Explicit Error Estimate for the Nonconforming Crouzeix-Raviart Finite Element

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Abstract—**In this paper, we study the explicit expressions of the constants in the error estimate of the nonconforming finite element method. We obtain an explicit relation between the consistency error estimate and the geometric characters of the triangle, which together with the interpolation error estimate implies the final finite element error estimate. Furthermore, such explicit a priori error estimate can be used as computable error bound, which is also consistent with the maximal angle condition for the optimal error estimate of the nonconforming finite element method.**

Index Terms—**Nonconforming finite element, Explicit error estimate, Maximal angle condition, Crouzeix-Raviart element.**

I. Introduction

As a very effective numerical method of partial differential equations, the finite element method (FEM) is widely applied to the engineering and scientific computation. Furthermore, it also has formed firm theoretical bases such as a priori and a posteriori error estimations, seeing, e.g. ([1], [2], [3]). Let u, u_h denote the exact solution of the model problem and the associated discrete solution, respectively. The convergence analysis of the finite element method is typical of the form

$$\|u - u_h\| \leq Ch^k|u|, \tag{I.1}$$

where h denotes the maximal diameter of the triangulation, $\|\cdot\|$ and $|\cdot|$ stand for some appropriate norm and semi-norm in certain function spaces, respectively.

The constant C in (I.1) is independent of element size, but may dependent on the sine of the minimal angle of the triangulation for the 2-D case, which is equivalent to the well-known non-degenerate assumption or regular assumption of finite element meshes, seeing, e.g. ([4]). In fact, the minimal angle condition for the finite elements can be relaxed, which results in the so-called anisotropic elements or degenerate elements found long time before in [5]. Since late 1980's the anisotropic elements have been extensively studied, ref. ([6], [7]) and references therein.

As is known that there also appear various constants in the process to derive the final error estimates. It is good to evaluate or bound these constants explicitly for a quantitative error bound purpose. Actually, there are some works on estimation of the error constants for the finite elements, see, e.g., ([8], [9], [10]) for linear triangular finite elements and ([11], [12]) for bilinear quadrilateral finite elements. However, almost all of them are concentrated on the standard conforming finite element method, which only involves an explicit interpolation error estimate. For the nonconforming finite element method, an explicit error estimate for the Crouzeix-Raviart element is given in [13] by its close relation to the mixed finite element method. Note that Kikuchi and Liu [14] have derived an explicit interpolation error bound for the Crouzeix-Raviart element, but it can not imply an explicit bound for the finite element error.

In this paper, we still study explicit error estimate for the nonconforming Crouzeix-Raviart element for the Poisson problem. The difference between our method and that in [13] is that we obtain the explicit error bounds for the Crouzeix-Raviart element by the standard nonconforming finite element method in stead of its close relation to the mixed finite element method. Therefore, our method is more suitable for general nonconforming case. In this process, the finite element error of the nonconforming element can be explicitly estimated by the sum of an interpolation error and a consistency error, seeing section II. The interpolation error has been obtained by Kikuchi and Liu, seeing section III . In section IV, the consistency error is considered based on some anisotropic results given by Apel et al. in [15]. In section V, the results in sections III and IV are applied to obtain the explicit error estimate for the Poisson problem. Furthermore, our explicit error estimate for the nonconforming element is consistent with the maximal angle condition. In section VI, we show by a numerical experiment that the experimentally determined constants are below the theoretical estimate given in section V. In the final section of the paper, we give some comments and extensions of the results.

II. Discretization of the model problem

In this paper, let Ω be a bounded convex polygonal domain, and the semi-norms in Sobolev space $H^2(\Omega)$ be define by

$$|v|_{1,\Omega} = (\sum_{i=1}^{2} \|\frac{\partial v}{\partial x_i}\|_{0,\Omega}^2)^{\frac{1}{2}},$$

$$|v|_{2,\Omega} = (\sum_{i,j=1}^{2} \|\frac{\partial^2 v}{\partial x_i \partial x_j}\|_{0,\Omega}^2)^{\frac{1}{2}}.$$

We consider the following Poisson problem: find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{II.1}$$

where $f \in L^2(\Omega)$.

It is known that for the convex domain with $\partial\Omega \in C^2$, the above problem has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, the following Miranda-Talente estimate holds, ref. ([16], [17]),

$$|u|_{2,\Omega} \leq \|f\|_{0,\Omega}, \tag{II.2}$$

which was extended to the general convex polygonal domain in [18].

Manuscript received June 17, 2019; revised September, 23, 2019. This work was supported in part by Higher Education Key Scientific Research Program Funded by Henan Province (No. 20A110011, 20B630002).

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The variational form of problem (II.1) is to find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{II.3}$$

Here, $(\cdot,\cdot)_\Omega$ denotes the inner products of both $L^2(\Omega)$ and $L^2(\Omega)^2$, where the subscript Ω is often omitted. The existence and uniqueness of variational solution u of (II.3) follows from Lax-Milgram lemma.

Let $\mathcal{T}_h$ be a finite element triangulation of Ω. Based on $\mathcal{T}_h$, we introduce the Crouzeix-Raviart finite element space

$$V_h \triangleq \Big\{ v_h \in L^2(\Omega); v_h|_T \in P_1(T), \\ \int_l [v_h] ds = 0, \forall l \subseteq \partial T,\ \forall T \in \mathcal{T}_h \Big\},$$

where $[v_h]$ is the jump of the function v_h on the edge. And for boundary edges, we identify $[v_h]$ with v_h.

We note that $V_h \not\subseteq H_0^1(\Omega)$, which means that the method is nonconforming. Thus ∇v_h is not defined on inter-element boundaries and we define the finite solution u_h by using the weaker inner product

$$(u, v)_h \triangleq \sum_T (u, v)_T,$$

that is to find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h)_h = (f, v_h),\ \forall v_h \in V_h.$$

Similar to the proof of the second Strang's lemma [1], the finite element error $u - u_h$ can be explicitly estimated in the norm $\|\cdot\|_h$, that is,

$$\|u - u_h\|_h \leq \inf_{v_h \in V_h} \|u - v_h\|_h \\ + \sup_{v_h \in V_h} \frac{|(\nabla u, \nabla v_h)_h - (f, v_h)|}{\|v_h\|_h}, \tag{II.4}$$

where $\|v_h\|_h = (\sum_T |v_h|_{1,T}^2)^{\frac{1}{2}}$. The two terms are called an approximation error and a consistency error, respectively. The approximation error is estimated by using $v_h = I_h v$ with a suitably defined interpolation operator I_h, seeing the next section. A general discussion of the consistency error is given in section IV .

III. Approximation error estimate

In this section, we mainly present the results given by Kikuchi and Liu, ref. [14]. To this end, let h_T, a_T and θ_T be positive constants such that

$$h_T > 0, \\ 0 < a_T \leq 1, \\ (\frac{\pi}{3} \leq) \cos^{-1}\frac{a_T}{2} \leq \theta_T < \pi. \tag{III.1}$$

Then we define the triangle T_{a_T,θ_T,h_T} by $\triangle OAB$ with three vertices $O(0,0), A(h_T,0)$ and $B(a_T h_T \cos\theta_T, a_T h_T \sin\theta_T)$. From (III.1), $\angle AOB = \theta_T$ is the maximum interior angle and AB the edge of maximum length, i.e., $|AB| \geq h_T \geq a_T h_T$, so that $h_T = |OA|$ denotes the medium edge length, although the notation h_T is often used as the largest edge length. Since we can configure any triangle T as T_{a_T,θ_T,h_T} by an appropriate congruent transformation in R^2, without loss of generality we suppose that T_{a_T,θ_T,h_T} is a general triangle T with the edges $l_1 = OA, l_2 = OB$ and $l_3 = AB$. Then l_2 is the minimal edge and l_1, l_3 are two long edges. We will use abbreviated notations $T = T_{a_T,\theta_T,h_T}, \bar{T} = T_{a_T,\frac{\pi}{2},h_T}$ and $\hat{T} = T_{1,\frac{\pi}{2},1}$, seeing Fig. 1.

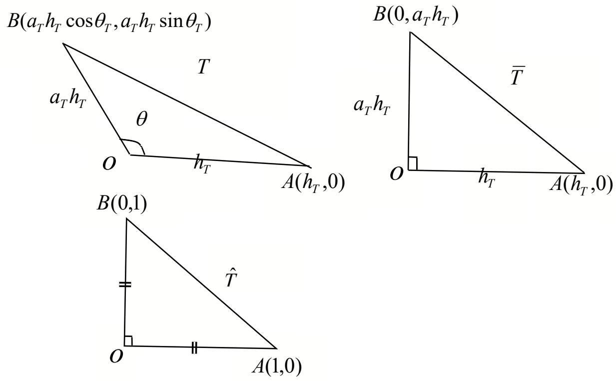

Fig. 1. Notations for triangles: $T = T_{a_T,\theta_T,h_T}$, $\bar{T} = T_{a_T,\frac{\pi}{2},h_T}$ and $\hat{T} = T_{1,\frac{\pi}{2},1}$

We define the following closed linear spaces for functions over T:

$$V_T^0 = \Big\{ v \in H^1(T);\ \int_T v \mathrm{d}x = 0 \Big\}, \tag{III.2}$$

$$V_T^1 = \Big\{ v \in H^1(T);\ \int_{l_1} v \mathrm{d}s = 0 \Big\}, \tag{III.3}$$

$$V_T^2 = \Big\{ v \in H^1(T);\ \int_{l_3} v \mathrm{d}s = 0 \Big\}. \tag{III.4}$$

To give the interpolation error estimate, we introduce the positive constants defined by

$$C_i(a_T, \theta_T, h_T) = \sup_{v \in V_T^i} \frac{\|v\|_{0,T}}{|v|_{1,T}},\ i = 0, 1, 2. \tag{III.5}$$

The existence of these constants follows from the standard compactness arguments. For convenience, we will use the abbreviated notations $C_i(a_T, h_T) = C_i(a_T, \frac{\pi}{2}, h_T)$ and $C_i = C_i(1, \frac{\pi}{2}, 1)$ for $i = 0, 1, 2$. We present the estimates of them as follows, ref. ([14], Theorems 1–4).

Lemma 3.1:

$$C_i(a_T, \theta_T, h_T) \leq C_i(a_T, h_T)(1 + |\cos\theta_T|)^{\frac{1}{2}} \\ \leq C_i(1 + |\cos\theta_T|)^{\frac{1}{2}} h_T, \tag{III.6}$$

where $i = 0, 1, 2$,

$$C_0 \leq \frac{1}{\pi}, \\ C_1 = C_2, \\ 0.49282 \leq C_1 \leq 0.49293. \tag{III.7}$$

The Crouzeix-Raviart interpolation $I_h : H^1(\Omega) \to V_h$ is defined by

$$\int_l I_h u \mathrm{d}s = \int_l u \mathrm{d}s,\ \forall l \subset \partial T,\ \forall T \in \mathcal{T}_h. \tag{III.8}$$

We present the local error estimate for the interpolation I_h which is given in [14].

Lemma 3.2: For any $v \in H^2(T)$, we have

$$\big| v - I_h v \big|_{1,T} \leq C_0 \big(1 + |\cos\theta_T|\big)^{\frac{1}{2}} h_T |v|_{2,T}. \tag{III.9}$$

Therefore, according to Lemma 3.2 and (II.2) we set $v_h = I_h u$ in the first term of (II.4), and then we get the approximation error estimate

$$\begin{aligned}\inf_{v_h\in V_h}\|u-v_h\|_h &\le \|u-I_hu\|_h\\ &=(\sum_T |u-I_hu|_{1,T}^2)^{\frac{1}{2}}\\ &\le \sqrt{2}C_0h|u|_{2,\Omega}\\ &\le \sqrt{2}C_0h\|f\|_{0,\Omega},\end{aligned}\tag{III.10}$$

where $h=\max\limits_T h_T$.

IV. Consistency error estimate

A. Simple triangulation $\bar{\mathcal{T}}_h=\{\bar{T}\}$

In this subsection, we discuss the explicit estimate of the consistency error over the simple triangulation $\bar{\mathcal{T}}_h$. Firstly we introduce a sharp trace theorem on $\hat{T}$.

Lemma 4.1: $\forall v\in H^1(\hat{T}),\ \forall\varepsilon>0$, we have

$$\|v\|_{0,\hat{l}_i}^2\le 2(1+\frac{1}{\varepsilon^2})\|v\|_{0,\hat{T}}^2+\varepsilon^2|v|_{1,\hat{T}}^2,\ i=1,2,\tag{IV.1}$$

$$\|v\|_{0,\hat{l}_3}^2\le 2\sqrt{2}(1+\frac{2}{\varepsilon^2})\|v\|_{0,\hat{T}}^2+2\sqrt{2}\varepsilon^2|v|_{1,\hat{T}}^2.\tag{IV.2}$$

Proof: The proof for (IV.1) can be found in [13]. Therefore we only need to give the proof of (IV.2).

By density, we assume that $v\in C^\infty(\hat{T})$. Let $\hat{K}$ be the unit square domain containing $\hat{T}$, i.e.,

$$\hat{K}=\{(x,y);\ 0\le x,y\le 1\}.$$

Then we can extend v to $\hat{K}$ by reflection with respect to the line $x+y=1$ as following:

$$\tilde{v}=\begin{cases} v(x,y), & (x,y)\in\hat{T},\\ v(1-y,1-x), & (x,y)\in\hat{K}/\hat{T}.\end{cases}\tag{IV.3}$$

Due to the symmetry of $\tilde{v}$, we have

$$\begin{aligned}\|\tilde{v}\|_{0,\hat{K}}^2&=2\|v\|_{0,\hat{T}}^2,\\ \|\frac{\partial\tilde{v}}{\partial x}\|_{0,\hat{K}}^2&=|v|_{1,\hat{T}}^2,\\ \|\frac{\partial\tilde{v}}{\partial y}\|_{0,\hat{K}}^2&=|v|_{1,\hat{T}}^2.\end{aligned}\tag{IV.4}$$

Here we only prove the second equation in (IV.4), i.e.,

$$\begin{aligned}\|\frac{\partial\tilde{v}}{\partial x}\|_{0,\hat{K}}^2&=\|\frac{\partial\tilde{v}}{\partial x}\|_{0,\hat{T}}^2+\|\frac{\partial\tilde{v}}{\partial x}\|_{0,\hat{K}/\hat{T}}^2\\ &=\|\frac{\partial v}{\partial x}\|_{0,\hat{T}}^2+\int_0^1\mathrm{d}x\int_{1-x}^1(\frac{\partial v(1-y,1-x)}{\partial x})^2\mathrm{d}y\\ &=\|\frac{\partial v}{\partial x}\|_{0,\hat{T}}^2+\int_0^1\mathrm{d}z\int_0^{1-z}(\frac{\partial v(t,z)}{\partial z})^2\mathrm{d}t\\ &=|v|_{1,\hat{T}}^2.\end{aligned}$$

The third one can be proved similarly and the first one is obvious. Consider the function $w\in C^\infty(\hat{K})$, then

$$\begin{aligned}w^2(\xi,\eta)&=w^2(x,y)+2\int_x^\xi w(t,\eta)\frac{\partial w(t,\eta)}{\partial t}\mathrm{d}t\\ &\quad+2\int_y^\eta w(x,z)\frac{\partial w(x,z)}{\partial z}\mathrm{d}z\\ &\le w^2(x,y)+2|\int_x^\xi w(t,\eta)\frac{\partial w(t,\eta)}{\partial t}\mathrm{d}t|\\ &\quad+2|\int_y^\eta w(x,z)\frac{\partial w(x,z)}{\partial z}\mathrm{d}z|.\end{aligned}\tag{IV.5}$$

Integrating both sides of (IV.5) on $\hat{l}_3$ and $\hat{K}$, by simple calculations we get

$$\begin{aligned}\int_{\hat{l}_3}w^2\mathrm{d}\hat{s}&\le\sqrt{2}\|w\|_{0,\hat{K}}^2+2\sqrt{2}\|w\|_{0,\hat{K}}\|\frac{\partial w}{\partial x}\|_{0,\hat{K}}\\ &\quad+2\sqrt{2}\|w\|_{0,\hat{K}}\|\frac{\partial w}{\partial y}\|_{0,\hat{K}}\\ &\le\sqrt{2}(1+\frac{2}{\varepsilon^2})\|w\|_{0,\hat{K}}^2+\sqrt{2}\varepsilon^2|w|_{1,\hat{K}}^2.\end{aligned}\tag{IV.6}$$

Since $C^\infty(\hat{K})$ is dense in $H^1(\hat{K})$, according to (IV.3–IV.4) and (IV.6) we obtain

$$\begin{aligned}\int_{\hat{l}_3}v^2\mathrm{d}\hat{s}&=\int_{\hat{l}_3}\tilde{v}^2\mathrm{d}\hat{s}\\ &\le\sqrt{2}(1+\frac{2}{\varepsilon^2})\|\tilde{v}\|_{0,\hat{K}}^2+\sqrt{2}\varepsilon^2|\tilde{v}|_{1,\hat{T}}^2\\ &\le 2\sqrt{2}(1+\frac{2}{\varepsilon^2})\|v\|_{0,\hat{T}}^2+2\sqrt{2}\varepsilon^2|v|_{1,\hat{T}}^2,\end{aligned}$$

which is (IV.2). ∎

Let $P_{\bar{T}}:L^1(\bar{T})\to P_0(\bar{T})$ and $P_l:L^1(l)\to P_0(l)$ be the averaging operator on the triangle $\bar{T}$ and the edge l which preserve polynomials of degree zero, respectively. For the convenience of the subsequent discussion, we present the following estimates.

Lemma 4.2: For any $v\in H^1(\bar{T})$ and $v_h\in P_1(\bar{T})$, we have

$$\begin{aligned}&\left|\int_{l_i}(v-P_{\bar{T}}v)(v_h-P_{l_i}v_h)\mathrm{d}s\right|\\ &\le\sqrt{(C_0^2+\sqrt{2}C_0)(C_1^2+\sqrt{2}C_1)}\\ &\quad\cdot\frac{|l_i|}{|\bar{T}|}\Big(\sum_{j=1,2}|l_j|^2\|\frac{\partial v}{\partial x_j}\|_{0,\bar{T}}^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1,2}|l_j|^2\|\frac{\partial v_h}{\partial x_j}\|_{0,\bar{T}}^2\Big)^{\frac{1}{2}},\\ &\quad i=1,2,\end{aligned}\tag{IV.7}$$

$$\begin{aligned}&\left|\int_{l_3}(v-P_{\bar{T}}v)(v_h-P_{l_3}v_h)\mathrm{d}s\right|\\ &\le\sqrt{(C_0^2+\sqrt{2}C_0)(\sqrt{2}C_1^2+4C_1)}\\ &\quad\cdot\frac{|l_3|}{|\bar{T}|}\Big(\sum_{j=1,2}|l_j|^2\|\frac{\partial v}{\partial x_j}\|_{0,\bar{T}}^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1,2}|l_j|^2\|\frac{\partial v_h}{\partial x_j}\|_{0,\bar{T}}^2\Big)^{\frac{1}{2}}.\end{aligned}\tag{IV.8}$$

Proof: By using transformation to the reference edge $\hat{l}_i$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\int_{l_i}(v-P_{\bar{T}}v)(v_h-P_{l_i}v_h)\mathrm{d}s\\ &=|l_i|\int_{\hat{l}_i}(\hat{v}-P_{\hat{T}}\hat{v})(\hat{v}_h-P_{\hat{l}_i}\hat{v}_h)\\ &\le|l_i|\left\|\hat{v}-P_{\hat{T}}\hat{v}\right\|_{0,\hat{l}_i}\left\|\hat{v}_h-P_{\hat{l}_i}\hat{v}_h\right\|_{0,\hat{l}_i}.\end{aligned}\tag{IV.9}$$

For $i=1,2$, combining (III.6) and (IV.1), we have

$$\begin{aligned}\|\hat{v}-P_{\hat{T}}\hat{v}\|_{0,\hat{l}_i}^2&\le 2(1+\frac{1}{\varepsilon^2})\left\|\hat{v}-P_{\hat{T}}\hat{v}\right\|_{0,\hat{T}}^2+\varepsilon^2|\hat{v}|_{1,\hat{T}}^2\\ &\le(2C_0^2+\frac{2C_0^2}{\varepsilon^2}+\varepsilon^2)|\hat{v}|_{1,\hat{T}}^2\\ &\xlongequal{\varepsilon^2=\sqrt{2}C_0}(2C_0^2+2\sqrt{2}C_0)|\hat{v}|_{1,\hat{T}}^2.\end{aligned}\tag{IV.10}$$

Since the transformation from $\hat{T}$ to $\bar{T}$ leads to

$$(\frac{\partial\hat{v}}{\partial\hat{x}_1})^2+(\frac{\partial\hat{v}}{\partial\hat{x}_2})^2=|l_1|^2(\frac{\partial v}{\partial x_1})^2+|l_2|^2(\frac{\partial v}{\partial x_2})^2.\tag{IV.11}$$

Then (IV.10-IV.11) imply that

$$\|\hat{v}-P_{\hat{T}}\hat{v}\|_{0,\hat{l}_i}^2 \le \frac{(C_0^2+\sqrt{2}C_0)}{|\bar{T}|} \cdot \Big(|l_1|^2\|\frac{\partial v}{\partial x_1}\|_{0,\bar{T}}^2+|l_2|^2\|\frac{\partial v}{\partial x_2}\|_{0,\bar{T}}^2\Big), \quad i=1,2. \tag{IV.12}$$

Similarly we also have

$$\Big\|\hat{v}_h-P_{\hat{l}_i}\hat{v}_h\Big\|_{0,\hat{l}_i}^2 \le \frac{(C_1^2+\sqrt{2}C_1)}{|\bar{T}|} \cdot \Big(|l_1|^2\|\frac{\partial v_h}{\partial x_1}\|_{0,\bar{T}}^2+|l_2|^2\|\frac{\partial v_h}{\partial x_2}\|_{0,\bar{T}}^2\Big), \quad i=1,2. \tag{IV.13}$$

Combining (IV.9) and (IV.12–IV.13), we obtain (IV.7). For $i=3$, combining (III.5) and (IV.2), we proceeding along the same lines above and also obtain (IV.8). ■

Suppose $w=\nabla u$. In the sense of (II.4) it is our aim to derive an explicit estimate for

$$\sup_{v_h\in V_h}\frac{|(w,\nabla v_h)_h-(f,v_h)|}{\|v_h\|_h}. \tag{IV.14}$$

To this end, we introduce an auxiliary finite element space

$$\tilde{V}_h \triangleq \Big\{\tilde{v}_h\in L^2(\Omega);\ \tilde{v}_h|_{\bar{T}}\in \mathrm{span}\{1,x_2\}, \int_{l_i}[\tilde{v}_h]\mathrm{d}s=0,\ i=1,3,\ \forall \bar{T}\in\bar{\mathcal{T}}_h\Big\},$$

which is sufficiently close to V_h, ref. [15].

For any fixed $v_h\in V_h$, we define $\tilde{v}_h\in\tilde{V}_h$ such that

$$\int_{l_i}\tilde{v}_h\mathrm{d}s=\int_{l_i}v_h\mathrm{d}s,\ i=1,3,\ \forall\bar{T}\in\bar{\mathcal{T}}_h. \tag{IV.15}$$

Both $\frac{\partial v_h}{\partial x_2}$ and $\frac{\partial \tilde{v}_h}{\partial x_2}$ are constants. Even better on $\bar{T}$, we have

$$\frac{\partial v_h}{\partial x_2}=\frac{\partial \tilde{v}_h}{\partial x_2}, \tag{IV.16}$$

which is given in [15].

Combining (III.5–III.6) and (IV.15–IV.16) we have the following estimate

$$\|v_h-\tilde{v}_h\|_{0,\bar{T}}\le C_1h_T\|\frac{\partial v_h}{\partial x_1}\|_{0,\bar{T}}. \tag{IV.17}$$

We are now prepared to prove an explicit estimate for the consistency error over $\bar{\mathcal{T}}_h$.

Lemma 4.3: For the simple triangulation $\bar{\mathcal{T}}_h$ and $w=\nabla u$, we have

$$\sup_{v_h\in V_h}\frac{|(w,\nabla v_h)_h-(f,v_h)|}{\|v_h\|_h} \le \Big[C_1+2\sqrt{2}\sqrt{C_0^2+\sqrt{2}C_0} \cdot \big(\sqrt{C_1^2+\sqrt{2}C_1}+\sqrt{\sqrt{2}C_1^2+4C_1}\ \big)\Big]h\|f\|_{0,\Omega}. \tag{IV.18}$$

Proof: Since $w=\nabla u$, that is, $-\nabla w=f$, we introduce $\tilde{v}_h$ as above by using (IV.16) and Green's formula as follows,

$$\begin{aligned}
&(w,\nabla v_h)_h-(f,v_h)\\
&=\sum_{\bar{T}}\int_{\bar{T}}\big(w_1\frac{\partial v_h}{\partial x_1}+w_2\frac{\partial \tilde{v}_h}{\partial x_2}-fv_h\big)\mathrm{d}x\\
&=-\sum_{\bar{T}}\int_{\bar{T}}\big(\frac{\partial w_1}{\partial x_1}v_h+\frac{\partial w_2}{\partial x_2}\tilde{v}_h+fv_h\big)\mathrm{d}x\\
&\quad+\sum_{\bar{T}}\int_{\partial\bar{T}}w_1v_hn_1\mathrm{d}s+\sum_{\bar{T}}\int_{\partial\bar{T}}w_2\tilde{v}_hn_2\mathrm{d}s\\
&=\sum_{\bar{T}}\int_{\bar{T}}\frac{\partial w_2}{\partial x_2}(v_h-\tilde{v}_h)\mathrm{d}x+\sum_{\bar{T}}\int_{\partial\bar{T}}w_1v_hn_1\mathrm{d}s\\
&\quad+\sum_{\bar{T}}\int_{\partial\bar{T}}w_2\tilde{v}_hn_2\mathrm{d}s.
\end{aligned} \tag{IV.19}$$

Combining the Cauchy-Schwarz inequality and (IV.17), we obtain the estimate of the first term of (IV.19)

$$\begin{aligned}
&\sum_{\bar{T}}\int_{\bar{T}}\frac{\partial w_2}{\partial x_2}(v_h-\tilde{v}_h)\mathrm{d}x\\
&\le\sum_{\bar{T}}\|\frac{\partial w_2}{\partial x_2}\|_{0,\bar{T}}\|v-\tilde{v}_h\|_{0,\bar{T}}\\
&\le C_1h\sum_{\bar{T}}\|\frac{\partial w_2}{\partial x_2}\|_{0,\bar{T}}\|\frac{\partial v_h}{\partial x_1}\|_{0,\bar{T}}\\
&\le C_1h|w_2|_{1,\Omega}\|v_h\|_h.
\end{aligned} \tag{IV.20}$$

Consider the second term of (IV.19). Since

$$\begin{aligned}
&\sum_{\bar{T}}\sum_{l\in\partial\bar{T}\setminus\partial\Omega}\int_l w_1n_1\mathrm{d}s=0,\\
&P_lv_h=\frac{1}{|l|}\int_l v_h\mathrm{d}s=0,\ \forall l\subset\partial\Omega,
\end{aligned} \tag{IV.21}$$

we reformulate the second term by

$$\begin{aligned}
&\sum_{\bar{T}}\int_{\partial\bar{T}}w_1v_hn_1\mathrm{d}s\\
&=\sum_{\bar{T}}\sum_{l\subset\partial\bar{T}}\int_l w_1(v_h-P_lv_h)n_1\mathrm{d}s.
\end{aligned} \tag{IV.22}$$

Furthermore, since $\int_l(v_h-P_lv_h)\mathrm{d}s=0$ for all l, we continue with

$$\begin{aligned}
&\sum_{\bar{T}}\int_{\partial\bar{T}}w_1v_hn_1\mathrm{d}s\\
&=\sum_{\bar{T}}\sum_{l\subset\partial\bar{T}}\int_l(w_1-P_{\bar{T}}w_1)(v_h-P_lv_h)n_1\mathrm{d}s.
\end{aligned} \tag{IV.23}$$

Since the outward unit normal to $l_i\subset\partial\bar{T}$ is

$$n=\begin{cases}(0,-1), & i=1,\\ (-1,0), & i=2,\\ \big(\frac{a_T}{\sqrt{1+a_T^2}},\frac{1}{\sqrt{1+a_T^2}}\big), & i=3,\end{cases} \tag{IV.24}$$

according to (IV.23) and Lemma 4.2, we obtain

$$\begin{aligned}
&\sum_{\bar{T}}\int_{\partial\bar{T}} w_1 v_h n_1 \mathrm{d}s \\
&\le \sum_{\bar{T}}\sum_{l\subset\partial\bar{T}} \Big|n_1\int_l (w_1 - P_{\bar{T}}w_1)(v_h - P_l v_h)\mathrm{d}s\Big| \\
&\le \sum_{\bar{T}}\Big[\sqrt{(C_0^2+\sqrt{2}C_0)(C_1^2+\sqrt{2}C_1)}\cdot\frac{|l_2|}{|\bar{T}|}h_T^2|w_1|_{1,\bar{T}}|v_h|_{1,\bar{T}} \\
&\quad + \frac{a_T}{\sqrt{1+a_T^2}}\cdot\sqrt{(C_0^2+\sqrt{2}C_0)(\sqrt{2}C_1^2+4C_1)} \\
&\quad \cdot\frac{|l_3|}{|\bar{T}|}h_T^2|w_1|_{1,\bar{T}}|v_h|_{1,\bar{T}}\Big] \\
&= 2\sqrt{C_0^2+\sqrt{2}C_0}\sum_{\bar{T}}\Big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)h_T|w_1|_{1,\bar{T}}|v_h|_{1,\bar{T}} \\
&\le 2\sqrt{C_0^2+\sqrt{2}C_0}\Big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)h|w_1|_{1,\Omega}\|v_h\|_h.
\end{aligned} \tag{IV.25}$$

The third term of (IV.19) can also be explicitly estimated in the same way. We note only two new points. The first is that $P_{l_i}\tilde{v}_h = 0$ is in general only satisfied for the long edges $l_1, l_3 \in \partial\Omega$, compare with the second term of (IV.21). For the short edges $l_2 \in \partial\Omega$, we have to use $n_2 = 0$. Secondly, since $\frac{\partial\tilde{v}_h}{\partial x_1} = 0$ the term $|l_1|^2\|\frac{\partial\tilde{v}_h}{\partial x_1}\|_{0,\bar{T}}^2$ vanishes. Hence the estimate reads

$$\begin{aligned}
&\sum_{\bar{T}}\int_{\partial\bar{T}} w_2\tilde{v}_h n_2\mathrm{d}s \\
&\le \sum_{\bar{T}}\sum_{l\subset\bar{T}}\Big|n_2\int_l (w_2 - P_{\bar{T}}w_2)(\tilde{v}_h - P_l\tilde{v}_h)\mathrm{d}s\Big| \\
&\le \sum_{\bar{T}}\Big[\sqrt{(C_0^2+\sqrt{2}C_0)(C_1^2+\sqrt{2}C_1)} \\
&\quad \cdot\frac{|l_1|}{|\bar{T}|}a_T h_T^2|w_2|_{1,\bar{T}}\|\frac{\partial\tilde{v}_h}{\partial x_2}\|_{0,\bar{T}} \\
&\quad + \frac{\sqrt{(C_0^2+\sqrt{2}C_0)(\sqrt{2}C_1^2+4C_1)}}{\sqrt{1+a_T^2}} \\
&\quad \cdot\frac{|l_3|}{|\bar{T}|}a_T h_T^2|w_2|_{1,\bar{T}}\|\frac{\partial\tilde{v}_h}{\partial x_2}\|_{0,\bar{T}}\Big] \\
&= 2\sqrt{C_0^2+\sqrt{2}C_0}\sum_{\bar{T}}\Big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)h_T|w_2|_{1,\bar{T}}\|\frac{\partial v_h}{\partial x_2}\|_{0,\bar{T}} \\
&\le 2\sqrt{C_0^2+\sqrt{2}C_0}\Big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)h|w_2|_{1,\Omega}\|v_h\|_h,
\end{aligned} \tag{IV.26}$$

where we have also used (IV.16).

Combining (IV.19–IV.20), (IV.25–IV-A) and Hölder inequality, we obtain

$$\begin{aligned}
&\big|(w,\nabla v_h) - (f, v_h)\big| \\
&\le C_1 h|w_2|_{1,\Omega}\|v_h\|_h + 2\sqrt{C_0^2+\sqrt{2}C_0}\Big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)h\big(|w_1|_{1,\Omega}+|w_2|_{1,\Omega}\big)\|v_h\|_h \\
&\le C_1 h|w_2|_{1,\Omega}\|v_h\|_h + 2\sqrt{C_0^2+\sqrt{2}C_0}\Big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)h|w_1|_{1,\Omega}\|v_h\|_h \\
&\le \Big[C_1 + 2\sqrt{2}\sqrt{C_0^2+\sqrt{2}C_0}\big(\sqrt{C_1^2+\sqrt{2}C_1} \\
&\quad + \sqrt{\sqrt{2}C_1^2+4C_1}\big)\Big]h|w|_{1,\Omega}\|v_h\|_h,
\end{aligned}$$

which together with (II.2) can imply (IV.18). ■

B. General triangulation $\mathcal{T}_h = \{T\}$

In this subsection, we explicitly give the consistency error over $\mathcal{T}_h$ based on that over $\bar{\mathcal{T}}_h$.

Lemma 4.4: For the general triangulation $\mathcal{T}_h$ and $w = \nabla u$, put $\theta = \max_T \theta_T$, we have

$$\begin{aligned}
\sup_{v_h\in V_h}\frac{|(w,\nabla v_h)_h - (f,v_h)|}{\|v_h\|_h} &\le \Big(\frac{\sqrt{2}C_1}{\sqrt{\sin\theta}} + \frac{4\sqrt{C_0^2+\sqrt{2}C_0}}{\sqrt{\sin\theta}} \\
&\quad \cdot\big(\sqrt{C_1^2+\sqrt{2}C_1}+\sqrt{\sqrt{2}C_1^2+4C_1}\big)\Big)h\|f\|_{0,\Omega}.
\end{aligned} \tag{IV.27}$$

Proof: Suppose $\bar{F}$ is the transformation from $\bar{T}$ to T and $\bar{v} = v\circ\bar{F}$, then we have

$$\begin{aligned}
&\big(w,\nabla v_h\big)_h - \big(f, v_h\big) \\
&= \sum_T\int_T\big(w_1\frac{\partial v_h}{\partial x_1} + w_2\frac{\partial v_h}{\partial x_2} - f v_h\big)\mathrm{d}x \\
&= \sum_T\sin\theta_T\int_{\bar{T}}\Big[\bar{w}_1\frac{\partial\bar{v}_h}{\partial\bar{x}_1} + \bar{w}_2\big(-\frac{\cos\theta_T}{\sin\theta_T}\frac{\partial\bar{v}_h}{\partial\bar{x}_1} \\
&\quad + \frac{1}{\sin\theta_T}\frac{\partial\bar{v}_h}{\partial\bar{x}_2}\big) - \bar{f}\bar{v}_h\Big]\mathrm{d}\bar{x} \\
&= \sum_T\int_{\bar{T}}\Big[\big(\bar{w}_1\sin\theta_T - \bar{w}_2\cos\theta_T\big)\frac{\partial\bar{v}_h}{\partial\bar{x}_1} \\
&\quad + \bar{w}_2\frac{\partial\bar{v}_h}{\partial\bar{x}_2} - \bar{f}\bar{v}_h\sin\theta_T\Big]\mathrm{d}\bar{x}.
\end{aligned} \tag{IV.28}$$

Let $\bar{\eta}|_{\bar{T}} = (\sin\theta_T\bar{w}_1 - \cos\theta_T\bar{w}_2, \bar{w}_2)$ and $\bar{g}|_{\bar{T}} = \bar{f}\sin\theta_T$, then

$$-\bar{\nabla}\bar{\eta} = -\big(\frac{\partial\bar{\eta}_1}{\partial\bar{x}_1} + \frac{\partial\bar{\eta}_2}{\partial\bar{x}_2}\big) = \bar{g}, \tag{IV.29}$$

$$(w,\nabla v_h)_h - (f,v_h) = (\bar{\eta},\bar{\nabla}\bar{v}_h)_{\bar{h}} - (\bar{g},\bar{v}_h), \tag{IV.30}$$

where $(\bar{\eta},\bar{\nabla}\bar{v}_h)_{\bar{h}} = \sum_{\bar{T}}\int_{\bar{T}}(\bar{\eta}_1\frac{\partial\bar{v}_h}{\partial\bar{x}_1} + \bar{\eta}_2\frac{\partial\bar{v}_h}{\partial\bar{x}_2})\mathrm{d}\bar{x}$.

Combining (IV.29–IV.30) and Lemma 4.3, we get that

$$\begin{aligned}
\big(w,\nabla v_h\big)_h - \big(f,v_h\big) &\le \Big[C_1 + 2\sqrt{2}\sqrt{C_0^2+\sqrt{2}C_0} \\
&\quad \cdot\big(\sqrt{C_1^2+\sqrt{2}C_1}+\sqrt{\sqrt{2}C_1^2+4C_1}\big)\Big]h\|\bar{g}\|_{0,\bar{\Omega}}\|\bar{v}_h\|_{\bar{h}},
\end{aligned} \tag{IV.31}$$

where $\bar{\Omega} = \bigcup\bar{T}$ and $\|\bar{v}_h\|_{\bar{h}} = (\sum_{\bar{T}}|\bar{v}_h|_{1,\bar{T}}^2)^{\frac{1}{2}}$.

Since

$$\begin{aligned}\|\bar{g}\|^2_{0,\bar{\Omega}} &= \sum_{\bar{T}} \sin^2\theta_T \|\bar{f}\|^2_{0,\bar{T}} \\ &= \sum_{T} \sin\theta_T \|f\|^2_{0,T} \\ &\le \|f\|_{0,\Omega},\end{aligned} \tag{IV.32}$$

$$\begin{aligned}\|\bar{v}_h\|^2_{\bar{h}} &= \sum_{\bar{T}} |\bar{v}_h|^2_{1,\bar{T}} = \sum_{\bar{T}} \int_{\bar{T}} [(\frac{\partial \bar{v}_h}{\partial \bar{x}_1})^2 + (\frac{\partial \bar{v}_h}{\partial \bar{x}_2})^2] \mathrm{d}\bar{x} \\ &= \sum_T \frac{1}{\sin\theta_T} \int_T \Big[(\frac{\partial v_h}{\partial x_1})^2 + \big(\cos\theta_T \frac{\partial v_h}{\partial x_1} + \sin\theta_T \frac{\partial v_h}{\partial x_2}\big)^2 \Big] \mathrm{d}x \\ &= \sum_T \frac{1}{\sin\theta_T} \int_T \Big[(1+\cos^2\theta_T)(\frac{\partial v_h}{\partial x_1})^2 \\ &\quad + 2\sin\theta_T\cos\theta_T \frac{\partial v_h}{\partial x_1}\frac{\partial v_h}{\partial x_2} + \sin^2\theta_T (\frac{\partial v_h}{\partial x_2})^2 \Big] \mathrm{d}x \\ &= \sum_T \frac{2}{\sin\theta_T} |v_h|^2_{1,T} \\ &\le \frac{2}{\sin\theta_T} \|v_h\|^2_h.\end{aligned} \tag{IV.33}$$

Thus (IV.31–IV.33) imply (IV.27). ■

V. Explicit error estimate for the model problem

In sections III and IV, we have explicitly estimate the approximation error and the consistency error for the Crouzeix-Raviart element, respectively, seeing (III.10) and (IV.27). Therefore, according to (II.4) we have the following error estimate for the model problem.

Theorem 5.1: Let u be the solution of (II.3) and u_h the finite element solution defined by (II). Assume θ is the maximal angle. Then the finite element error can be explicitly estimated by

$$\begin{aligned}\big\|u-u_h\big\|_h \le \Big[2\sqrt{2}C_0 + \frac{\sqrt{2}C_1}{\sqrt{\sin\theta}} + \frac{4\sqrt{C_0^2+\sqrt{2}C_0}}{\sqrt{\sin\theta}} \\ \cdot \Big(\sqrt{C_1^2+\sqrt{2}C_1} + \sqrt{\sqrt{2}C_1^2+4C_1}\Big)\Big] h\big\|f\big\|_{0,\Omega},\end{aligned}$$

where $C_0 \le \dfrac{1}{\pi}$ and $0.49282 \le C_1 \le 0.49293$.

VI. Numerical experiment

In this section, we test the error bound studied in the above sections by numerical computation. Consider the following Dirichlet problem:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = [0,1]\times[0,1]$ and $f(x,y) = 2\pi^2\sin(\pi x)\sin(\pi y)$. The exact solution of this problem is $u(x,y) = \sin(\pi x)\sin(\pi y)$.

We firstly subdivide Ω into n^2 small rectangles, where each edge of Ω is divided into n segments with $n+1$ points $(1-\cos(\frac{i\pi}{n}))/2,\ i = 0,1,\ldots,n$. By dividing each small rectangle into two triangles, we obtain an anisotropic triangular mesh, seeing Fig. 2 ($n = 16$). Let u_h be the numerical solution.

Numerical calculations are carried out by employing the non-conforming Crouzeix-Raviart finite element method. Numerical results are listed in Table 1. Herein,

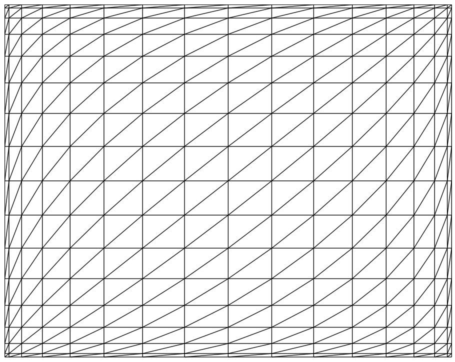

Fig. 2. An mesh for $n = 16$.

TABLE I
NUMERICAL RESULTS FOR THE NON-CONFORMING CROUZEIX-RAVIART FINITE ELEMENT METHOD

n	h	c	$\|u-u_h\|_h$	$\|f\|_{0,\Omega}$	C
8	0.270598	5.685837	0.395260	9.869604	0.147999
16	0.137950	10.729399	0.200933	9.869604	0.147581
32	0.069309	20.892908	0.100884	9.869604	0.147481
64	0.034696	41.254045	0.050495	9.869604	0.147458
128	0.017353	81.992482	0.025256	9.869604	0.147462

$$h = \max_T h_T,\ C = \frac{\|u-u_h\|_h}{h\|f\|_{0,\Omega}},\ c = \max_T \frac{h_T}{\rho_T},$$

where h_T is the diameter of T and ρ_T the diameter of the biggest ball contained in T.

From the numerical results we can easily see that the experimentally determined constants are below the theoretical estimate (≤ 8.997472183) given in section V and the error constant is independent of the ratio h_T/ρ_T.

VII. Conclusion

We have developed the explicit error estimate for the non-conforming Crouzeix-Raviart finite element based on a careful exploration. The explicit constants of some inequalities, together with the consistency error constant, have been obtained. These estimates allow us to derive some computable upper bounds of the nonconforming finite element errors, which can serve as a posteriori error estimate. Another feature of our error estimates is that we do not need to assume any mesh condition on the triangulation.

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