Exact Traveling Wave Solutions to the Fifth-order KdV Equation Using the Exponential Expansion Method

Riadh Hedli, and Abdelouahab Kadem

Abstract—**In this paper, an analytical method called an $exp(-\varphi(\xi))$-expansion method is applied to find new exact traveling wave solution of a class of the fifth-order Korteweg-de Vries equation (fKdV), and some particular cases of this class have an aspect interesting physical that are the Lax, Sawada-Kotera, Caudrey-Dodd-Gibbon, Kaup- Kupershmidt and Ito equations.**

Index Terms—**Fifth-order Korteweg-de Vries equation, $exp(-\varphi(\xi))$-expansion method, nonlinear evolution equation, traveling wave solution, soliton solution.**

I. Introduction

Nonlinear evolution equations (NLEEs) are applied in many fields of science and engineering, especially in fluid mechanics, laser optics, plasma physics, and others. The study of the traveling wave solutions of the NLEEs allows us to know the internal structure of nonlinear phenomena.

Several methods can be used to find these solutions, such as Hirota's bilinear method [1], tanh method [2], [3], sine-cosine method [1], [4], tanh-coth method [1], [5], Painlevé analysis [6], homogeneous balance method [7], [8], Darboux transformation [9], Fan sub-equation method [10], exp-function method [11], [12], $(\frac{G'}{G})$-expansion method [13], first integral method [14], trial equation method [15], extended trial equation method [16] and so on.

In this work, we present a technique called $exp(-\varphi(\xi))$-expansion method [17]–[20], it's described in section II. This method allows to find new exact traveling wave solutions of a class of the fifth-order Korteweg-de Vries equation (fKdV) which has the formula [1], [21]

$$u_t + u_{xxxxx} + auu_{xxx} + bu_xu_{xx} + cu^2u_x = 0 \qquad (1)$$

Where a, b and c are non-zero real parameters, and $u = u(x,t)$ is an unknown function.

In accordance with the values of a, b and c, we found new exact solutions of the most well-known equations of fKdV [1], like the Lax equation for $a = 10$, $b = 20$ and $c = 30$, the Sawada-Kotera (SK) equation for $a = b = c = 5$, the Caudrey-Dodd-Gibbon (CDG) equation for $a = b = 30$ and $c = 180$, the Kaup-Kupershmidt (KK) equation for $a = 10$, $b = 25$ and $c = 20$, and the Ito equation for $a = 3$, $b = 6$ and $c = 2$.

Manuscript received June 9, 2019; revised August 29, 2019.

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The fKdV equation attracted an important field of research in mathematical physics, with various applications in nonlinear optics and quantum mechanics. It represents movements of long waves in shallow water surfaces. In recent years, many analytical and numerical methods have been used to solve several forms of this equation. Wazwaz [1]–[5] studied the equation using Hirota's bilinear, tanh, sine-cosine and extended tanh methods. In [1], the multiple-soliton solutions are determined for different forms of the fKdV equation by using the Hirota's bilinear formalism. In [2], [3], he used the tanh method for solving various forms of the fKdV equation, that include the Lax, SK, KK, Ito and the CDG forms and other related special cases. Two main criteria are defined to create effective strategies that regulate the relation between the parameters of the equation. Abundant solitons solutions are derived. In [4], the sine–cosine and the tanh methods are used to present an analytic study of the fKdV equation, that provided exact periodic and solitons solutions. In [5], the extended tanh method is used to derive new solitons solutions for many forms of the fKdV equation, which contain the Lax, SK, Sawada–Kotera–Parker–Dye (SKPD), KK, Kaup–Kupershmidt–Parker–Dye (KKPD), and the Ito equations. The criteria established in [2] are confirmed by using this new approach.

Many other authors as Sierra and Salas [22] used a generalization of the tanh-coth method in order to obtain new periodic and soliton solutions to various important forms of the fKdV equation. Salas et al. [12], [23] found exact solutions to the general fKdV equation by using the exp function, the generalized projective Riccati equations methods and the Cole-Hopf transformation. Zayed and Alurrfi [7] applied the homogeneous balance method to find the exact solutions of the Lax, SK, KK, Ito and CDG forms. Khan and Akbar [24] used the modified simple equation (MSE) method to solve the generalized fKdV equation that gives exact traveling wave solutions. Jaradat et al. [25] investigated the multiple soliton solutions and multiple singular soliton solutions of a class of the fifth-order nonlinear evolution equation with variable coefficients. They used the simplified bilinear method based on a transformation method combined with the Hirota's bilinear sense.

This article is organized as follows: In Section II, we give the steps of the method. In Section III, we apply this method to solve a class of the fKdV equation. In Section IV, we illustrated many solutions graphically for some particular values of parameters. Also a conclusion is given in Section V. Finally some references are given at the end of this paper.

II. The $exp(-\varphi(\xi))$-expansion method

The general nonlinear PDE

$$P\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, u_{xxx} \ldots\ldots\right) = 0 \tag{2}$$

is transformed into ODE

$$R\left(u, u', u'', u''', \ldots\ldots\right) = 0 \tag{3}$$

by the transformation $u(x,t) = u(\xi),\ \xi = x \pm \omega t$, where ω is the speed of traveling wave.

Basic ideas of this method:

Step 1. Suppose that the solution of (3) can be described as follows

$$u(\xi) = \sum_{i=0}^{m} \alpha_i (\exp(-\varphi(\xi)))^i \tag{4}$$

Where m is a positive integer and $\alpha_i\ (i = 0, 1, ..., m)$ are constants to be established, such that $\alpha_m \neq 0$ and $\varphi(\xi)$ satisfies the following ODE:

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda \tag{5}$$

where λ and μ are constants.

We pose $\Theta = \lambda^2 - 4\mu$, the solutions of (5) are:

When $\Theta > 0,\ \mu \neq 0$

$$\varphi(\xi) = \ln\left(\frac{-1}{2\mu}\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}(\xi + k_1)\right) + \lambda\right)\right) \tag{6}$$

or

$$\varphi(\xi) = \ln\left(\frac{-1}{2\mu}\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}(\xi + k_1)\right) + \lambda\right)\right) \tag{7}$$

When $\Theta < 0,\ \mu \neq 0$

$$\varphi(\xi) = \ln\left(\frac{1}{2\mu}\left(\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}(\xi + k_1)\right) - \lambda\right)\right) \tag{8}$$

or

$$\varphi(\xi) = \ln\left(\frac{1}{2\mu}\left(\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}(\xi + k_1)\right) - \lambda\right)\right) \tag{9}$$

When $\Theta > 0,\ \mu = 0,\ \lambda \neq 0$

$$\varphi(\xi) = \ln\left(\frac{1}{\lambda}\left(\exp(\lambda(\xi + k_1)) - 1\right)\right) \tag{10}$$

When $\Theta = 0,\ \mu \neq 0,\ \lambda \neq 0$

$$\varphi(\xi) = \ln\left(-\frac{2\lambda(\xi + k_1) + 4}{\lambda^2(\xi + k_1)}\right) \tag{11}$$

When $\Theta = 0,\ \mu = 0,\ \lambda = 0$

$$\varphi(\xi) = \ln(\xi + k_1) \tag{12}$$

where k_1 is a constant of integration.

Step 2. We find m that appeared in (4) by balancing the higher order derivatives and the higher order nonlinear terms occurred in (3). Then, we replace $u(\xi)$ and its derivatives in (3), we obtain a polynomial of $exp(-\varphi(\xi))$. This provides a system of algebraic equations that implies the parameters α_i and ω. Finally, we get the exact solutions of (2).

III. Application of the fifth-order KdV equation

Let us the fifth-order KdV equation, that has the form

$$u_t + u_{xxxxx} + auu_{xxx} + bu_x u_{xx} + cu^2 u_x = 0 \tag{13}$$

Using $u(x,t) = u(\xi)$ and $\xi = x - \omega t$, we obtain

$$-\omega u' + u''''' + auu''' + bu'u'' + cu^2 u' = 0 \tag{14}$$

we integrate (14) and ignoring the constant of integration, we have

$$-\omega u + u'''' + auu'' + \frac{b-a}{2}\left(u'\right)^2 + \frac{c}{3}u^3 = 0 \tag{15}$$

Based on the finite expansion (4), we balance u'''' with u^3, we find $m = 2$, thus, the solution of (15) is of the form

$$u(\xi) = \alpha_0 + \alpha_1 \exp(-\varphi(\xi)) + \alpha_2 (\exp(-\varphi(\xi)))^2 \tag{16}$$

where $\alpha_i\ (i = 0, 1, 2)$ are constants to be established and $\alpha_2 \neq 0$.

According to the step 2, we get the following system

$$\begin{aligned}
&a\lambda\mu\alpha_0\alpha_1 - \tfrac{1}{2}a\mu^2\alpha_1^2 + 8\lambda\mu^2\alpha_1 + 16\mu^3\alpha_2 + 2a\mu^2\alpha_0\alpha_2 \\
&+14\lambda^2\mu^2\alpha_2 + \lambda^3\mu\alpha_1 - \omega\alpha_0 + \tfrac{1}{3}c\alpha_0^3 + \tfrac{1}{2}b\mu^2\alpha_1^2 = 0 \\
&6a\lambda\mu\alpha_0\alpha_2 + a\lambda^2\alpha_0\alpha_1 + 2b\mu^2\alpha_1\alpha_2 + b\lambda\mu\alpha_1^2 + 30\lambda^3\mu\alpha_2 \\
&+2a\mu\alpha_0\alpha_1 + \lambda^4\alpha_1 + c\alpha_0^2\alpha_1 - \omega\alpha_1 + 120\lambda\mu^2\alpha_2 \\
&+22\lambda^2\mu\alpha_1 + 16\mu^2\alpha_1 = 0 \\
&8a\mu\alpha_0\alpha_2 + a\mu\alpha_1^2 + 2b\mu^2\alpha_2^2 + 60\lambda\mu\alpha_1 + \tfrac{1}{2}b\lambda^2\alpha_1^2 \\
&+232\lambda^2\mu\alpha_2 + c\alpha_0\alpha_1^2 + 4a\lambda^2\alpha_0\alpha_2 + 3a\lambda\alpha_0\alpha_1 - \omega\alpha_2 \\
&+4b\lambda\mu\alpha_1\alpha_2 + 15\lambda^3\alpha_1 + 3a\lambda\mu\alpha_1\alpha_2 + c\alpha_0^2\alpha_2 \\
&+136\mu^2\alpha_2 + 16\lambda^4\alpha_2 + \tfrac{1}{2}a\lambda^2\alpha_1^2 + b\mu\alpha_1^2 = 0 \\
&2a\lambda\alpha_1^2 + 6a\mu\alpha_1\alpha_2 + b\lambda\alpha_1^2 + 10a\lambda\alpha_0\alpha_2 + 2a\lambda\mu\alpha_2^2 \\
&+4b\lambda\mu\alpha_2^2 + 50\lambda^2\alpha_1 + 2c\alpha_0\alpha_1\alpha_2 + 2a\alpha_0\alpha_1 + 40\mu\alpha_1 \\
&+2b\lambda^2\alpha_1\alpha_2 + 440\lambda\mu\alpha_2 + \tfrac{1}{3}c\alpha_1^3 + 4b\mu\alpha_1\alpha_2 \\
&+3a\lambda^2\alpha_1\alpha_2 + 130\lambda^3\alpha_2 = 0 \\
&2b\lambda^2\alpha_2^2 + c\alpha_1^2\alpha_2 + 330\lambda^2\alpha_2 + 4b\mu\alpha_2^2 + 9a\lambda\alpha_1\alpha_2 \\
&+240\mu\alpha_2 + c\alpha_0\alpha_2^2 + 6a\alpha_0\alpha_2 + 4b\lambda\alpha_1\alpha_2 + 2a\lambda^2\alpha_2^2 \\
&+\tfrac{3}{2}a\alpha_1^2 + 60\lambda\alpha_1 + \tfrac{1}{2}b\alpha_1^2 + 4a\mu\alpha_2^2 = 0 \\
&6a\lambda\alpha_2^2 + 4b\lambda\alpha_2^2 + c\alpha_1\alpha_2^2 + 6a\alpha_1\alpha_2 + 2b\alpha_1\alpha_2 \\
&+336\lambda\alpha_2 + 24\alpha_1 = 0 \\
&\tfrac{1}{3}c\alpha_2^3 + 2b\alpha_2^2 + 120\alpha_2 + 4a\alpha_2^2 = 0
\end{aligned} \tag{17}$$

Solving the above system we find the following results:

Case A: If $b = \frac{10c}{a} - a$

We obtain the two sets

Family 1:

$$\begin{gathered}\omega = \left(\lambda^2 - 4\mu\right)^2 \\ \alpha_0 = \frac{-6a\mu}{c},\ \alpha_1 = \frac{-6a\lambda}{c},\ \alpha_2 = \frac{-6a}{c}\end{gathered} \tag{18}$$

Family 2:

$$\begin{aligned}
\omega =& \frac{1}{8c}\left(a^2 - 4c \mp a\sqrt{9a^2 - 24c}\right)\left(\lambda^2 - 4\mu\right)^2 \\
\alpha_0 =& \frac{1}{4c}\left(\pm\sqrt{9a^2 - 24c}(\lambda^2 - 4\mu) - 3a\lambda^2 - 12a\mu\right) \\
\alpha_1 =& \frac{-6a\lambda}{c},\ \alpha_2 = \frac{-6a}{c}
\end{aligned} \tag{19}$$

Case B: If $b \neq \frac{10c}{a} - a$

We obtain

$$\begin{aligned}
&\omega = 0,\ \mu = \frac{1}{4}\lambda^2\\
&\alpha_0 = \frac{-3\lambda^2}{2c}\frac{\beta_0\delta \pm \beta_1 \mp \beta_2}{\gamma_0\delta \pm \gamma_1 \mp \gamma_2}\\
&\alpha_1 = \frac{-6\lambda}{c}\frac{\beta_3\delta \pm \beta_4}{(3a+2b)\,\delta \pm (\beta_3 - 28c)}\\
&\alpha_2 = \frac{-3}{c}\,(2a+b\pm\delta)
\end{aligned} \tag{20}$$

Such as

$$\begin{aligned}
\delta =&\sqrt{(2a+b)^2-40c}\\
\beta_0 =&(b+a)\,(3a+2b)^2\,(2a+b)^3\\
&-2c\,(3a+2b)\,(2a+b)\left(116a^2+169ab+58b^2\right)\\
&+2c\left(3754a^2c+5136abc+1712b^2c-3920c^2\right)\\
\beta_1 =&(b+a)\,(3a+2b)^2\,(2a+b)^4\\
\beta_2 =&2c\,(3a+2b)\,(2a+b)^2\left(146a^2+219ab+78b^2\right)\\
&-2c\left(13568a^3c+26706a^2bc+17200ab^2c\right)\\
&-2c\left(3632b^3c-32480ac^2-22960bc^2\right)\\
\beta_3 =&(3a+2b)\,(2a+b)-28c\\
\beta_4 =&(3a+2b)\,(2a+b)^2-116ac-68bc\\
\gamma_0 =&(b+a)\,(3a+2b)^2\,(2a+b)^2\\
&-2c\,(3a+2b)\left(101a^2+144ab+48b^2\right)\\
&+2c\,(1232ac+952bc)\\
\gamma_1 =&(b+a)\,(3a+2b)^2\,(2a+b)^3\\
\gamma_2 =&2c\,(3a+2b)\,(2a+b)\left(131a^2+194ab+68b^2\right)\\
&-2c\left(5044a^2c+7136abc+2472b^2c-7840c^2\right)
\end{aligned} \tag{21}$$

A. The first condition $b = \frac{10c}{a} - a$

Family 1:

Substituting (18) into (16) we obtain

$$u(\xi) = -\frac{6a\mu}{c} - \frac{6a\lambda}{c}\exp(-\varphi(\xi)) - \frac{6a}{c}(\exp(-\varphi(\xi)))^2 \tag{22}$$

According to the solutions of (5), we discuss the following cases:

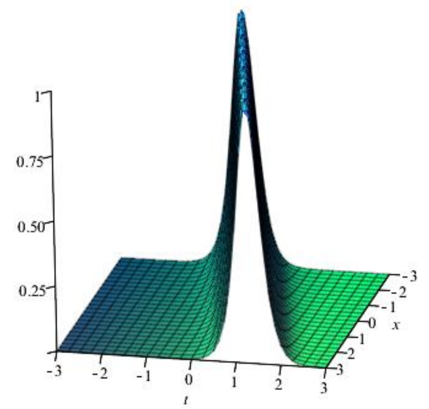
Case 1.1: When $\Theta > 0,\ \mu \neq 0$

$$\begin{aligned}
u_1(x,t) = &-\frac{6a\mu}{c} + \frac{12a\lambda\mu}{c\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\,(\xi+k_1)\right)+c\lambda}\\
&-\frac{24a\mu^2}{c\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}\,(\xi+k_1)\right)+\lambda\right)^2}
\end{aligned} \tag{23}$$

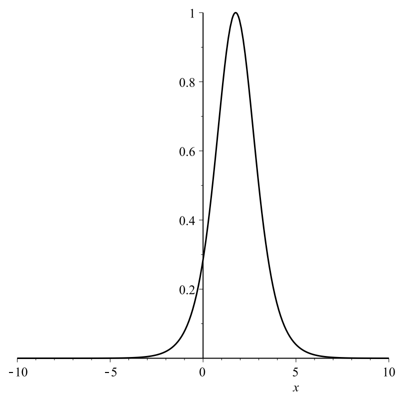
or

$$\begin{aligned}
u_2(x,t) = &-\frac{6a\mu}{c} + \frac{12a\lambda\mu}{c\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\,(\xi+k_1)\right)+c\lambda}\\
&-\frac{24a\mu^2}{c\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}\,(\xi+k_1)\right)+\lambda\right)^2}
\end{aligned} \tag{24}$$

where $\Theta = \lambda^2 - 4\mu$, $\xi = x - \Theta^2 t$ and k_1 is a constant of integration.



(a) u_1 in 3D with $-3 \le x \le 3$, $-3 \le t \le 3$



(b) u_1 in 2D with $-10 \le x \le 10$, $t = 1$

Fig. 1: Bell shape soliton solution of u_1 for $a = 10$, $b = 20$, $c = 30$, $\lambda = 2$, $\mu = 0.5$, $k_1 = 1$

Case 1.2: When $\Theta < 0,\ \mu \neq 0$

$$\begin{aligned}
u_3(x,t) = &-\frac{6a\mu}{c} + \frac{12a\lambda\mu}{c\lambda - c\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\,(\xi+k_1)\right)}\\
&-\frac{24a\mu^2}{c\left(\lambda-\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}\,(\xi+k_1)\right)\right)^2}
\end{aligned} \tag{25}$$

or

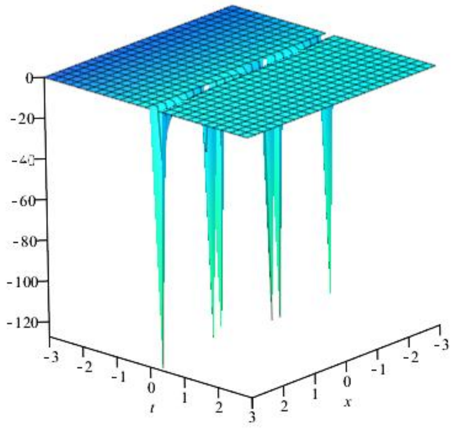
$$\begin{aligned}
u_4(x,t) = &-\frac{6a\mu}{c} + \frac{12a\lambda\mu}{c\lambda - c\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\,(\xi+k_1)\right)}\\
&-\frac{24a\mu^2}{c\left(\lambda-\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}\,(\xi+k_1)\right)\right)^2}
\end{aligned} \tag{26}$$

where $\xi = x - \Theta^2 t$.

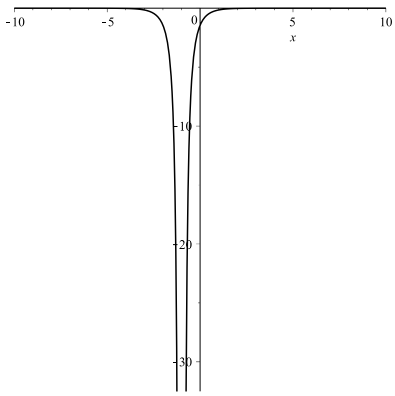
Case 1.3: When $\Theta > 0,\ \mu = 0,\ \lambda \neq 0$

$$u_5(x,t) = -\frac{6a\lambda^2\exp(\lambda(\xi+k_1))}{c\,(\exp(\lambda(\xi+k_1))-1)^2} \tag{27}$$

where $\xi = x - \lambda^4 t$.



(a) u_5 in 3D with $-3 \le x \le 3$, $-3 \le t \le 3$



(b) u_5 in 2D with $-10 \le x \le 10$, $t = 0$

Fig. 2: Dark solitary wave solution of u_5 for $a = 10$, $b = 20$, $c = 30$, $\lambda = 2$, $\mu = 0$, $k_1 = 1$

Case 1.4: When $\Theta = 0,\ \mu \neq 0,\ \lambda \neq 0$

$$u_6(x,t) = -\frac{3a\lambda^2}{2c} + \frac{3a\lambda^3\,(x+k_1)}{c\lambda(x+k_1)+2c} - \frac{3a\lambda^4(x+k_1)^2}{2c(\lambda(x+k_1)+2)^2} \tag{28}$$

Case 1.5: When $\Theta = 0,\ \mu = 0,\ \lambda = 0$

$$u_7(x,t) = -\frac{6a}{c\,(x+k_1)^2} \tag{29}$$

Family 2:
Substituting (19) into (16) we find

$$\begin{aligned} u(\xi) =& \frac{1}{4c}\left(\pm\Theta\sqrt{9a^2-24c} - 3a\lambda^2 - 12a\mu\right) \\ &- \frac{6a\lambda}{c}\exp(-\varphi(\xi)) - \frac{6a}{c}(\exp(-\varphi(\xi)))^2 \end{aligned} \tag{30}$$

Based on the solutions of (5), we discuss the following cases:

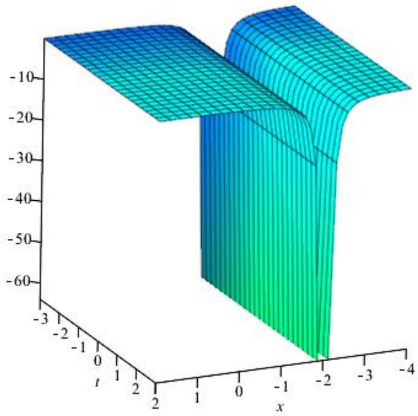
Case 2.1: When $\Theta > 0,\ \mu \neq 0$

$$\begin{aligned} u_{8,9}(x,t) =& \frac{1}{4c}\left(\pm\Theta\sqrt{9a^2-24c} - 3a\lambda^2 - 12a\mu\right) \\ &+ \frac{12a\lambda\mu}{c\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}(\xi+k_1)\right)+c\lambda} \\ &- \frac{24a\mu^2}{c\left(\sqrt{\Theta}\tanh\left(\frac{\sqrt{\Theta}}{2}(\xi+k_1)\right)+\lambda\right)^2} \end{aligned} \tag{31}$$

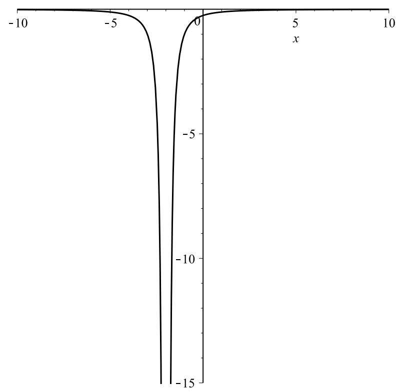
or

$$\begin{aligned} u_{10,11}(x,t) =& \frac{1}{4c}\left(\pm\Theta\sqrt{9a^2-24c} - 3a\lambda^2 - 12a\mu\right) \\ &+ \frac{12a\lambda\mu}{c\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}(\xi+k_1)\right)+c\lambda} \\ &- \frac{24a\mu^2}{c\left(\sqrt{\Theta}\coth\left(\frac{\sqrt{\Theta}}{2}(\xi+k_1)\right)+\lambda\right)^2} \end{aligned} \tag{32}$$

where $\xi = x - \frac{1}{8c}\left(3a^2 - 4c \mp a\sqrt{9a^2-24c}\right)\Theta^2 t$.



(a) u_6 in 3D with $-4 \le x \le 2$, $-3 \le t \le 3$

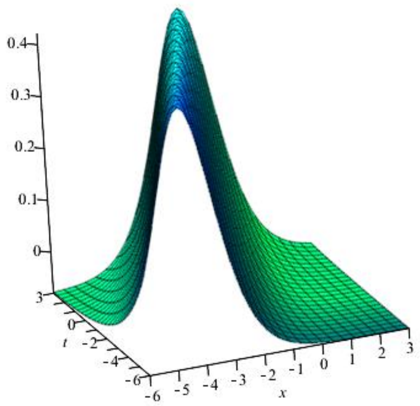


(b) u_6 in 2D with $-10 \le x \le 10$

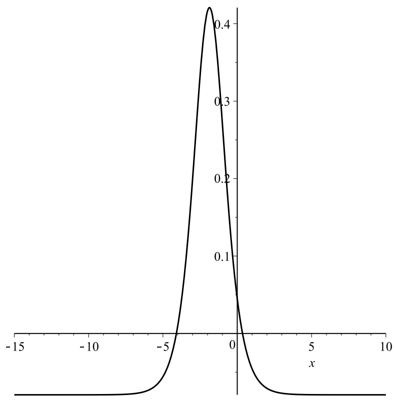
Fig. 3: Rational solution of u_6 for $a = 30$, $b = 30$, $c = 180$, $\lambda = 2$, $\mu = 1$, $k_1 = 1$

Case 2.2: When $\Theta < 0,\ \mu \neq 0$

$$\begin{aligned} u_{12,13}(x,t) =& \frac{1}{4c}\left(\pm\Theta\sqrt{9a^2-24c} - 3a\lambda^2 - 12a\mu\right) \\ &+ \frac{12a\lambda\mu}{c\lambda - c\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}(\xi+k_1)\right)} \\ &- \frac{24a\mu^2}{c\left(\lambda-\sqrt{-\Theta}\tan\left(\frac{\sqrt{-\Theta}}{2}(\xi+k_1)\right)\right)^2} \end{aligned} \tag{33}$$



(a) u_8 in 3D with $-6 \leq x \leq 3,\ -6 \leq t \leq 3$



(b) u_8 in 2D with $-15 \leq x \leq 10,\ t = 1$

Fig. 4: Bell shape soliton solution of u_8 for $a = 30,\ b = 30,\ c = 180,\ \lambda = 2,\ \mu = 0.5,\ k_1 = 1$

or

$$u_{14,15}(x,t) = \frac{1}{4c}\left(\pm\Theta\sqrt{9a^2-24c} - 3a\lambda^2 - 12a\mu\right) + \frac{12a\lambda\mu}{c\lambda - c\sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}(\xi + k_1)\right)} - \frac{24a\mu^2}{c\left(\lambda - \sqrt{-\Theta}\cot\left(\frac{\sqrt{-\Theta}}{2}(\xi+k_1)\right)\right)^2} \tag{34}$$

where $\xi = x - \frac{1}{8c}\left(3a^2 - 4c \mp a\sqrt{9a^2-24c}\right)\Theta^2 t$.

Case 2.3: When $\Theta > 0,\ \mu = 0,\ \lambda \neq 0$

$$u_{16,17}(x,t) = \frac{1}{4c}\left(\pm\sqrt{9a^2-24c} - 3a\right)\lambda^2 - \frac{6a\lambda^2\exp(\lambda(\xi+k_1))}{c\left(\exp(\lambda(\xi+k_1)) - 1\right)^2} \tag{35}$$

where $\xi = x - \frac{1}{8c}\left(3a^2 - 4c \mp a\sqrt{9a^2-24c}\right)\lambda^4 t$.

According to the values of parameters a, b and c, the first condition (III-A) is satisfied for the Lax, SK and CDG equations. Therefore, each of them has the seventeen solutions indicated above (23)–(29), (31)–(35).

B. The second condition $b \neq \frac{10c}{a} - a$

Substituting (20) into (16), four solutions are obtained.

Case 1: When $\Theta = 0,\ \mu \neq 0,\ \lambda \neq 0$

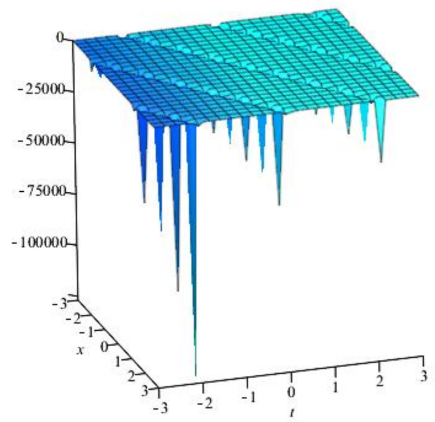
$$u_{18,19}(x,t) = \frac{-3\lambda^2}{2c}\frac{\beta_0\delta \pm \beta_1 \mp \beta_2}{\gamma_0\delta \pm \gamma_1 \mp \gamma_2} + \frac{3\lambda^3(\beta_3\delta \pm \beta_4)(x+k_1)}{c\left((3a+2b)\delta \pm (\beta_3 - 28c)\right)(\lambda(x+k_1)+2)} - \frac{3\lambda^4}{4c}\frac{(2a+b\pm\delta)(x+k_1)^2}{(\lambda(x+k_1)+2)^2} \tag{36}$$

Case 2: When $\Theta = 0,\ \mu = 0,\ \lambda = 0$

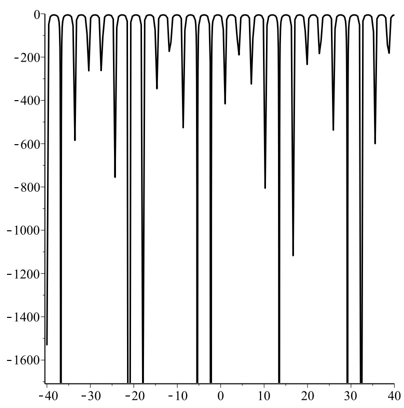
$$u_{20,21}(x,t) = -\frac{3(2a+b\pm\delta)}{c(x+k_1)^2} \tag{37}$$

The KK and Ito equations check the second condition (III-B), then, each one admits the four solutions (36), (37).

Remark 1: All solutions obtained in this paper verify the entire equations indicated above.



(a) u_{12} in 3D with $-3 \leq x \leq 3,\ -3 \leq t \leq 3$



(b) u_{12} in 2D with $-40 \leq x \leq 40,\ t = 7$

Fig. 5: Dark periodic cusp solution of u_{12} for $a = 5,\ b = 5,\ c = 5,\ \lambda = 2,\ \mu = 2,\ k_1 = 1$

IV. Graphical Illustration of the Solutions

Some traveling wave solutions of the fKdV equation are graphically illustrated using Maple 17 in the figures 1 to 5, which present the shapes of the solutions u_1, u_5, u_6, u_8 and u_{12} respectively.

Solutions u_1, u_8 and u_9 describe the bell shape soliton solution. The figures 1 and 4 present the bell shape soliton solution obtained from u_1 and u_8 respectively. Figure of u_9 is similar to that of u_8.

Solutions u_2, u_5, u_{10}, u_{11}, u_{16} and u_{17} represent the dark solitary wave solution. The figure 2 shows the shape of dark solitary wave solution of u_5. Shapes of u_2, u_{10}, u_{11}, u_{16} and u_{17} are similar to the figure of u_5.

Solutions u_6, u_7, u_{18}, u_{19}, u_{20} and u_{21} represent the rational solution. In figure 3, we present the rational solution of u_6. The figures of u_7, u_{18}, u_{19}, u_{20} and u_{21} are similar to that of u_6.

Solutions u_3, u_4, u_{12}, u_{13}, u_{14} and u_{15} describe the dark periodic cusp solution. The figure 5 illustrates the shape of dark periodic cusp solution obtained from u_{12}. The figures of u_3, u_4, u_{13}, u_{14} and u_{15} are similar to that of u_{12}.

V. Conclusion

In this work, we have found new exact traveling wave solutions of a class of the fifth-order Korteweg-de Vries equation, and several forms of this class using the $exp(-\varphi(\xi))$-expansion method with the help of Maple 17. We have defined two main criteria to obtain the traveling wave solutions of fKdV equation according to the parameters values of the equation. We can observe that there are different types of traveling wave solutions, which are soliton type solutions, dark solitary wave solutions, dark periodic cusp solutions and rational solutions. This study demonstrates that the method proved its efficiency for application to NLEEs.

Acknowledgment

The authors would like to express their sincere thanks to the editor and reviewers for their valuable comments and suggestions to improve the quality of this paper.

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