 1 \quad \text{and} \quad W_0''(0) = \frac{1}{2} \end{cases} \tag{17}$$

Thus, the Formula (15) becomes :

$$\begin{cases} W_p(L_n) = 1 + \frac{1}{2} L_n + \frac{1}{2} L_n^2 + \mathcal{O}(e_n^3), \text{ for p} \in \mathbb{N}^* \\ W_0(L_n) = 1 + \frac{1}{2} L_n + \frac{1}{4} L_n^2 + \mathcal{O}(e_n^3) \end{cases} \tag{18}$$

Using (14), we get

$$\begin{cases} W_p(L_n) = 1 + c_2 e_n + [-c_2^2 + 3c_3]e_n^2 + \mathcal{O}(e_n^3), \text{ for p} \in \mathbb{N} \\ W_0(L_n) = 1 + c_2 e_n + [-2c_2^2 + 3c_3]e_n^2 + \mathcal{O}(e_n^3) \end{cases} \tag{19}$$

Substituting (13) and (19) in formula (9), we obtain the error equation

$$\begin{cases} e_{n+1} = (c_2^2 - c_3)e_n^3 + \mathcal{O}(e_n^4), & \text{for p=0} \\ e_{n+1} = -c_3 e_n^3 + \mathcal{O}(e_n^4), & \text{for p} \neq 0 \end{cases}$$

which confirms that the order of convergence of the methods (Hp) are three, whatever the natural integer p. This terminates the demonstration of theorem.

□

3.2 Global convergence of new methods

The following two lemmas will be used to study of the monotonic convergence of methods (Hp).

Lemma 3.2. *Let us write the iterative functions of methods (H_p) :*

$F_p(x) = x - \frac{f(x)}{f'(x)}.W_p(L_f(x)) \qquad$ *for p=0,1,2, ...*

Then, the derivatives of F_p is given by:

$$F_p'(x) = 1 - L_f(x)[1 + L_f(x)(L_{f'}(x) - 2)]W_p'(L_f(x)) - W_p(L_f(x))(1 - L_f(x)) \tag{20}$$

Lemma 3.3. *Let x a real number such as $0 \leqslant x < 2$ and (V_p) the sequence defined by :*

$V_0 = \frac{2}{2-x}$, *and* $V_{p+1} = 1 + \frac{x}{2}V_p^2$, *for* $p = 0, 1, 2 \ldots$

Then, we have:

Then, (V_p) is increasing sequence with strictly positive terms.

Proof. For a given value of x chosen in the interval $[0\,,2)$, it easy to prove by induction that $V_p \geqslant 1$ for all $p \in \mathbb{N}$. Let us prove by induction that (V_p) is a increasing sequence. We have:

$V_1 - V_0 = \dfrac{x^2}{(2-x)^2}$, then $V_1 \geqslant V_0$. Let us suppose that $V_{p+1} \geqslant V_p$, since $V_p > 0$ for all $p \in \mathbb{N}$, then $V_{p+1}^2 \geqslant V_p^2$, and since we also have $x \geqslant 0$, we deduce that $V_{p+2} \geqslant V_{p+1}$ and the prove is terminated.

□

Theorem 3.4. *Let $f \in C^m[a\,,b]$, $m \geqslant 4$, $f^{'} \neq 0$, $f^{"} \neq 0$, $0 \leqslant L_f < 1$ and the iterative functions $F_p^{'}$ of f, defined by (10) for a natural integer p, be increasing function on an interval $[a\,,b]$ containing the root α of f. Then sequence given by (9) is decreasing (resp. increasing) and converges to α from any point $x_0 \in [a\,,b]$ checking $f(x_0)f^{'}(x_0) > 0$ (resp. $f(x_0)f^{'}(x_0) < 0$)*

Proof. Let's choose an x_0 such that $f(x_0)f^{'}(x_0) > 0$, then $x_0 > \alpha$. Applying the Mean Value Theorem to functions F_p where p is a natural integer, we obtain:

$$x_1 - \alpha = F_p(x_0) - F_p(\alpha) = F_p^{'}(\eta)(x_0 - \alpha)$$

for some $\eta \in (\alpha\,, x_0)$. As the derivative of iterative function given by (20) checks $F_p^{'}(x) \geqslant 0$ in $[\alpha\,,b]$ so $x_1 \geqslant \alpha$. By induction, we obtain $x_n \geqslant \alpha$ for all $n \in \mathbb{N}$.

Furthermore, from (9), we have:

$$x_1 - x_0 = -W_p(L_0)\frac{f(x_0)}{f^{'}(x_0)}$$

As $0 \leqslant L_0 < 2$ then from lemma 2 and by posing $x = L_0$ we have $V_p = W_p(L_0) > 0$ for all $p \in \mathbb{N}$, and since $\frac{f(x_0)}{f^{'}(x_0)} > 0$, we deduce that $x_1 \leqslant x_0$. Now it is easy to prove by induction that $x_{n+1} \leqslant x_n$ for all $n \in \mathbb{N}$.

As a consequence, the sequences (9) are decreasing and converges to a limit $\mu \in [a\,,b]$ where $\mu \geqslant \alpha$, by making the limit of (9) we get:

$$\mu = \mu - \frac{f(\mu)}{f^{'}(\mu)}W_p(L_f(\mu))$$

We have $W_p(L_f(\mu) > 0)$ for all $p \in \mathbb{N}$ and for every real $L_f(\mu) \in [0\,,2)$ so $W_p(L_f(\mu)) \neq 0$ and consequently $f(\mu) = 0$. As α is unique zero of f in $[a\,,b]$ therefore $\mu = \alpha$. This completes proof of the theorem. Similarly, we can prove that the sequence (9) is increasing and converges to under the same hypotheses of theorem 3.4, but for $f(x_0)f^{'}(x_0) < 0$.

□

4 Advantage of new family

To show the peculiarity of present family (Hp), we make an analytical comparison of the convergence speeds of its methods between them.

Theorem 4.1. *Let $f \in C^m[a\,,b]$, $m \geq 4$, $f^{'} \neq 0$, $f^{''} \neq 0$, $0 \leq L_f(x) < 2$, and the iterative functions F_p and F_{p+1} of f, defined by (10) for a natural integer p, be increasing functions on an interval $[a\,,b]$ containing the root α of f. Starting from the same initial point $x_0 \in [a\,,b]$, the rate of convergence of method $(H_{p+1})$is higher than one of method (Hp).*

Proof. Assume that initial value x_0 satisfies $f(x_0)f^{'}(x_0) > 0$, so $x_0 > \alpha$. The theorem 3.4 announces that the case where $f^{'} \neq 0$, $f^{''} \neq 0$, $0 \leq L_f(x) < 2$, and F_p and F_{p+1} are increasing functions an interval $[a\,,b]$, then the sequences (x_n^p) and (x_n^{p+1}) given by (9), are decreasing and converges to α from any point $x_0 \in [a\,,b]$

Let (a_n) and (b_n) be two sequences defined by (x_n^{p+1}) and (x_n^p) respectively. Since $a_0 = b_0 = x_0$ and the two sequences are decreasing, we expect that $a_n \leqslant b_n$ for all $n \in \mathbb{N}$. This can be proved by induction. Let $n = 1$, then:

$$a_1 - b_1 = -\frac{f(x_0)}{f^{'}(x_0)}(W_{p+1}(L_0) - W_p(L_0))$$

We have $0 \leq L_0 = L_f(x_0) < 2$, so from lemma 3.3: $W_{p+1}(L_0) \geqslant W_p(L_0)$ and as $\frac{f(x_0)}{f^{'}(x_0)} > 0$, we derive that : $a_1 \leq b_1$

Let's suppose that $a_n \leq b_n$. Since, under the above hypotheses, F_{p+1} is increasing function in $[a\,,b]$, we obtain $F_{p+1}(a_n) \leq F_{p+1}(b_n)$

Furthermore, we have :

$$F_{p+1}(b_n) - F_p(b_n) = -\frac{f(b_n)}{f^{'}(b_n)}\left(W_{p+1}(L_n) - W_p(L_n)\right) \leq 0$$

so $F_{p+1}(a_n) \leq F_p(b_n)$. Finally $a_{n+1} \leq b_{n+1}$ and th induction is completed.

□

The present theorem is of great importance because it clearly illustrates the power of proposed family. Indeed, we have analytically justified that, under certain conditions, the convergence speed of its methods increases with the parameter p. As, in addition, the integer p can take very large values, then the convergence speed can always be improved with p. Conversely, Halley's method, which is a particular case of this family obtained for $p = 0$, will have a lower convergence rate than other new methods in the same family, having higher parameters.

5 Numerical results

In this section, we exhibit numerical results which show the behavior of methods in the proposed family for some arbitrary chosen test functions.

Computations have been performed using MATLAB R2015b and the stopping criterion has been taken as $|x_{n+1} - x_n| \leq 10^{-15}$ and $|f(x_n)| \leq 10^{-15}$.

We give the number of iterations (N) and/or the number of function evaluations (NOFE) necessary to check the stopping criterion, CU designates that method converge to undesired root.

The tests functions, used in Table II, III and IV, and their roots α, are displayed in Table I.

5.1 Comparison between some methods of Halley's family (Hp)

We consider the function f_{11} defined in table I on interval $I = [2\,, 10]$. By taking $x_0 = 9$, we have $f(x_0)f^{'}(x_0) > 0$. The table II presents a numerical comparison between some methods from the proposed family (Hp) obtained for $p = 1, 4, 11, 20$ and 21.

The table II shows that :

- All the selected sequences (H1, H4, H11, H20 and H21) defined by (9) are decreasing and converges to the root $\alpha = 2$ of the function f_{11} on interval I;
- The increase of parameter (p), leads to a great improvement of the convergence speed of methods (H_p) and to a clear decrease of their number of iterations.
- The convergence rate of Halley's method (H1) is lower than that of the other new methods which have higher values of parameter p (H4,H11,H20 and H21).

5.2 Comparison with some third order methods

In table III, we present some numerical tests for Newton's method and various cubically convergent iterative methods. Compared are Newton's method (NM) defined by formula (3), Chebyshev's method (CB) defined by (13) in [15], Sharma's method (SR) defined by (17) with $\alpha = 0.5$ in [19], Chun's method (CH) defined by (23) with $a_n = 1$ in [15], Sharma's method (SM) defined by (20) with $a_n = 1$ in [16], Jiang-Han's method (JH) [17] defined by (19) with parameter $\alpha = 1$ in [19], Ostrowski's method (OT) defined by (26) in [19], Halley's method (HL) defined by (4) given before and Super-Halley's method (SH) of Gutiérrez and Hernández [15]. To represent new scheme (9), we choose four methods designated as (H2), (H8), (H10) and (H20).

Table III shows that in the majority of selected examples, the results obtained with the four proposed methods of new family are better or similar to the other third order methods used, because they converge more quickly and with lesser number of iterations.

5.3 Comparison with higher order methods

In Table IV, we give the number of iterations (N) and the number of function evaluations (NOFE) sufficient to meet the stopping criterion. We compare Four methods of proposed family (H2, H4, H7 and H21), with some higher order methods. (FG) denotes fifth order method of Fang for Fang and al. (formula (2) in [30]), (NR) denotes fifth order method of Noor and al. method (algorithm 2.4. in [33]). (CC) represents Chun and Ham's method (formula (10), (11), (12) in [28]), (TK) denotes Thukral's method (formula (27) in [35]), (TR) designates Fernandez's method (formula (14) and (15) in [12]), Three sixth-order methods.

Comparison with several fifth and sixth order methods illustrates the performance and efficiency of the proposed new family. Indeed, the table IV illustrate that new proposed methods behaves either similarly or even better on the most of examples considered, as require an equal or smaller number of function evaluations.

6 Conclusion

In this paper, we have introduced a new family of iterative methods for solving nonlinear equations with simple roots. The family is generated by using, in the beginning, the Halley's approximation in second-order Taylor polynomial and then repeating the same scenario by applying, each time, the new correction. We have proven that, under some conditions, all methods of proposed family has third-order convergence. The main characteristics of this family reside, on the one hand, in the fact that these sequences are derived from each other via a recurring formula dependent on a natural integer p and, on the other hand, under certain conditions, the convergence speed of its methods improves when the value of p increase. To illustrate new techniques, several numerical examples are presented. The performances of ours methods are

TABLE I
Test Functions and their Roots

Test functions	Root (α)
$f_1(x) = e^x - 3x^2$	-0.4589622675369485
$f_2(x) = x^3 - 10$	2.154434690031884
$f_3(x) = (x-2)^2 - \ln x$	3.057103549994738
$f_4(x) = x^2 - 5x + 6$	3.000000000000000 2.000000000000000
$f_5(x) = x\exp(x^2) - (\sin x)^2 + 3\cos x + 5$	-1.207647827130919
$f_6(x) = (x-1)^3 - 1$	2.000000000000000
$f_7(x) = x^{12} - 2x^3 - x + 1$	0,5903344367965851
$f_8(x) = (\sin x)^2 - x^2 + 1$	1.404491648215341
$f_9(x) = 2\sin x - 1$	0,5235987755982989
$f_{10}(x) = x^2 + x - 12$	3.00000000000000
$f_{11}(x) = 2x^2 - 6x + 4$	2.00000000000000

TABLE II
Comparison between some Methods of Halley's Family (Hp)

H1	H4	H11	H20	H21
9.0000000000000	9.0000000000000	9.0000000000000	9.0000000000000	9.0000000000000
3.619665511244937	2.968400627720842	2.421183688334962	2.186540486312487	2.171784630356075
2.213887974995401	2.021767459559896	2.000004603595781	2.0	2.0
2.001626826786968	2.000000000029282	2.0		
2.000000000013896	2.0			
2.0				

TABLE III
Comparison with some Third Order Methods

		N: Number of iterations												
Test function	x0	NM	CB	SR	CH	SM	JH	OT	HL	SH	H2	H8	H10	H20
f_1	-1,5	6	4	4	4	4	5	4	4	3	3	3	3	3
f_2	1,5	6	4	4	4	4	4	3	4	3	3	3	3	3
f_3	5	6	4	4	4	4	5	4	4	4	3	3	3	3
f_4	4,5	6	4	4	4	4	5	4	4	3	3	2	2	2
f_4	0	7	5	4	5	5	5	4	4	4	3	3	2	2
f_5	-0,8	7	5	5	5	5	5	4	3	4	3	4	4	4
f_9	1,05	5	4	4	4	4	4	3	4	3	4	3	3	3
f_7	0,8	5	5	3	4	4	4	4	4	3	3	4	3	3
f_8	1,05	6	4	4	4	4	4	3	4	3	3	3	3	3

compared with some known methods of similar or higher order. The computational results have confirmed the robust and efficient nature of the techniques of new family constructed in this paper.

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TABLE IV
Comparison with Some Higher Order Methods

		N: Number of iterations									NOFE: Number of functions evaluations								
Function	x0	FG	CC	TK	TR	NR	H2	H4	H7	H21	FG	CC	TK	TR	NR	H2	H4	H7	H21
f_9	1.05	3	3	3	3	3	4	3	3	3	12	12	12	12	12	12	9	9	9
f_8	1	3	3	4	5	3	3	3	3	3	12	12	16	20	12	9	9	9	9
f_4	4,5	3	3	3	6	3	3	3	2	2	12	12	12	24	12	9	9	6	6
f_{10}	4,2	2	2	2	7	2	2	2	2	1	8	8	8	28	8	6	6	6	3
f_2	1,5	3	3	4	6	3	3	3	3	3	12	12	16	24	12	9	9	9	9
f_5	-1	3	2	3	5	3	3	3	3	3	12	8	12	20	12	9	9	9	9
f_3	2,6	3	3	4	5	3	3	3	3	4	12	12	16	20	12	9	9	9	12
f_1	-1,5	3	3	3	4	3	3	3	3	3	12	12	12	16	12	9	9	9	9
f_7	0,2	3	3	CU	3	3	3	3	4	4	12	12	CU	12	12	9	9	12	12
f_6	1,7	3	3	3	8	3	3	3	3	3	12	12	12	32	12	9	9	9	9

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