Generalized Rational Multi-step Method for Delay Differential Equations

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Abstract- **This paper presents the generalized rational multi-step method for solving delay differential equations (DDEs). Here, we develop the r-step p-th order generalized multi-step method which is based on rational function approximation technique. The local truncation error and stability analysis are given. The delay argument is approximated using Lagrange interpolation. The applicability of this method has been demonstrated by numerical examples of DDEs with constant delay (HIV-1 infection model), time dependent delay and state dependent delays.**

Index Terms-**Lagrange interpolation, Multi-step Method, Rational function, Stability polynomial and Stability region, Delay differential equations, HIV-1 infection model.**

I. INTRODUCTION

Delay differential equations (DDEs) are a type of differential equations in which the derivative of the unknown function depends not only at its present time but also at the previous times. In ordinary differential equations (ODEs), a simple initial condition is given. But to specify DDEs, additional information is needed. Because the derivative depends on the solution at the previous times, an initial history function which gives information about the solution in the past needs to be specified. A general form of the first order DDE is

$$y'(t) = f\big(t, y(t), y(t-\tau)\big), \quad t > t_0$$

$$y(t) = \Phi(t), \quad t \le t_0 \tag{1}$$

where Φ(t) is the initial function and τ is the delay term. The function Φ(t) is known as the 'history function', as it gives information about the solution in the past. If the delay term τ is a constant, then it is called constant delay. If it is function of time t, then it is called time dependent delay. If it is a function of time t and y(t), then it is called state dependent delay.

These equations arise in population dynamics [1], control systems [2], chemical kinetics [3], and in several areas of science and engineering. Recently there has been a growing interest in obtaining the numerical solutions of stiff and non-stiff DDEs. Rostann [4] et al. implemented Adomian Decomposition Method for solving system of DDEs. Radzi [5] et al. developed the two and three point one-step block method for solving DDEs. Fuziyah Ishak et al. [6] described block method for solving Pantograph type functional DDEs. Toheeb et al. [7] obtained the exact/approximate solution of DDEs by using the combination of Laplace and the variational iteration method. Emimal and Vinci [8] proposed RK method based on Harmonic Mean for solving DDEs with constant lags.

Several numerical methods have been constructed for solving stiff DDEs. Bocharov [9] applied linear multi-step methods for the numerical solution of stiff delay differential systems modelling an immune response. Vinci and Emimal [10, 11] proposed composite RK methods and a new one step method based on polynomial scheme for solving stiff and non-stiff DDEs.

Several multi-step techniques using variety of interpolating polynomials and functions have been developed to solve ODEs. Simon Ola Fatunla [12] developed the nonlinear multistep methods based on inverse polynomial for solving IVPs. Okosun and Ademiluyi [13] derived two-step second order inverse polynomial methods for integration of differential equations with singularities. Abolarin and Akingbade [14] derived the fourth stage inverse polynomial scheme IVPs. Teh Yuan Ying et al. [15] proposed a new class of rational multi-step methods for solving IVPs.

In this paper we present the r-step p-th order generalized rational multi-step method for solving DDEs which is based on rational function approximation technique. This method has been referred here as GRMM (r, p). The organization of this paper is as follows:

In section II, the derivation of GRMM (r, p) is given. In section III, the implementation of GRMM for 2-step and of orders 2, 3 and 4 is presented. In section IV, the stability analysis of GRMM (2, p), (p = 2, 3, 4) has been given. In section V, numerical illustrations of DDEs are provided to demonstrate the efficiency of this method.

II. DERIVATION OF GENERALISED RATIONAL MULTI-STEP METHOD (GRMM (r, p))

Assume that the analytical solution $y_{n+r} = y(t_{n+r})$ to the initial value problem (1) can be given by

$$y_{n+r} = a_0 + \frac{a_1 h}{1+\sum_{j=1}^{K} b_j h^j}, \qquad 1 + \sum_{j=1}^{K} b_j h^j \neq 0 \tag{2}$$

where $a_0, a_1, b_j, (j = 1,2,\ldots K)$ are coefficients that need to be determined.

With the GRMM (r, p) in (2), we associate the difference operator L defined by

$$L[y(t);h]_{GRMM(r,p)} = (y(t+rh) - a_0) \times \left(1 + \sum_{j=1}^{K} b_j h^j\right) - a_1 h \tag{3}$$

where y(t) is an arbitrary, continuous and differentiable function.

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Expanding $y(t+rh)$ as Taylor series and collecting terms in (3), we have

$$L[y(t);h]_{GRMM} = C_0h^0 + C_1h^1 + \cdots + C_Kh^K + C_{K+1}h^{K+1} + \cdots \quad (4)$$

where C_i, $i = 0,1,\ldots,K,K+1$ are the coefficients that need to be determined.
Taking K=p-1 in (2) and expanding $y(t+rh)$ into Taylor series, we get

$$\begin{aligned} L[y(t);h]_{GRMM} = & -a_0 + y(t) + h(-a_1 - a_0b_1 + b_1y(t) + ry'(t)) \\ & + h^2\left(-a_0b_2 + b_2y(t) + \frac{r^2}{2!}y''(t) + rb_1y'(t)\right) \\ & + \cdots \\ & + h^p\left(\frac{r^p}{p!}y^{(p)}(t) + \frac{r^{p-1}}{(p-1)!}b_1y^{(p-1)}(t) + \cdots + rb_{p-1}y'(t)\right) \\ & + h^{p+1}\left(\frac{r^{p+1}}{(p+1)!}y^{(p+1)}(t) + \frac{r^p}{p!}b_1y^{(p)}(t) + \cdots + \frac{r^2}{2!}b_{p-1}y''(t)\right) \\ & + O(h^{p+2}) \end{aligned} \quad (5)$$

Comparing (4) and (5), we get

$$\begin{aligned} C_0 &= -a_0 + y(t), \\ C_1 &= -a_1 - a_0b_1 + b_1y(t) + ry'(t), \\ C_2 &= rb_1y'(t) + \frac{r^2}{2!}y''(t) + b_2y(t) - a_0b_2, \\ &\ldots, \\ C_p &= \frac{r^p}{p!}y^{(p)}(t) + \frac{r^{p-1}}{(p-1)!}b_1y^{(p-1)}(t) + \cdots + rb_{p-1}y'(t), \\ C_{p+1} &= \frac{r^{p+1}}{(p+1)!}y^{(p+1)}(t) + \frac{r^p}{p!}b_1y^{(p)}(t) + \cdots + \frac{r^2}{2!}b_{p-1}y''(t) \end{aligned} \quad (6)$$

For p-th order GRMM, we put $C_0 = C_1 = C_2 = \ldots = C_p = 0$ in (6) which gives the following solutions

$$\begin{aligned} a_0 &= y(t), \\ a_1 &= ry'(t), \\ b_1 &= -\frac{ry''(t)}{2!y'(t)}, \\ &\ldots, \\ b_{p-1} &= -\left[\frac{r^{p-1}}{p!}\frac{y^{(p)}(t)}{y'(t)} + \frac{r^{p-2}}{(p-1)!}b_1\frac{y^{(p-1)}(t)}{y'(t)} + \frac{r^{p-3}}{(p-2)!}b_2\frac{y^{(p-2)}(t)}{y'(t)} + \cdots + \frac{r}{2!}b_{p-2}\frac{y''(t)}{y'(t)}\right] \end{aligned} \quad (7)$$

and

$$C_{p+1} = \frac{r^{p+1}}{(p+1)!}y^{(p+1)}(t) + \frac{r^p}{p!}b_1y^{(p)}(t) + \cdots + \frac{r^2}{2!}b_{p-1}y''(t) \quad (8)$$

When $t = t_n$, we can write $y_n = y(t_n)$. Then (7) becomes

$$\begin{aligned} a_0 &= y_n, \\ a_1 &= ry'_n, \\ b_1 &= -\frac{ry''_n}{2!y'_n}, \ldots, \\ b_{p-1} &= -\left[\frac{r^{p-1}}{p!}\frac{y_n^{(p)}}{y'_n} + \frac{r^{p-2}}{(p-1)!}b_1\frac{y_n^{(p-1)}}{y'_n} + \frac{r^{p-3}}{(p-2)!}b_2\frac{y_n^{(p-2)}}{y'_n} + \cdots + \frac{r}{2!}b_{p-2}\frac{y''_n}{y'_n}\right] \end{aligned} \quad (9)$$

where $y_n = y(t_n)$ and $y_n^{(m)} = y^{(m)}(t_n)$, m = 1,2,… by the localizing assumption. Taking K = p-1 in (2),

$$y_{n+r} = a_0 + \frac{a_1h}{1 + b_1h + b_2h^2 + \cdots + b_{p-2}h^{p-2} + b_{p-1}h^{p-1}},$$

where $1 + b_1h + b_2h^2 + \cdots + b_{p-1}h^{p-1} \neq 0$ (10)

Substituting (9) in (10),

$$y_{n+r} = y_n + \frac{rhy'_n}{1 + \left(\frac{-ry''_n}{2!y'_n}\right)h + \ldots + \left(-\left[\frac{r^{p-1}y_n^{(p)}}{p!\,y'_n} + \frac{r^{p-2}}{(p-1)!}b_1\frac{y_n^{(p-1)}}{y'_n} + \frac{r^{p-3}}{(p-2)!}b_2\frac{y_n^{(p-2)}}{y'_n} + \cdots + \frac{r}{2!}b_{p-2}\frac{y''_n}{y'_n}\right]\right)h^{p-1}} \quad (11)$$

The local truncation error (LTE) of GRMM (r, p) is given by

$$LTE_{GRMM(r,p)} = h^{p+1}\left(\frac{r^{p+1}}{(p+1)!}y^{(p+1)}(t) + \frac{r^p}{p!}b_1y^{(p)}(t) + \ldots + \frac{r^2}{2!}b_{p-1}y''(t)\right) + O(h^{p+2}) \quad (12)$$

III. IMPLEMENTATION STRATEGY OF GRMM (r, p)

In this section, GRMM (r, p) is implemented for various steps r and any order p.
For example, here we discuss the implementation of GRMM for 2-step and of orders 2, 3 and 4.

Taking r = 2 and p = 2, 3, 4 in (11) respectively and on simplification, we get

For GRMM (2, 2):

$$y_{n+2} = y_n + \frac{2h(y'_n)^2}{y'_n - hy''_n} \quad (13)$$

For GRMM (2, 3):

$$y_{n+2} = y_n + \frac{6h(y'_n)^3}{3(y'_n)^2 - 3hy'_ny''_n + 3h^2(y''_n)^2 - 2h^2y'_ny'''_n} \quad (14)$$

For GRMM (2, 4):

$$y_{n+2} = y_n + \frac{6h(y'_n)^4}{3(y'_n)^3 - 3h(y'_n)^2y''_n + 3h^2y'_n(y''_n)^2 - 3h^3(y''_n)^3 - 2h^2(y'_n)^2y'''_n + 4h^3y'_ny''_ny'''_n - h^3(y'_n)^2y_n^{(4)}} \quad (15)$$

From (12), the local truncation error of GRMM (2, p), where p = 2, 3, 4 are given respectively by

$$LTE_{GRMM(2,2)} = h^3\left(-\frac{2(y_n'')^2}{y_n'} + \frac{4}{3}y_n'''\right) + o(h^4) \quad (16)$$

$$LTE_{GRMM(2,3)} = h^4\left(\frac{2(y_n'')^3}{(y_n')^2} - \frac{8y_n''y_n'''}{3y_n'} + \frac{2}{3}y_n^{(4)}\right) + o(h^5) \quad (17)$$

$$LTE_{GRMM3(2,4)} = h^5\left(\frac{-2\left(9(y_n'')^4\right) - 18y_n'(y_n'')^2y_n''' + 4(y_n')^2(y_n''')^2 + 6(y_n')^2y_n''y_n^{(4)}}{9(y_n')^3} + \frac{4}{15}y_n^{(5)}\right) + o(h^6) \quad (18)$$

In a similar manner, we can implement GRMM (r, p) for various steps r and any order p.

IV. STABILITY ANALYSIS OF GRMM (r, p)

In this section, we derived the stability polynomials of GRMM (2, p), (p = 2, 3, 4) and obtained their corresponding stability regions.

We consider a commonly used linear test equation with a constant delay $\tau = mh$ where m is a positive integer,

$$y'(t) = \lambda y(t) + \mu y(t-\tau), \quad t > t_0$$
$$y(t) = \phi(t), \quad t \le t_0 \quad (19)$$

where $\lambda, \mu \in C,\ \tau > 0$ and Φ is continuous.

When p = 2, a slight rearrangement of (13) can be written as

$$y_{n+2} = y_n + 2hy_n' + 2h^2y_n'' \quad (20)$$

Using (20) in (19), we get

$$y_{n+2} = y_n + 2h\big(\lambda y_n + \mu y(t_n - \tau)\big) + 2h^2\big(\lambda y'_n + \mu y'(t_n-\tau)\big) \quad (21)$$

$$y(t_n - mh) = y(t_{n-m}) = \sum_{l=-r_1}^{s_1} L_l(c_i)\, y_{n-m+l}$$

with

$$L_l(c_i) = \prod_{j=-r_1}^{s_1} \frac{c_i - j_1}{l - j_1}, \quad j_1 \ne l \ \text{ and } \ r_1, s_1 > 0$$

Taking $y(t_n - \tau) = \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}$ and

$$y'(t_n-\tau) = \lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l} \quad (22)$$

Then (21) becomes

$$y_{n+2} = y_n + 2h\Big(\lambda y_n + \mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\Big) + 2h^2\begin{pmatrix} \lambda\big(\lambda y_n + \mu\sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\big) \\ +\mu\lambda \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} \\ +\mu \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\end{pmatrix}$$

$$y_{n+2} = y_n + 2\lambda h y_n + 2\lambda^2h^2y_n + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\,(2\mu h + 4h^2\mu\lambda) + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}\,(2h^2\mu^2)$$

$$y_{n+2} = y_n(1 + 2\lambda h + 2(\lambda h)^2) + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l}\,\big(\mu h(2 + 4\lambda h)\big) + \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}(2(\mu h)^2)$$

Let $\alpha = \lambda h$ and $\beta = \mu h$ then the above equation becomes

$$y_{n+2} = y_n(1 + 2\alpha + 2\alpha^2) + \big(\beta(2+4\alpha)\big)\sum_{l=-r_1}^{s_1} L_l(c) y_{n-m+l} + 2\beta^2 \sum_{l=-r_1}^{s_1} L_l(c) y_{n-2m+l}$$

To obtain the stability polynomial, the delay term is approximated using three points Lagrange interpolation.

By putting $n - m + l = 0$ and $n - 2m + l = 0$ and by taking l = -1, 0, 1, the stability polynomial will be in the standard form.

The recurrence is stable if the zeros of ζ_i of the stability polynomial

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - (1 + \alpha + 2\alpha^2)\zeta^n - \beta(2 + 4\alpha)(L_{-1}(c) + L_0(c)\zeta + L_1(c)\zeta^2) - 2\beta^2(L_{-1}(c) + L_0(c)\zeta + L_1(c)\zeta^2)$$

satisfies the root condition $|\zeta_i| \le 1$. From this, the stability polynomial for the method GRMM (2, 2) with $\tau = 1$ is given as

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - (1 + \alpha + 2\alpha^2)\zeta^n - (2\beta + 2\beta^2 + 4\alpha\beta)$$

Similarly, by considering suitable number of points in Lagrange interpolation according to the order of the method, we can obtain the corresponding stability polynomials of GRMM (2, p).

When p = 3, the stability polynomial for GRMM (2, 3) is given as

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - \left(1 + \alpha + 2\alpha^2 + \frac{4}{3}\alpha^3\right)\zeta^n - \left(2\beta + 2\beta^2 + \frac{4}{3}\beta^3 + 4\alpha\beta + 4\alpha^2\beta + 4\alpha\beta^2\right)$$

When p = 4, the stability polynomial for GRMM (2, 4) is given as

$$S(\alpha, \beta : \zeta) = \zeta^{n+2} - \left(1 + \alpha + 2\alpha^2 + \frac{4}{3}\alpha^3 + \frac{2}{3}\alpha^4\right)\zeta^n - \left(2\beta + 2\beta^2 + \frac{4}{3}\beta^3 + \frac{2}{3}\beta^4 + 2\alpha\beta + 4\alpha^2\beta + 4\alpha\beta^2 + \frac{8}{3}\alpha\beta^3 + \frac{8}{3}\alpha^3\beta + 4\alpha^2\beta^2\right)$$

The stability regions of GRMM (2, 2), GRMM (2, 3) and GRMM (2, 4) are given in Fig. 1 -3.

In a similar manner, we can obtain the stability polynomials and their corresponding regions of GRMM with r-step and of any order p.

V. NUMERICAL EXAMPLES

Example 1:

Consider the stiff linear system of DDEs with multiple delays

$$y_1'(t) = -\frac{1}{2}y_1(t) - \frac{1}{2}y_2(t-1) + f_1(t),$$

$$y_2'(t) = -y_2(t) - \frac{1}{2}y_1\left(t-\frac{1}{2}\right) + f_2(t), \quad 0 \le t \le 1$$

with initial conditions

$$y_1(t) = e^{-t/2}, \qquad \frac{-1}{2} \le t \le 0,$$

$$y_2(t) = e^{-t}, \qquad -1 \le t \le 0$$

and $f_1(t) = \frac{1}{2}e^{-(t-1)}, \quad f_2(t) = \frac{1}{2}e^{-(t-1/2)/2}$

The exact solution is

$$y_1(t) = e^{-t/2}, \qquad y_2(t) = e^{-t}$$

By taking *h* = 0.01 in the above examples, the absolute errors of GRMM (2, p) where p = 2, 3, 4 are given in Tables 1 – 2 and the graphs are shown in Fig. 4.

Example 2:

Consider the time dependent DDE

$$y'(t) = \frac{t-1}{t}y(\ln(t)-1)y(t), \qquad t \ge 1$$

with initial condition

$$y(t) = 1, \qquad t \le 1$$

and the exact solution is

$$y(t) = \exp(t - \ln(t) - 1), \qquad t \ge 1$$

By taking *h* = 0.01 in the above examples, the absolute errors of GRMM (2, p) where p = 2, 3, 4 are given in Table 3 and the graphs are shown in Fig. 5.

Example 3:

Consider the state dependent DDE

$$y'(t) = \cos(t)y(\mathrm{y(t)}-2), \qquad t \ge 0$$

with initial condition

$$y(t) = 1, \qquad t \le 0$$

and the exact solution is

$$y(t) = \sin(t) + 1, \qquad 0 \le t \le 1$$

By taking *h* = 0.01 in the above examples, the absolute errors of GRMM (2, p) where p = 2, 3, 4 are given in Table 4 and the graphs are shown in Fig. 6.

Example 4: (HIV-1 infection model)

Consider a mathematical model of HIV-1 infection to CD4+ T cells including the inhibitor drug discussed in [16]. Let x(t) be the number of infected cells and y(t) be the number of virus producing cells and z(t) be the density of the Cytotoxic T-Lymphocyte (CTL) responses against virus-infected cells.

Model 1:

In this basic delay HIV-1 infection model, we assume that the virus producing cells are killed by CTL instantaneously. When the delay τ is small, this model can be represented by the following set of equations

$$\frac{dx}{dt} = \lambda - dx - \beta x(t-\tau)y(t-\tau)$$

$$\frac{dy}{dt} = \beta x(t-\tau)y(t-\tau) - ay - pyz$$

$$\frac{dz}{dt} = ky - bz$$

Model 2:

In reality, there is a latency period during the process of killing of virus-producing cells by CTL. (i.e. not instantaneous as in Model 1). Hence we include a delay in the terms representing killing of virus-producing cells by CTL and in the stimulation of CTL. The model equations are given by

$$\frac{dx}{dt} = \lambda - dx - \beta xy$$

$$\frac{dy}{dt} = \beta xy - ay - py(t-\tau)z$$

$$\frac{dz}{dt} = ky(t-\tau) - bz$$

Model 3:

In this model, we include the delays exist in the process of infection of healthy T cells and also in the terms representing killing of virus-producing cells by CTL and in the stimulation of CTL together. The model equations can be represented by the following set of equations

$$\frac{dx}{dt} = \lambda - dx - \beta x(t-\tau_1)y(t-\tau_1)$$

$$\frac{dy}{dt} = \beta x(t-\tau_1)y(t-\tau_1) - ay - py(t-\tau_2)z$$

$$\frac{dz}{dt} = ky(t-\tau_2) - bz$$

The variables and parameters used in these three models are given in Table 5.

For the Models 1 and 2, the initial conditions are taken as

$x(\theta) = 280.0$, $y(\theta) = 18.5189$ and $z(\theta) = 185.1893$

and for the Model 3 as

$x(\theta) = 230.0$, $y(\theta) = 18.5189$ and $z(\theta) = 185.1893$

where $\theta \in (-\tau, 0]$.

The numerical simulations of these models by GRMM (2, 4) are given in Fig. 7 - 9.

VI. CONCLUSION

In this paper, the generalized rational multi-step method of r-step and p-th order by means of rational interpolating function is presented for solving DDEs. The local truncation errors have been determined. The stability polynomials of GRMM (2, p) where p = 2, 3, 4 are derived and their corresponding stability regions are obtained. The delay argument is approximated using Lagrange interpolation.

Numerical examples of DDEs with constant delay, time dependent delay and state dependent delays have been considered to demonstrate the efficiency of the proposed method. From the Tables 1 – 4, it is evident that the proposed method gives results with good accuracy. In HIV-1 infection model, the solution graphs are well comparable with the numerical simulations given in [16]. Hence, it is concluded that the proposed GRMM (r, p) is suitable for solving DDEs.

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TABLE 1

ABSOLUTE ERRORS IN y_1 OF EXAMPLE 1

Time T	Absolute error in GRMM(2,2)	Absolute error in GRMM(2,3)	Absolute error in GRMM(2,4)
0.2	7.540422e-07	1.256772e-12	1.256772e-12
0.4	1.364571e-06	2.274514e-12	2.274514e-12
0.6	1.852071e-06	3.087086e-12	3.086975e-12
0.8	2.234430e-06	3.724465e-12	3.724354e-12
1.0	2.527244e-06	4.212519e-12	4.213074e-12

TABLE 2

ABSOLUTE ERRORS IN y_2 OF EXAMPLE 1

Time t	Absolute error in GRMM(2,2)	Absolute error in GRMM(2,3)	Absolute error in GRMM(2,4)
0.2	3.313668e-04	3.258823e-04	3.258823e-04
0.4	5.706615e-04	5.616804e-04	5.616803e-04
0.6	7.377051e-04	7.266749e-04	7.266748e-04
0.8	8.484124e-04	8.363706e-04	8.363705e-04
1.0	9.155322e-a04	9.032078e-04	9.032077e-04

TABLE 3
ABSOLUTE ERRORS IN y OF EXAMPLE 2

Time T	Absolute error in GRMM(2,2)	Absolute error in GRMM(2,3)	Absolute error in GRMM(2,4)
1.1	3.696129e-06	3.030344e-06	3.780994e-07
1.2	4.314730e-06	3.072100e-06	3.903088e-08
1.3	4.693366e-06	3.133922e-06	9.991727e-07
1.4	5.001597e-06	3.222089e-06	2.729523e-06
1.5	5.289875e-06	3.318091e-06	3.626918e-06

TABLE 4
ABSOLUTE ERRORS IN y OF EXAMPLE 3

Time T	Absolute error in GRMM(2,2)	Absolute error in GRMM(2,3)	Absolute error in GRMM(2,4)
0.2	1.347972e-05	1.882367e-08	6.554333e-09
0.4	2.798915e-05	8.214616e-08	8.342122e-09
0.6	4.478309e-05	2.044308e-07	8.551573e-09
0.8	6.555183e-05	4.206720e-07	1.155962e-08
1.0	9.305806e-05	2.850904e-06	3.236336e-08

TABLE 5
VARIABLES AND PARAMETERS USED IN THE MODELS

Parameters	Definition	Default values assigned
λ	production rate of CD4+ T cells	$10.0\text{mm}^{-3}\ \text{day}^{-1}$
d	Death rate of susceptible CD4+ T cells	0.01day^{-1}
β	Rate of contact between x and y	$0.002\text{mm}^{-3}\ \text{day}^{-1}$
a	Death rate of virus-producing cells	0.24day^{-1}
k	Rate of stimulation of CTL	0.2day^{-1}
b	Death rate of CTL	0.02day^{-1}
p	Killing rate of virus-producing cells by CTL	$0.001\text{mm}^{-3}\ \text{day}^{-1}$

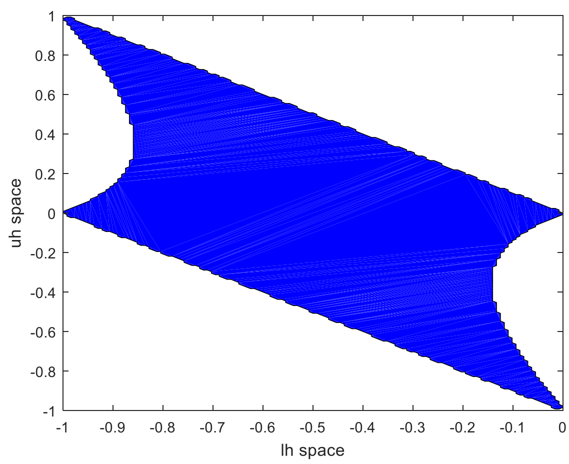


Fig. 1 Stability Region of GRMM (2, 2)

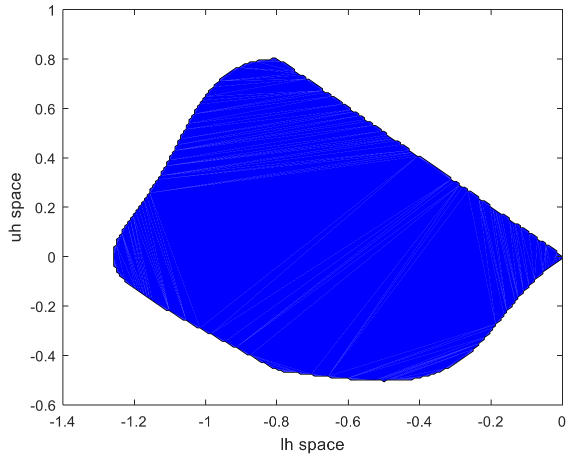


Fig. 2 Stability Region of GRMM (2, 3)

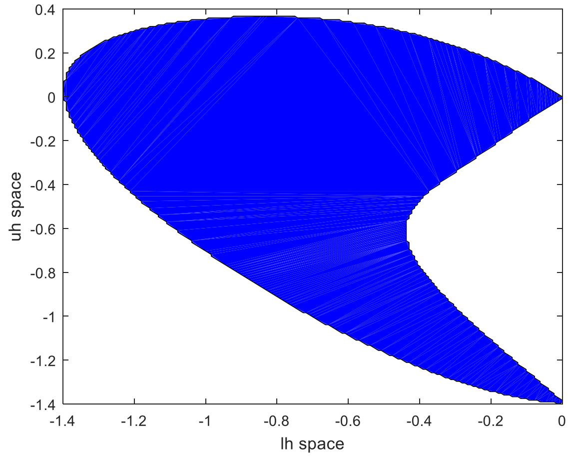


Fig. 3 Stability Region of GRMM (2, 4)

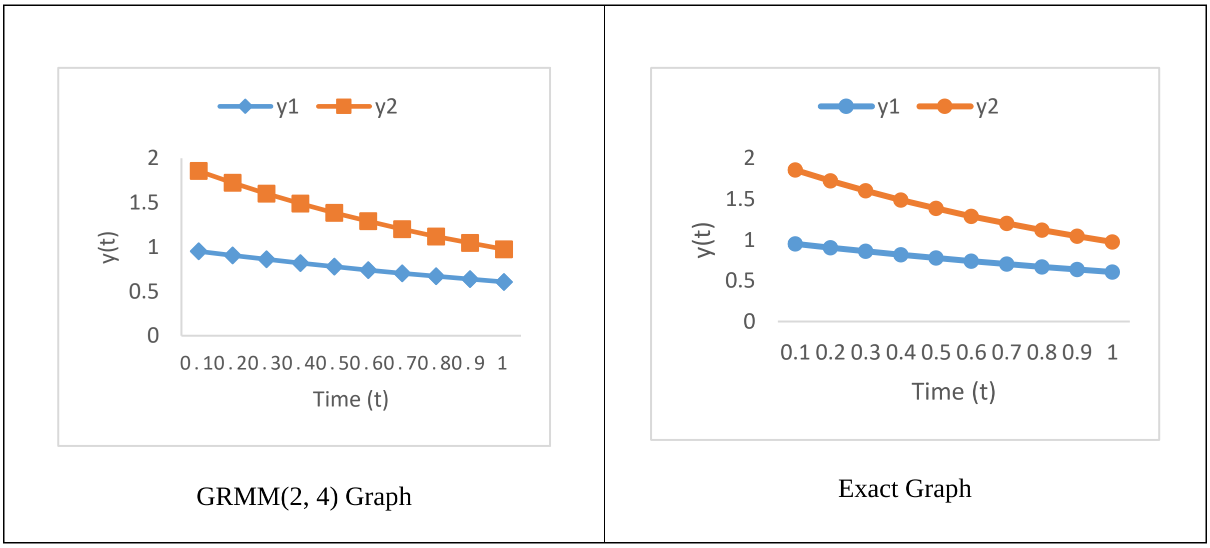
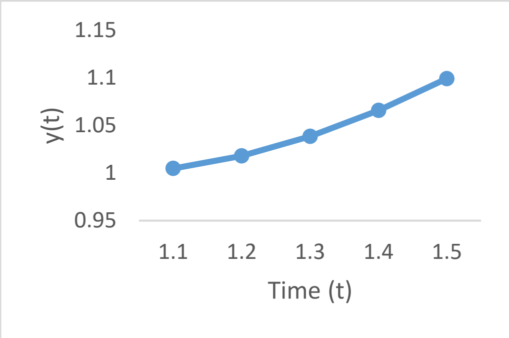
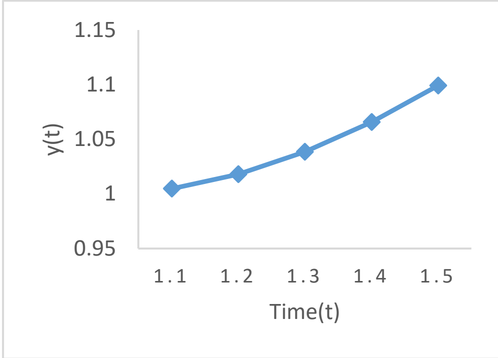


Fig. 4. Solution Graphs of Example 1

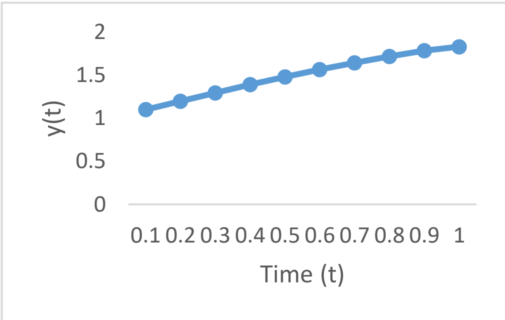


GRMM(2, 4) Graph

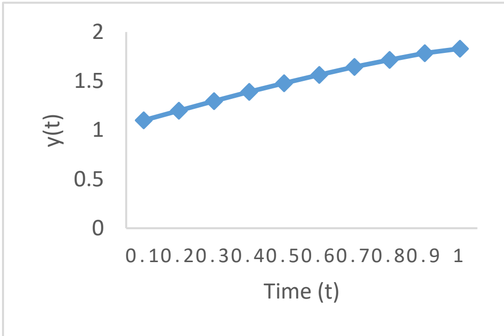


Exact Graph

Fig.5. Solution Graphs of Example 2



GRMM(2, 4) Graph



Exact Graph

Fig.6. Solution Graphs of Example 3

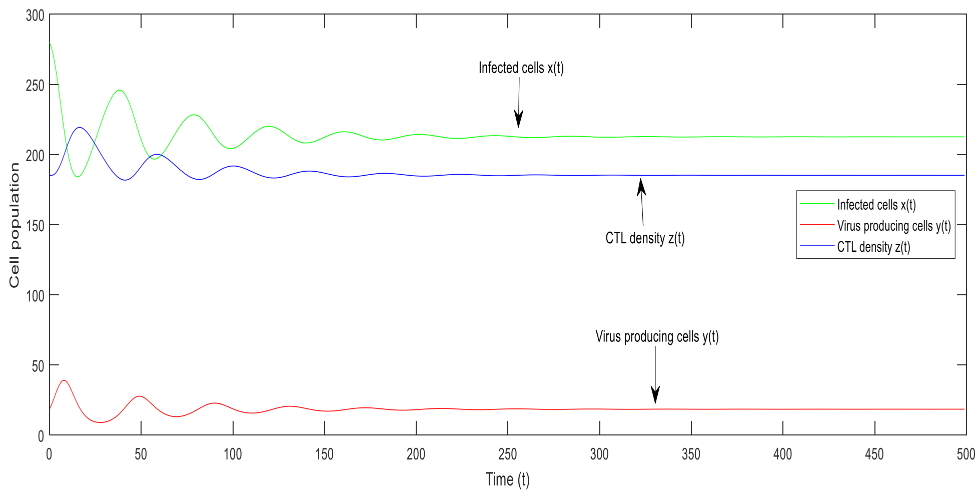


Fig. 7 Solution Graphs of GRMM (2, 4) in Example 4 (Model 1: $\tau = 1$)

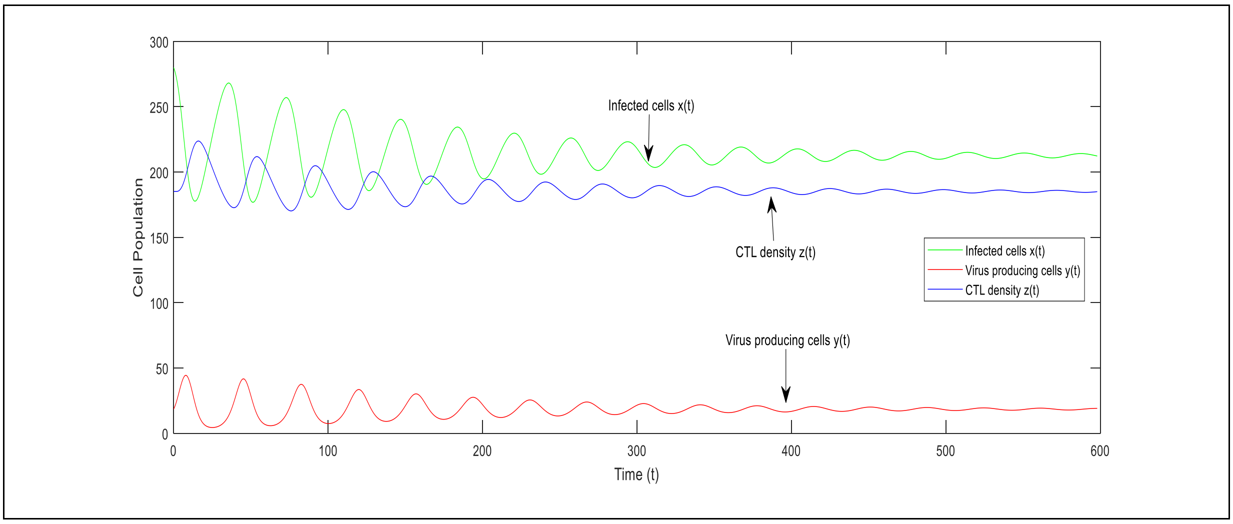


Fig. 8 Solution Graphs of GRMM (2, 4) in Example 4 (Model 2: $\tau = 1$)

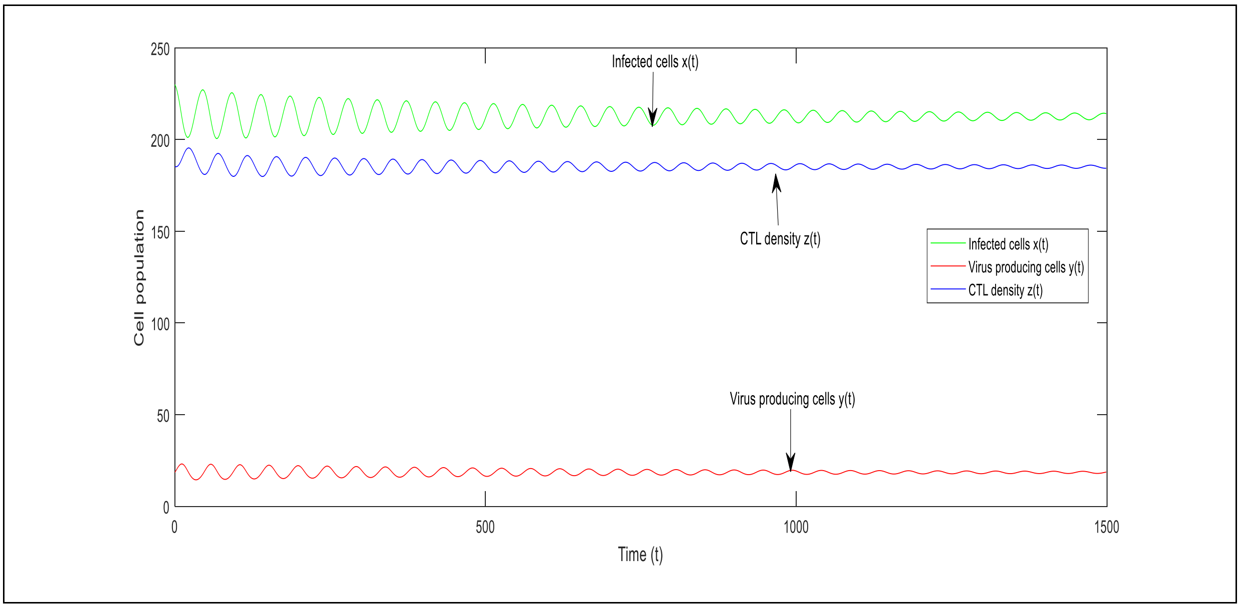


Fig. 9 Solution Graphs of GRMM (2, 4) in Example 4 (Model 3: $\tau_1 = 1, \tau_2 = 2$)