Jacobi Collocation Methods for Solving the Fractional Bagley-Torvik Equation

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Abstract—**In this work, a numerical method based on the Jacobi collocation approximation is extended to the fractional Bagley-Torvik equation. The fractional derivative is described in the Caputo sense. First, the differential equation is equivalently restated as the Volterra integral equation. Then, the Jacobi collocation method is used to solve the integral equation. Convergence analysis of the proposed method is investigated in terms of the L^∞ norm and the weighted L^2 norm. In addition, numerical results are presented to confirm our analysis.**

Index Terms—**Bagley-Torvik equation, Jacobi polynomial, Collocation method, convergence analysis.**

I. Introduction

OVER the past few decades, many mathematical, physical and engineering phenomena have been successfully described using fractional calculus, which is the theory of integrals and derivatives of non-integer orders. For example, Torvik and Bagley [1] formulated a fractional differential equation that simulates the motion of a rigid plate immersed in a Newtonian fluid as follows:

$$Ay''(t) + BD^{\frac{3}{2}}y(t) + Cy(t) = f(t) \tag{1}$$

subject to

$$y(0) = y_0, \quad y'(0) = y'_0. \tag{2}$$

Here, $y(t)$ represents the displacement of the plate of mass M and surface area S. The constants A, B and C are given by

$$A = M, \quad B = 2S\sqrt{\mu\rho},$$

where μ and ρ are the viscosity and density, respectively, of the fluid in which the plate is immersed, and

$$C = k,$$

where k is the stiffness of the spring to which the plate is attached. Finally, $f(t)$ represents the loading force. The operator D^α is the Caputo fractional derivative. The equation of existence and the uniqueness of the solution to this initial value problem have been discussed in [2], [3].

Fractional differential equations have been found to be more realistic in modeling a variety of physical phenomena, engineering processes, biological systems and financial products, such as signal identification and image processing, optical systems, thermal system materials and control systems [4]–[6]. A considerable amount of work has been invested in determining numerical solutions of the Bagley-Torvik equation. Podlubny [7] also investigated the solution of this problem and proposed a numerical method in his book. Ray and Bera [8] applied the Adomian decomposition method to solve the Bagley-Torvik equation and obtained the same solution as Podlubny's solution by the Green's function. Diethelm and Ford [9] solved the problem with the Adams predictor and corrector method. Cenesiz et al. [10], [11] suggested a new generalization of the Taylor collocation method for obtaining a numerical solution of a class of fractional order differential equations. Li and Ray [12], [13] derived the Haar wavelet operational matrix of the fractional order integration and applied the matrix to the Bagley-Torvik equation. In [14]–[17], Chebyshev polynomial and hybrid functions were considered to find an approximate solution for the problem. Most recently, Raja [18] introduced fractional neural networks to solve fractional order differential equations, including the Bagley-Torvik equation. Sakar et al. applied a new reproducing kernel Hilbert space to the fractional Bagley–Torvik equation [19].

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II. Basic definitions of fractional calculus theory

There are various definitions of fractional integrals and derivatives. The widely used definition of a fractional integral is the Riemann-Liouville definition, and that of a fractional derivative is the Caputo definition.

Definition 1: The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0.$$

Definition 2: The fractional derivative D^α of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d(\tau),$$

for $n-1 < \alpha \leq n, \quad n \in N, \quad t > 0$, and $\quad f(t) \in C^n_{-1}$. For the Caputo definition, we have

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0)\frac{t^i}{i!}. \tag{3}$$

III. Existence and Uniqueness of Solutions

The existence and uniqueness of the solution to linear and nonlinear fractional differential equations have been thoroughly investigated in [20], [21] and [22]. Before the description of existence and uniqueness of solutions, we find it convenient to rewrite the original Bagley-Torvik equation in the form of the integral equations. Without loss of generality, we consider $y_0 = 0, y_0' = 0$ and $A = 1$. We convert (1) and (2) to the forms of

$$\tilde{y}''(t) + BD^{\frac{3}{2}}\tilde{y}(t) + C\tilde{y}(t) = \tilde{f}(t), \quad t \in [0, T] \tag{4}$$

and

$$\tilde{y}(0) = 0, \quad \tilde{y}'(0) = 0, \tag{5}$$

where $\tilde{y}(t) = y(t) - y_0 - y_0't$, and $\tilde{f}(t) = f(t) - y_0 - y_0't$. Integrating (4) and making use of (3), we obtain the integrated form of (4) and (5):

$$\tilde{y}(t) = \frac{-B}{\sqrt{\pi}}\int_0^t (t-s)^{-\frac{1}{2}}\tilde{y}(s)ds - C\int_0^t (t-s)\tilde{y}(s)ds + g(t), \tag{6}$$

where $t \in [0, T]$ and $g(t) = \int_0^t (t-s)\tilde{f}(s)ds$. The Bagley-Torvik equation is equivalent to the integral equations (6) For simplicity, the equation (6) that we shall consider is of the form

$$\tilde{y}(t) = \int_0^t K(t,s)\tilde{y}(s)ds + g(t) \tag{7}$$

where

$$K(t,s) = \frac{-B}{\sqrt{\pi}}(t-s)^{\frac{1}{2}} - C(t-s).$$

Theorem 3: Assume the function $\tilde{f}(t) \in L^1[0,T]$ is bounded. Then the problem (4) and (5) has a unique continuous solution.

Proof: Consider the iterations

$$\tilde{y}_k(t) = \int_0^t K(t,s)\tilde{y}_{k-1}(s)ds + g(t), k = 1, 2, \cdots, \tag{8}$$

and $\tilde{y}_0(t) = g(t)$. Moreover, letting

$$\phi_k(t) = \tilde{y}_k(t) - \tilde{y}_{k-1}(t), \quad \phi_0(t) = g(t) \tag{9}$$

we have

$$\phi_k(t) = \int_0^t K(t,s)\phi_{k-1}(s)ds, \quad k = 1, 2, \cdots. \tag{10}$$

From (8),(9), we obtain

$$y_k(t) = \sum_{i=0}^{k} \phi_i(t).$$

Next, since $\tilde{f}(t) \in L^1[0,T]$ is bounded, we deduce that the function $g(t)$ is continuous on $[0,T]$ and $|g(t)| \leq M, M > 0$. From (9), we have

$$|\phi_1(t)| \leq M\int_0^t K(t,s)ds \leq \frac{MT^2}{2}, \tag{11}$$

which, together with (10), leads to

$$|\phi_k(t)| \leq M\int_0^t K(t,s)ds \leq \frac{MT^{2k}}{2^k}.$$

Therefore the sequence $\phi_k(t)$ converges uniformly to a limit $\tilde{y}(t)$. That is

$$\tilde{y}(t) = \lim \tilde{y}_k(t) = \sum_{i=0}^{\infty} \phi_i(t).$$

In addition, by summing the equations in (10)

$$\sum_{i=0}^{\infty} \phi_{i+1}(t) = \sum_{i=0}^{\infty} \int_0^t K(t,s)\phi_i(s)ds.$$

Using (9) (10) one verifies easily that

$$\sum_{i=0}^{\infty} \phi_i(t) - g(t) = \int_0^t K(t,s)\sum_{i=0}^{\infty} \phi_i(s)ds,$$

and

$$\tilde{y}(t) = \int_0^t K(t,s)\tilde{y}(s)ds - g(t).$$

Hence $\tilde{y}(t)$ is a solution of (7).

Now, we prove that $\tilde{y}(t)$ is a unique solution. Assume there exists another solution $\tilde{u}(t)$, then

$$|\tilde{y}(t) - \tilde{u}(t)| \leq \int_0^t K(t,s)|\tilde{y}(s) - \tilde{u}(s)|ds.$$

Since $y(t) - u(t)$ is continuous function on $[0,T]$. There exist a constant $L > 0$ such that $|y(t) - u(t)| < L$ for all $t \in [0,T]$. Thus the above equation becomes

$$|\tilde{y}(t) - \tilde{u}(t)| \leq L\int_0^t K(t,s)ds \leq \frac{LT^2}{2}. \tag{12}$$

Repeated application of (12) gives

$$|\tilde{y}(t) - \tilde{u}(t)| \leq \frac{LT^{2n}}{2^n}.$$

Thus as $n \to \infty$, we have $\tilde{y}(t) = \tilde{u}(t)$.

■

IV. Solution of the Bagley-Torvik equation

Let $\omega^{\alpha,\beta}(x)$ be a weight function in the usual sense, for $\alpha, \beta > -1$. It is well known that the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1) = \{u : u \quad \text{is measurable and} \quad \|u\|_{\omega^{\alpha,\beta}} < \infty\}$$

and

$$\|u\|_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} |u(x)|^2\omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

and the inner product

$$(u,v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx,$$

where $u, v \in L^2_{\omega^{\alpha,\beta}}(-1,1)$.

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^{N}$, which is the set of $N+1$ Jacobi Gauss, Jacobi Gauss-Radau or Jacobi Gauss-Lobatto points, and by $\{\omega_i\}_{i=0}^{N}$, which are the corresponding weights. Let P_N denote the space of all polynomials of degree not exceeding N. For any $u \in C[-1,1]$, we define the Lagrange interpolating polynomial $I_N^{\alpha,\beta}u \in P_N$ satisfying

$$I_N^{\alpha,\beta}u(x_i) = u(x_i), \quad 0 \leq i \leq N.$$

It can be written in the form

$$I_N^{\alpha,\beta}u(x) = \sum_{i=0}^{N} u(x_i)F_i(x), \quad 0 \leq i \leq N,$$

where $F_i(x)$ is the Lagrange interpolation basis associated with the Jacobi collocation points $\{x_i\}_{i=0}^{N}$.

Now, we present the Jacobi collocation method for (6). For ease of analysis, we make the change of variables

$$t = T(1+x)/2, \quad x = 2t/T - 1, \quad x \in [-1,1], \quad t \in [0,T]$$

and

$$s = T(1+\tau)/2, \quad \tau \in [-1,x].$$

Equation (6) is transformed into

$$\begin{aligned} u(x) =& B_1 \int_{-1}^{x} (x-\tau)^{-\frac{1}{2}} u(\tau)d\tau \\ &+ B_2 \int_{-1}^{x} (x-\tau)u(\tau)d\tau + G(x), \end{aligned} \tag{13}$$

where $u(x) = \tilde{y}(T(1+x)/2)$, $G(x) = \tilde{g}(T(1+x)/2)$, $B_1 = -B/\sqrt{2\pi T}$ and $B_2 = -TC/4$.

First, equation (13) holds at the collocation points $\{x_i\}_{i=0}^N$ on $[-1,1]$:

$$\begin{aligned} u(x_i) =& B_1 \int_{-1}^{x_i} (x_i-\tau)^{-\frac{1}{2}} u(\tau)d\tau \\ &+ B_2 \int_{-1}^{x_i} (x_i-\tau)u(\tau)d\tau + G(x_i) \end{aligned} \tag{14}$$

for $0 \le i \le N$. In obtaining high order accuracy for the above Volterra integral equation, the main difficulty is computing the integral term in (14). To overcome this difficulty, we transfer the interval $[-1,x_i]$ to a fixed interval $[-1,1]$. We rewrite the integral terms in (14) as

$$\begin{aligned} &\int_{-1}^{x_i} (x_i-\tau)^{-\frac{1}{2}} u(\tau)d\tau \\ &= \left(\frac{1+x_i}{2}\right)^{\frac{1}{2}} \int_{-1}^{1} (1-\theta)^{-\frac{1}{2}} u(\tau_i(\theta))d\theta \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\int_{-1}^{x_i} (x_i-\tau)u(\tau)d\tau \\ &= \left(\frac{1+x_i}{2}\right) \int_{-1}^{1} (1-\theta)u(\tau_i(\theta))d\theta \end{aligned} \tag{16}$$

by using the following variable change:

$$\tau = \tau_i(\theta) = \frac{x_i+1}{2}\theta + \frac{x_i-1}{2}, \quad \theta \in [-1,1].$$

Next, using the Jacobi-Gauss quadrature formula, (15) and (16) can be approximated by

$$\int_{-1}^{x_i} (x_i-\tau)^{-\frac{1}{2}} u(\tau)d\tau \approx \left(\frac{1+x_i}{2}\right)^{-\frac{1}{2}} \sum_{k=0}^{N} u(\tau_i(\theta_k))\omega_k^{-\frac{1}{2},0} \tag{17}$$

and

$$\int_{-1}^{x_i} (x_i-\tau)u(\tau)d\tau \approx \left(\frac{1+x_i}{2}\right) \sum_{k=0}^{N} u(\tau_i(\theta_k))\omega_k^{1,0}, \tag{18}$$

where the points $\{\theta_i\}_{i=0}^N$ coincide with the collocation points $\{x_i\}_{i=0}^N$. We use u_i to approximate the function value $u(x_i)$, $0 \le i \le N$, and expand the approximate solution $u_N(x)$ as

$$u_N(x) = \sum_{j=0}^{N} u_i F_j(x) \tag{19}$$

to approximate the function $u(x)$, and

$$u(\tau_i(\theta_k)) \approx \sum_{j=0}^{N} F_j(\tau_i(\theta_k))u_j. \tag{20}$$

Then, inserting (19) and (20) into (14) leads to

$$\begin{aligned} u_i =& B_1 \left(\frac{1+x_i}{2}\right)^{-\frac{1}{2}} \sum_{j=0}^{N} \left(\sum_{k=0}^{N} F_j(\tau_i(\theta_k))\omega_k^{-\frac{1}{2},0}\right) u_j \\ &+ B_2 \left(\frac{1+x_i}{2}\right) \sum_{j=0}^{N} \left(\sum_{k=0}^{N} F_j(\tau_i(\theta_k))\omega_k^{1,0}\right) u_j + G(x_i) \end{aligned} \tag{21}$$

for $0 \le i \le N$. By solving the system of linear equations and obtaining the approximation $u_N(x)$, we can find that

$$y(t) \approx y_N(t) = u_N\left(\frac{2}{T}t - 1\right).$$

V. Some Useful Lemmas

In this section, we will recall some elementary lemmas that will be used in the convergence analysis.

Lemma 1: [23] If $u \in H^m(I)$ with $I := (-1,1)$ for some $m \ge 1$ and $\phi \in P_N$, then for the Gauss quadrature formula relative to the Jacobi weight, we have

$$\left| \int_{-1}^{1} u(x)\phi(x) - (u,\phi)_N \right| \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} \|\phi\|_{L^2_{\omega^{\alpha,\beta}}(I)}, \tag{22}$$

where

$$|u|_{H^{m,N}_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{i=\min(m,N+1)}^{m} \|u^{(i)}\|^2_{L^2_{\omega^{\alpha,\beta}}(I)} \right)^{\frac{1}{2}}, \tag{23}$$

$$(u,\phi)_N = \sum_{i=0}^{N} u(x_i)\phi(x_i)\omega_i. \tag{24}$$

Lemma 2: For any function $u \in H^{m,N}_{\omega^{\alpha,\beta}}(I)$, denoting its interpolation polynomial by $I_N^{\alpha,\beta}u$, the error estimates hold (see [23], [24]):

$$\|u - I_N^{\alpha,\beta}u\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}, \tag{25}$$

$$\begin{aligned} &\|u - I_N^{\alpha,\beta}u\|_\infty \le \\ &\begin{cases} CN^{\frac{1}{2}-m}\log N|u|_{H^{m;N}_{\omega^c}(I)}, & -1 < \alpha,\beta \le -\frac{1}{2}, \\ CN^{1+\gamma-m}|u|_{H^{m;N}_{\omega^c}(I)}, & \gamma = \max\{\alpha,\beta\}, otherwise, \end{cases} \end{aligned} \tag{26}$$

where $\omega^c = \omega^{-\frac{1}{2},-\frac{1}{2}}$ denotes the Chebyshev weight function.

Lemma 3: [25] Let $\{F_i\}_{i=0}^N$ be the Lagrange basis polynomials associated with the Gauss points of Jacobi polynomials. Then,

$$\begin{aligned} \|I_N^{\alpha,\beta}\|_\infty &= \max_{x\in[-1,1]} \sum_{i=0}^{N} |F_i(x)| \\ &\le \begin{cases} O(\log N), & -1 < \alpha,\beta \le \frac{1}{2}, \\ O(N^{\gamma+\frac{1}{2}}), & \gamma = \max(\alpha,\beta), otherwise. \end{cases} \end{aligned}$$

Lemma 4: (Gronwall inequality, see [26]) Suppose that u and v are nonnegative, locally integrable functions on $[-1,1]$ satisfying

$$u(x) \le v(x) + L\int_{-1}^{x}(x-\tau)^{-\mu}u(\tau)d\tau,$$

where $L \ge 0$ and $0<\mu<1$. Then, there exists a constant $C=C(\mu)$ such that

$$u(x) \le v(x) + CL\int_{-1}^{x}(x-\tau)^{-\mu}v(\tau)d\tau \quad -1\le x<1.$$

Lemma 5: [27] Let a and b be two real numbers $-\infty \le a < b \le \infty$, and let $1<p\le q<\infty$. Then, for all measurable functions $f \ge 0$, we have that the generalized Hardy inequality

$$\left(\int_a^b |(Tf)(x)|^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_a^b |f(x)|^p v(x)dx\right)^{\frac{1}{p}}$$

holds if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{\frac{1}{q}}\left(\int_a^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}}<\infty,$$
$$p'=\frac{p}{p-1}.$$

Here, T is an operator of the form $(Tf)(x) = \int_a^x k(x,t)f(t)dt$, with $k(x,t)$ as the given kernel, and u and v are nonnegative weight functions.

Lemma 6: [28] For a bounded function $u(x)$, there exists a constant C independent of $u(x)$ such that

$$\sup_N\left\|\sum_{i=0}^N u(x_i)F_i(x)\right\|_{L^2_{\omega^{\alpha,\beta}}(I)} \le C\max_{x\in[-1,1]}|u(x)|.$$

VI. Convergence Analysis

Theorem 4: Let $u(x)$ be the solution of (13), which is assumed to be sufficiently smooth. The approximated solution $u_N(x)$ is obtained by using numerical scheme (21). If $u\in H^m_{\omega^{-\alpha,-\beta}}(I)$ for some $m\ge 1$, then we have the estimate

$$\|u-u_N\|_\infty \le \begin{cases} CN^{\frac{1}{2}-m}\log NU_1, & -1<\alpha,\beta\le -\frac{1}{2};\\ CN^{1+\gamma-m}U_1, & \gamma=\max\{\alpha,\beta\}<0,\end{cases} \tag{27}$$

where $U_1 = |u|_{H^{m;N}_{\omega^c}} + N^{-\frac{1}{2}}\|u\|_\infty$.

$$\|u-u_N\|_{\omega^{\alpha,\beta}} \le \begin{cases} CN^{-m}(U_2+U_3), & -1<\alpha,\beta\le -\frac{1}{2};\\ CN^{-m}(U_2+U_4), & \gamma=\max\{\alpha,\beta\}<0,\end{cases} \tag{28}$$

where $U_2 = |u|_{H^{m,N}_{\omega^{\alpha,\beta}}}+\|u\|_\infty$, $U_3 = N^{\frac{1}{2}-\kappa}\log N|u|_{H^{m,N}_{\omega^c}(I)}$ and $U_4 = N^{1+\gamma-\kappa}|u|_{H^{m,N}_{\omega^c}(I)}$.

Proof: For ease of analysis, we rewrite numerical scheme (21) as

$$\begin{aligned} u_i =& B_1\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}u_N(\tau)d\tau+\\ & B_2\int_{-1}^{x_i}(x_i-\tau)u_N(\tau)d\tau + G(x_i)+I_1+I_2\end{aligned} \tag{29}$$

for $0\le i\le N$, where u_N is defined in (19),

$$\begin{aligned} I_1 =& B_1\left(\frac{1+x_i}{2}\right)^{-\frac{1}{2}}\sum_{j=0}^N\left(\sum_{k=0}^N F_j(\tau_i(\theta_k))\omega_k^{-\frac{1}{2},0}\right)u_j\\ & -B_1\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}u_N(\tau)d\tau,\end{aligned} \tag{30}$$

and

$$\begin{aligned} I_2 =& B_2\left(\frac{1+x_i}{2}\right)\sum_{j=0}^N\left(\sum_{k=0}^N F_j(\tau_i(\theta_k))\omega_k^{1,0}\right)u_j\\ & -B_2\int_{-1}^{x_i}(x_i-\tau)\,u_N(\tau)d\tau.\end{aligned} \tag{31}$$

Using (30) and (31) and the integration error estimates in (22), we obtain

$$|I_1| \le CN^{-m}|u_N|_{H^{m;N}_{\omega^{\alpha,\beta}}}, |I_2| \le CN^{-m}|u_N|_{H^{m;N}_{\omega^{\alpha,\beta}}}. \tag{32}$$

Subtracting (29) from (14) gives

$$\begin{aligned} u(x_i)-u_i =& B_1\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}e_N(\tau)d\tau+\\ & B_2\int_{-1}^{x_i}(x_i-\tau)e_N(\tau)d\tau - I_1 - I_2,\end{aligned} \tag{33}$$

where $e(x)=u(x)-u_N(x)$. Multiplying $F_i(x)$ on both sides of equation (33) and summing from $i=0$ to $i=N$ yields

$$\begin{aligned} I_Nu-u_N =& B_1I_N\left(\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}e(\tau)d\tau\right)\\ & + B_2I_N\left(\int_{-1}^{x_i}(x_i-\tau)e(\tau)d\tau\right)\\ & -\sum_{i=1}^N F_i(x)I_1 - \sum_{i=1}^N F_i(x)I_2.\end{aligned} \tag{34}$$

Consequently,

$$\begin{aligned} e(x) =& B_1\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}e(\tau)d\tau+\\ & B_2\int_{-1}^{x}(x-\tau)e(\tau)d\tau + \sum_{i=1}^5 J_i,\end{aligned} \tag{35}$$

where

$$\begin{aligned} J_1 &= u - I_Nu,\\ J_2 &= \sum_{i=1}^N F_i(x)I_1,\\ J_3 &= \sum_{i=1}^N F_i(x)I_2,\\ J_4 &= I_N\left(\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}e(\tau)d\tau\right)\\ &\quad -\int_{-1}^{x_i}(x_i-\tau)^{-\frac{1}{2}}e(\tau)d\tau,\end{aligned}$$

and

$$J_5 = I_N\left(\int_{-1}^{x_i}(x_i-\tau)\,e(\tau)d\tau\right) - \int_{-1}^{x_i}(x_i-\tau)\,e(\tau)d\tau.$$

It follows from the Gronwall inequality in Lemma (4) that

$$\|e_N(x)\|_\infty \leq C\sum_{i=1}^{5}\|J_i\|_\infty. \tag{36}$$

First, by (26),

$$\|J_1\|_\infty \leq \begin{cases} CN^{\frac{1}{2}-m}\log N|u|_{H^{m,N}_{\omega^c}(I)}, & -1<\alpha,\beta\leq-\frac{1}{2}, \\ CN^{1+\gamma-m}|u|_{H^{m,N}_{\omega^c}(I)}, & \gamma=\max\{\alpha,\beta\}, otherwise \end{cases}. \tag{37}$$

Next, using (30), (31), (32) and Lemma (3), we obtain

$$\|J_2\|_\infty = \left\|\sum_{i=0}^{N} I_1(x_i)F_i(x)\right\|_\infty$$
$$\leq \max_{0\leq i\leq N}|I_1(x_i)|\max_{x\in(-1,1)}\sum_{i=0}^{N}|F_i(x)| \leq$$
$$\begin{cases} CN^{-m}\log N(\|u_N\|_\infty), & -1\leq\alpha,\beta\leq-\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m}(\|u_N\|_\infty), & \gamma=\max\{\alpha,\beta\}, otherwise, \end{cases} \tag{38}$$

and

$$\|J_3\|_\infty = \left\|\sum_{i=0}^{N} I_2(x_i)F_i(x)\right\|_\infty \leq$$
$$\begin{cases} CN^{-m}\log N(\|u_N\|_\infty), & -1\leq\alpha,\beta\leq-\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m}(\|u_N\|_\infty), & \gamma=\max\{\alpha,\beta\}, otherwise, \end{cases} \tag{39}$$

for sufficiently large N. Finally, applying the results in [24], [29], we obtain

$$\|J_4\|_\infty \leq$$
$$\begin{cases} CN^{-\kappa}\log N(\|e\|_\infty), & -1\leq\alpha,\beta\leq-\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-\kappa}(\|e\|_\infty), & \gamma=\max\{\alpha,\beta\}, otherwise. \end{cases} \tag{40}$$

By virtue of (26) with $m=2$, we obtain

$$\|J_5\|_\infty \leq$$
$$\begin{cases} CN^{-\frac{3}{2}-m}\log N\|e\|_\infty, & -1<\alpha,\beta\leq-\frac{1}{2}; \\ CN^{-\gamma-1}\|e\|_\infty, & \gamma=\max\{\alpha,\beta\}, otherwise. \end{cases} \tag{41}$$

Combining (37), (38), (39), (40) and (41) gives the desired estimate (27).

Considering (35), we have

$$|e(x)| \leq B_1\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}|e(\tau)|d\tau + H(x), \tag{42}$$

where

$$H(x) = B_2\int_{-1}^{x}(x-\tau)e(\tau)d\tau + \sum_{i=1}^{5}J_i.$$

Applying the Gronwall inequality in Lemma 4, we have

$$|e(x)| \leq C\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}|H(\tau)|d\tau + H(x). \tag{43}$$

Using the Hardy inequality Lemma 5, we obtain

$$\begin{aligned} \|e(x)\|_{\omega^{\alpha,\beta}} &\leq C\left\|\int_{-1}^{x}(x-\tau)^{-\frac{1}{2}}|H(\tau)|d\tau\right\|_{\omega^{\alpha,\beta}} \\ &\quad + C\|H(x)\|_{\omega^{\alpha,\beta}} \\ &\leq C\|H(x)\|_{\omega^{\alpha,\beta}} \leq C\sum_{i=1}^{5}\|J_i\|_{\omega^{\alpha,\beta}}. \end{aligned} \tag{44}$$

Now, using (26), we have

$$\|J_1\|_{\omega^{\alpha,\beta}} = \|u - I_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \leq CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}. \tag{45}$$

Similarly, we obtain the estimates

$$\|J_2\|_{\omega^{\alpha,\beta}} \leq C\|I_1\|_\infty \leq CN^{-m}(\|e\|_\infty + \|u\|_\infty) \tag{46}$$

and

$$\|J_3\|_{\omega^{\alpha,\beta}} \leq C\|I_2\|_\infty \leq CN^{-m}(\|e\|_\infty + \|u\|_\infty). \tag{47}$$

Similarly, applying the results in [24], [29] and (27) yields

$$\|J_4\|_{\omega^{\alpha,\beta}} \leq CN^{-\kappa}\|e\|_\infty$$
$$\begin{cases} CN^{\frac{1}{2}-m-\kappa}\log NU_1, & -1\leq\alpha,\beta\leq-\frac{1}{2}, \\ CN^{1+\gamma-m-\kappa}U_1, & \gamma=\max\{\alpha,\beta\}<0. \end{cases} \tag{48}$$

By virtue of (26) with $m=2$, we obtain

$$\|J_5\|_{\omega^{\alpha,\beta}} \leq CN^{-2}\|e\|_{\omega^{\alpha,\beta}}. \tag{49}$$

Combining (45), (46), (47), (48) and (49) gives the desired estimate (28).

■

VII. Numerical examples

In this section, we demonstrate the effectiveness and simplicity of the proposed method using two examples.

Example 7.1: First, we consider the following Bagley-Torvik equation:

$$y''(t) + y^{\frac{3}{2}}(t) + y(t) = f(t),$$

with the initial conditions

$$y(0)=1, \quad y'(0)=1,$$

where $f(t)=\frac{15}{4}\sqrt{t}+\frac{15\sqrt{\pi}}{8}t+t^{\frac{5}{2}}+1$. In [30] the authors applied piecewise polynomial collocation method to deal with the equation on the intervals of $[0,1]$. We applied the present method to solve the problem on the intervals of $[0,1]$ and $[0,10]$. In Fig. 1, the numerical errors are plotted for various choices of N in terms of both the L^2 and L^∞ norms. The results of the piecewise polynomial collocation and our present methods, with various values of N, are listed in Table I. The numerical solutions using the present method are consistent with the solutions of the methods in [30].

TABLE I
Comparison of the maximum errors of the piecewise polynomial collocation method and present method for Example 7.1

N	Method in [30]	Our method
8	2.4382e-04	2.9572e-04
16	3.1586e-06	5.5642e-06
32	4.0640e-07	3.9324e-07
64	5.1963e-08	2.9746e-08
128	6.6429e-09	7.8351e-10
256	8.4399e-10	1.7682e-10

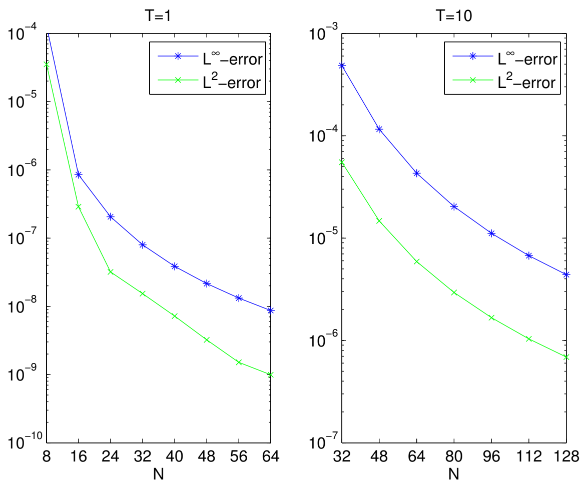

Fig. 1. The error $e(t)$ verse the number of collocation points in L^2 and L^∞ norms.

Example 7.2: Second, we consider the following Bagley–Torvik equation:

$$y''(t) + \frac{1}{2}D^{\frac{3}{2}}y(t) + \frac{1}{2}y(t) = g(t)$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$, where

$$g(t) = \begin{cases} 8, & 0 \leq t \leq 1; \\ 0, & t > 1. \end{cases}$$

The problem was considered in [10], [13], [16], [31]. We applied the present method to solve the problem with $N = 32$. The numerical solutions obtained by the present method and other numerical methods, such as the wavelet method [13] are given in Table II. Clearly, the numerical results show that the present method is effective and its accuracy is comparable to that of existing methods. The numerical results with $N = 32$ and the exact solution are plotted in Fig.2. The approximate solutions obtained using the present method show excellent agreement with the exact solutions.

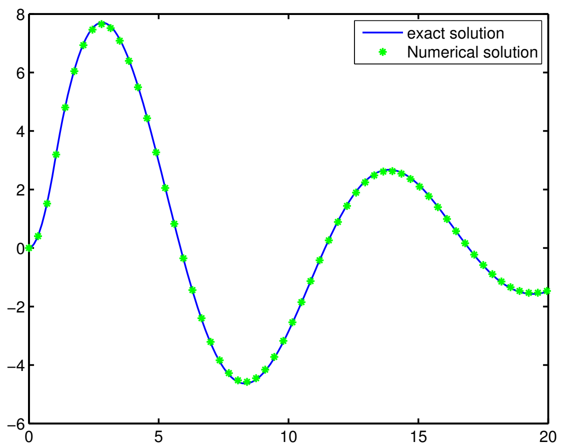

Fig. 2. Exact solution and numerical solution for $N = 32$

VIII. Conclusion

This work has addressed the extension of the Jacobi collocation method to the Bagley-Torvik equation. We have implemented the proposed method based on the Jacobi polynomial approximation and Gauss quadrature formula. First, we transformed the differential equation to the Volterra integral equation with a weakly singular kernel. Next, we presented a numerical scheme for the new Volterra integral equation. We proved the convergence of the method and obtained the error estimates in terms of the L^∞ norm and weighted L^2 norm for the approximated solution. These results were confirmed by some numerical examples. In our future work, the Jacobi collocation spectral analysis will be extended to nonlinear fractional differential equations and nonlinear fractional integro-differential equations.

TABLE II
Comparison of the numerical results of the wavelet and present method for Example 7.2

t	Wavelet method [13]	Our method $N = 32$	Exact solutions
1	3.53856	2.952482792	2.952583880
2	7.53718	6.760087332	6.760110396
3	8.28540	7.666174786	7.666141755
4	6.26126	6.077230168	6.077249465
5	2.53055	2.943928811	2.943935566
6	-1.49195	-0.525196957	-0.525171420
7	-4.50898	-3.246325319	-3.246304280
8	-5.72074	-4.550282354	-4.550290680
9	-5.00085	-4.302851128	-4.302864780
10	-2.84029	-2.848382245	-2.848380860

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