# Finding Differential Transform Using Difference Equations

Shadi Al-Ahmad, Mustafa Mamat, and Rami AlAhmad

*Abstract*—**Differential transform is applied to solve linear and nonlinear ordinary differential/difference equations. Many properties of the differential transform are known. In this paper, we construct and prove new properties of the differential transform. We particularly look into the differential transform of certain quotients of functions. Also, we construct and prove interesting relations between the differential transform, the difference operator, and incomplete gamma functions. Finally, we present some numerical examples to illustrate the results.**

*Index Terms*—**differential-transform, difference-operators, incomplete-gamma-functions.**

## I. Introduction

In mathematics, it is an essential problem to find the image of products and quotients of two functions under a linear transform. For example, one of the disappointments of Laplace transform that the Laplace transform of a product (or a quotient) of two functions doesn't equal the product (or the quotient) of their Laplace transforms, see [1]. In this paper, we are interested in discussing this issue in the case of what so called differential transforms. The differential transform (DTM) was introduced by Zhou [2] in a study of electrical circuits which have since developed into an extensive, rigorous, and exciting disciplines. DTM has been presented as a new iterative method for solving differential equations, initial value problems, difference equations, and boundary value problems. The idea of DTM is based on the concept of Taylor series and it usually gets the solution in a series form. This method constructs an analytical solution in the form of a polynomial and uses it as the approximation to exact solutions which are sufficiently differentiable [3], [4], [5], [6], [7]. In the introduction section, we give the definitions of the differential transform. Also, we present some properties for the differential transform.

### A. Differential Transform

*Definition 1.1:* Let $f(x)$ be analytic at $x_0$, then the differential transform is defined as

$$F(k) = \frac{f^{(k)}(x_0)}{k!}. \tag{1}$$

The inverse differential transform of $F(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k. \tag{2}$$

Manuscript received December 26, 2018; revised June 06, 2019.

S. Al-Ahmad and M. Mamat are with Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu, Malaysia e-mail: (alahmad.shadi@yahoo.com).

R.AlAhmad is with Department of Mathematics, Yarmouk University, Irbid, Jordan and with Faculty of Engineering, Higher Colleges of Technology, Ras Alkhaimah, UAE, P.O. box 4793.

From (1) and (2), we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{3}$$

The linearity of the differential transform and some more properties can be deduced from equations (1) and (2) as follows

*Theorem 1.1:* If $u(x) = g(x)+h(x)$, then $U(k) = G(k)+H(k)$.

*Theorem 1.2:* If $u(x) = cg(x)$, then $U(k) = cG(k)$ ,where $c$ is any constant.

*Theorem 1.3:* If $u(x) = \frac{d^n g(x)}{dx^n}$, then $U(k) = \frac{(k+n)!}{k!}G(k+n)$.

*Theorem 1.4:* If $u(x) = g(x)h(x)$, then $U(k) = \sum_{i=0}^{k} G(i)H(k-i)$ .

*Theorem 1.5:* If $u(x) = x^n$, then $U(k) = \delta(k-n)$ .

*Theorem 1.6:* If $u(x) = \exp(cx)$, then $U(k) = \frac{c^k}{k!}$ .

*Theorem 1.7:* If $u(x) = \cos(\omega x)$, then $U(k) = \frac{\omega^k}{k!}\cos(\frac{k\pi}{2})$.

*Theorem 1.8:* If $u(x) = \sin(\omega x)$, then $U(k) = \frac{\omega^k}{k!}\sin(\frac{k\pi}{2})$.

For further properties of the one dimensional DTM, see [2], [8], [9], [10], [11], [12].

## II. Preliminaries

In the introduction section, we give the definitions for the difference operators and incomplete gamma functions. We present some properties for the difference operators and the incomplete gamma functions.

### A. Difference operators

An important tool in our calculations is the difference operator, it is defined as follows

*Definition 2.1:* Let $N$ be the set of the natural numbers and let $S(N)$ be the set of all sequences over $N$. Define the difference operator $\Delta : S(N) \to S(N)$ as

$$(\Delta u)(n) = u(n+1) - u(n).$$

It is easy to prove that the difference operator $\Delta$ is linear operator and it satisfies the following proposition

*Proposition 2.1:* For $u \in S(N)$

$$\sum_{n=m}^{n-1} (\Delta u)(n) = u(n) - u(m). \tag{4}$$

For further properties of the difference operators and the difference equations, see [13], [14], [15], [16].

*B. Incomplete gamma functions*

The incomplete gamma function will play an important role in this paper, it is defined as

*Definition 2.2:* For $\Re(s) > 0$, The lower incomplete gamma function is defined as

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt,$$

and the upper incomplete gamma function is defined as:

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\,e^{-t}\,\mathrm{d}t.$$

Clearly,

$$\Gamma(s,z) = \Gamma(s) - \gamma(s,z). \tag{5}$$

Moreover, $\gamma(s,x) \longrightarrow \Gamma(s)$ as $x \longrightarrow \infty$ and $\Gamma(s,0) = \Gamma(s)$.

*Proposition 2.2:* For $N = 0, 1, 2, ...$

$$\sum_{n=0}^{N} \frac{a^n}{n!} = e^a \frac{\Gamma(N+1,a)}{N!}.$$

*Proof:* By integration by parts

$$\Gamma(n+1,a) = \int_a^\infty t^n e^{-t}dt = a^n e^{-a} + n\Gamma(n,a).$$

Therefore,

$$\Gamma(n+1,a) - n\Gamma(n,a) = a^n e^{-a}.$$

Divide both sides by $n!$ to get

$$\frac{\Gamma(n+1,a)}{n!} - \frac{\Gamma(n,a)}{(n-1)!} = \frac{a^n e^{-a}}{n!}.$$

Thus,

$$\Delta\frac{\Gamma(n,a)}{(n-1)!} = \frac{a^n e^{-a}}{n!}.$$

Taking the sum for both sides from $n = 1$ to $N-1$, and using proposition 2.1 gives

$$\frac{\Gamma(N,a)}{(N-1)!} - e^{-a} = \sum_{n=1}^{N-1} \frac{a^n e^{-a}}{n!}.$$

Hence,

$$\frac{\Gamma(N,a)}{(N-1)!} = \sum_{n=0}^{N-1} \frac{a^n e^{-a}}{n!}.$$

Therefore,

$$e^a \frac{\Gamma(N+1,a)}{N!} = \sum_{n=0}^{N} \frac{a^n}{n!}.$$

■

For more properties of incomplete gamma functions and their applications, see [9], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32].

## III. MAIN RESULTS

In this section, we prove properties of the differential transform for the functions of the forms

$$\frac{f(x)}{ax+b}, \frac{f(x)}{(ax+b)^2}, \frac{f(x)}{(ax+b)^n}, f(x)\ln(ax+b),\ and\ \frac{f(x)}{P_r(x)},$$

where $r \in N$ and $P_r(x)$ as

$$P_r(x) = \sum_{k=0}^{r} a_k x^k.$$

To start with, we have the following theorem which gives partial solution for the problem that the differential transform of the quotient of two functions doesn't equal the quotient of their differential transforms.

*Theorem 3.1:* Let

$$g(x) = \frac{f(x)}{P_r(x)}$$

and let $G(n;a,b)$ be the differential transform of $g(x)$. Then $G(n;a,b)$ satisfies the difference equation

$$\sum_{k=0}^{r} a_k G(n-k;a,b) = F(n).$$

*Proof:* Since

$$g(x) = \frac{f(x)}{P_r(x)},$$

then

$$P_r(x)g(x) = f(x).$$

Therefore,

$$\sum_{k=0}^{r} a_k x^k g(x) = f(x). \tag{6}$$

Now, if $h(x) = x^m g(x)$ then

$$H(n) = \sum_{j=0}^{n} \delta(j-m)G(n-j) = G(n-m).$$

Using this fact and by taking the differential transform for (6), we get the desired result. ■

The following lemma is an application of Theorem 3.1.

*Lemma 3.2:* Let

$$g(x) = \frac{f(x)}{ax+b} \tag{7}$$

and let $G(n;a,b)$ be the differential transform of $g(x)$. Then $G(n;a,b)$ satisfies the difference equation

$$aG(n-1;a,b) + bG(n;a,b) = F(n) \quad for \quad n \geq 1.$$

Moreover, $G(n;a,b)$ is given explictly as

$$G(n;a,b) = \begin{cases} \frac{1}{b}F(n), & a = 0, b \neq 0; \\ \frac{1}{a}F(n+1), & a \neq 0, b = 0; \\ \frac{1}{b}(-\frac{a}{b})^n \sum_{i=0}^{n}(-\frac{b}{a})^i F(i), & a \neq 0, b \neq 0. \end{cases}$$

*Proof:*

Case1

If $a = 0, b \neq 0$, then (7) can be written as

$$g(x) = \frac{f(x)}{b}.$$

This implies that

$$G(n;a,b)=\frac{F(n)}{b}.$$

Case2

If $a\neq 0, b=0$, then (7) can be written as

$$g(x)=\frac{f(x)}{ax}.$$

This gives that

$$axg(x)=f(x).$$

Taking the Differential Transform for both sides to get that

$$G(n;a,b)=\frac{1}{a}F(n+1).$$

Case3

If $a\neq 0, b\neq 0$ then (7) can be written as

$$axg(x)+bg(x)=f(x).$$

Taking the differential transform for both sides gives

$$G(n+1;a,b)+\frac{a}{b}G(n;a,b)=\frac{1}{b}F(n+1).$$

Multiplying both sides by $(-\frac{b}{a})^{n+1}$ to get that

$$(-\frac{b}{a})^{n+1}G(n+1;a,b)-(-\frac{b}{a})^{n}G(n;a,b)=\frac{1}{b}(-\frac{b}{a})^{n+1}\times F(n+1).$$

Sum both sides from 0 to $n-1$, and use (7) to get

$$\sum_{i=0}^{n-1}\Delta(-\frac{b}{a})^{i}G(i;a,b)=\sum_{i=0}^{n-1}\frac{1}{b}(-\frac{b}{a})^{i+1}F(i+1)$$

$$(-\frac{b}{a})^{n}G(n;a,b)-G(0)=\frac{1}{b}\sum_{i=0}^{n-1}(-\frac{b}{a})^{i+1}F(i+1).$$

Solving for $G(n;a,b)$ to get that

$$G(n;a,b)=(-\frac{a}{b})^{n}(\frac{F(0)}{b}+\frac{1}{b}\sum_{j=1}^{n}(-\frac{b}{a})^{j}F(j)),$$

which can be simplified as

$$G(n;a,b)=\frac{1}{b}(-\frac{a}{b})^{n}\sum_{i=0}^{n}(-\frac{b}{a})^{i}F(i).$$

■

*Lemma 3.3:* Let $L(n;a,b)$ be the differential transform of $f(x)ln(ax+b)$.
Then $\frac{\partial}{\partial b}L(n;a,b)$ is the differential transform of $\frac{f(x)}{ax+b}$.

*Proof:* Since $L(n;a,b)$ is the differential transform of $f(x)ln(ax+b)$, then

$$f(x)ln(ax+b)=\sum_{i=0}^{\infty}L(i;a,b)(x-x_0)^{i}.$$

Differentiating both sides with respect to $b$, we get

$$\frac{f(x)}{ax+b}=\sum_{i=0}^{\infty}\frac{\partial}{\partial b}L(i;a,b)(x-x_0)^{i}.$$

This is proves that $\frac{\partial}{\partial b}L(n;a,b)$ is the differential transform of $\frac{f(x)}{ax+b}$. ■

*Example 3.1:* Since the differential transform of $e^{cx}$ is $\frac{c^n}{n!}$ for $n=0,1,2,....$, and by using (7), then the differential transform of

$$g(x)=\frac{e^{cx}}{ax+b}$$

is

$$G(n;a,b,c)=\frac{1}{b}(\frac{-a}{b})^{n}\sum_{i=0}^{n}\frac{(-\frac{cb}{a})^{i}}{i!}$$
$$=\frac{1}{b}(\frac{-a}{b})^{n}e^{-\frac{cb}{a}}\frac{\Gamma(n+1,-\frac{cb}{a})}{\Gamma(n+1)}.$$

*Lemma 3.4:* If $G(n;a,b)$ is the differential transform of

$$g(x)=\frac{f(x)}{ax+b},$$

then the differential transform of

$$h(x)=\frac{f(x)}{(ax+b)^{m+1}},\ m\geq 0$$

is

$$H(n;a,b)=\frac{(-1)^m}{m!}\frac{\partial^m G(n;a,b)}{\partial b^m}.$$

*Proof:* Assume $G(n;a,b)$ is the differential transform of $g(x)=\frac{f(x)}{ax+b}$, then

$$\frac{f(x)}{ax+b}=\sum_{i=0}^{\infty}G(i;a,b)x^{i}.$$

Differentiating both sides m-times with respect to $b$, we get

$$\frac{m!(-1)^m f(x)}{(ax+b)^{m+1}}=\sum_{i=0}^{\infty}\frac{\partial^m G(i;a,b)x^i}{\partial b^m}.$$

This proves our assertion. ■

*Definition 3.1:* [33] The falling factorial $(x)_m$ is defind by

$$(x)_m=x(x-1)(x-2).....(x-(m-1))=\frac{\Gamma(x+1)}{\Gamma(x+m-1)}. \tag{8}$$

*Theorem 3.5:* If $G(n;a,b)$ is the differential transform of

$$g(x)=\frac{f(x)}{ax+b},$$

then the differential transform of

$$h(x)=\frac{f(x)}{(ax+b)^{m+1}},\ m\geq 0$$

is

$$H(n;a,b)=\frac{1}{m!b}(\frac{-1}{b})^{m}\sum_{i=0}^{n}(\frac{-a}{b})^{n-i}F(i)(i-N-1)_m,$$

where $(x)_m$ is the falling factorial defined by (8).

*Proof:* Using Lemma 3.4, the differential transform of

$$h(x)=\frac{f(x)}{(ax+b)^{m+1}},\ m\geq 0$$

is

$$H(n;a,b)=\frac{(-1)^m}{m!}\frac{\partial^m G(n;a,b)}{\partial b^m},$$

where $G(n;a,b)$ is the differential transform of $g(x) = \frac{f(x)}{ax+b}$. Therefore,

$$\begin{aligned}H(n;a,b) &= \frac{(-1)^m}{m!}\frac{\partial^m}{\partial b^m}\frac{1}{b}(\frac{-a}{b})^n\sum_{i=0}^{n}(\frac{-b}{a})^i F(i)\\ &= \frac{(-1)^m}{m!}\frac{\partial^m}{\partial b^m}\sum_{i=0}^{n}(-a)^{n-i}b^{i-n-1}F(i)\\ &= \frac{(-1)^m}{m!}\sum_{i=0}^{n}(-a)^{n-i}(i-n-1)_m b^{i-n-1-m}F(i)\\ &= \frac{1}{m!b}(\frac{-1}{b})^m\sum_{i=0}^{n}(\frac{-a}{b})^{n-i}F(i)(i-n-1)_m.\end{aligned}$$

■

Now, an interesting relation between differential transform and the incomplete gamma is given in the following Corollary

*Corollary 3.6:* The differential transform $F(n;a,b,c)$ of

$$f(x) = \frac{e^{cx}}{(ax+b)^{m+1}}$$

is

$$F(n;a,b,c) = \frac{e^{-\frac{cb}{a}}(-\frac{a}{b})^n}{b^{m+1}n!}\sum_{j=0}^{m}\begin{pmatrix} n+m-j\\ n\end{pmatrix}\begin{pmatrix} n\\ j\end{pmatrix}$$
$$\times(\frac{cb}{a})^j\Gamma(n-j+1,-\frac{cb}{a}).$$

*Proof:* By Example 3.1, we know that the differential transform of

$$f(x) = \frac{e^{cx}}{ax+b}$$

is

$$G(n;a,b,c) = (-a)^n b^{(-n-1)}\sum_{i=0}^{n}\frac{(-\frac{c}{a})^i b^i}{i!}.$$

Now, by Lemma 3.4, we proved that the differential transform of

$$h(x) = \frac{f(x)}{(ax+b)^{m+1}},\ m\geq 0$$

is

$$H(n;a,b) = \frac{(-1)^m}{m!}\frac{\partial^m G(n;a,b)}{\partial b^m}.$$

Hence,

$$F(n;a,b,c) = \frac{(-1)^m}{m!}\frac{\partial^m}{\partial b^m}((-a)^n b^{(-n-1)}\sum_{i=0}^{n}\frac{(-\frac{c}{a})^i b^i}{i!}).$$

By using Leibniz Rule [34]

$$\frac{\partial^m}{\partial b^m}\Big((-a)^n b^{-n-1}\sum_{i=0}^{n}\frac{(-\frac{c}{a})^i b^i}{i!}\Big)$$
$$= (-a)^n\sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}\frac{\partial^{m-j}}{\partial b^{m-j}}b^{(-n-1)}\frac{\partial^j}{\partial b^j}\sum_{i=0}^{n}\frac{(-\frac{c}{a})^i b^i}{i!}.$$
$$= (-a)^n\sum_{j=0}^{m}\begin{pmatrix} m\\ j\end{pmatrix}(-1)^{(m-j)}(n+m-j)_{m-j}$$
$$\times b^{(-n-1-m+j)}\sum_{i=0}^{n}\frac{(-\frac{c}{a})^i}{i!})\frac{\partial^j}{\partial b^j}(b)^i.$$

Now,

$$\begin{aligned}\sum_{i=0}^{n}\frac{(-\frac{c}{a})^i}{i!})\frac{\partial^j}{\partial b^j}(b)^i &= \sum_{i=j}^{n}\frac{(-\frac{c}{a})^i b^{(i-j)}}{(i-j)!}\\ &= \sum_{i=0}^{n-j}\frac{(-\frac{c}{a})^{(i+j)}b^i}{i!}\\ &= (-\frac{c}{a})^j\sum_{i=0}^{n-j}\frac{(-\frac{cb}{a})^i}{i!}\\ &= (-\frac{c}{a})^j e^{-\frac{cb}{a}}\frac{\Gamma(n-j+1,-\frac{cb}{a})}{(n-j)!}.\end{aligned}$$

Therefore,

$$F(n;a,b,c) = \frac{e^{-\frac{cb}{a}}(-\frac{a}{b})^n}{b^{m+1}n!}\sum_{j=0}^{m}\begin{pmatrix} n+m-j\\ n\end{pmatrix}\begin{pmatrix} n\\ j\end{pmatrix}$$
$$\times(\frac{cb}{a})^j\Gamma(n-j+1,-\frac{cb}{a}).$$

■

## IV. Numerical examples

*Example 4.1:* To find the differential transform of $f(t) = \frac{t^7}{1-t}$, write this equation as

$$f(t) - tf(t) = t^7. \tag{9}$$

Taking the differential transform to both sides of (9) gives

$$(\Delta F)(n) = \delta(n-6). \tag{10}$$

Now, take the sum from 0 to $n-1$ for both sides of (10) and use Proposition 2.1, you will get

$$\begin{aligned}F(n) = F(n) - F(0) &= \sum_{k=0}^{n-1}(\Delta F)(k-1)\\ &= \sum_{k=1}^{n-1}\delta(k-6) = u(n-7).\end{aligned}$$

Therefore, $F(n) = u(n-7)$, where $u(t)$ is the unit step function defined as $u(t) = 1$ if $t\geq 0$ and 0 if $t<0$. This result cocides with the result using Lemma 3.2.

*Example 4.2:* To find an approximation for $\int_0^{0.25}\frac{x^3e^x}{1-x}dx$, we begin with finding the differential transform of

$$g(x) = \frac{x^3e^x}{1-x}.$$

Write this equation as

$$g(x) - xg(x) = x^3e^x. \tag{11}$$

Taking the differential transform to both sides of (11) gives

$$(\Delta G)(n-1) = G(n) - G(n-1) = \sum_{j=0}^{n}\frac{\delta(j-3)}{(n-j)!} = \frac{1}{(n-3)!}, n\geq 3. \tag{12}$$

Now, take the sum from 3 to $n$ for both sides of (10) and use Proposition 2.1, you will get

$$G(n) - G(2) = \sum_{k=3}^{n}\frac{1}{(k-3)!} = \sum_{k=0}^{n-3}\frac{1}{k!} = \frac{e\Gamma(n-2,1)}{(n-2)!}.$$

Therefore,

$$\frac{x^3 e^x}{1-x} = x^3 + 2x^4 + \frac{5}{2}x^5 + \frac{8}{3}x^6 + \frac{65}{24}x^7 + \cdots .$$

Using this expansion, the approximated value of the integral

$$\int_0^{0.25} \frac{x^3 e^x}{1-x} dx = 0.00149733.$$

The actual value is 0.00149882 and the resulted relative error is less than $10^{-3}$.

In the following two examples, one can figure out the steps should be followed to find the differential transform for quotient of two polynomials.

*Example 4.3:* To find the differential transform of

$$f(x) = \frac{2-3x}{(1-x)(1-2x)}. \tag{13}$$

Equation (13) gives

$$(1-3x+2x^2)f(x) = 2-3x. \tag{14}$$

Taking the differential transform for both sides of (14) to get that

$$F(n) - 3F(n-1) + 2F(n-2) = 0,\ n \geq 2. \tag{15}$$

Also, using Equation (14), we get

$$F(0) = 2 \ and \ F(1) = 3. \tag{16}$$

Since Equation (15) is a constant coefficient difference equation, we set $F(n) = \lambda^n$. This implies that the characteristic equation is

$$\lambda^n - 3\lambda^{n-1} + 2\lambda^{n-2} = 0. \tag{17}$$

By solving Equation (17), we get that $\lambda_1 = 2 \ or \ \lambda_2 = 1$. Thus,

$$F(n) = c_1 + c_2 2^n.$$

Apply the conditions (16) to get that $F(n) = 1 + 2^n$ is the differential transform of $f(x)$.

*Example 4.4:* In this example, we find the differential transform of

$$f(x) = \frac{5x^2-2x+3}{x^4-2x^2+1}. \tag{18}$$

Equation (18) gives

$$(x^4-2x^2+1)f(x) = 5x^2-2x+3. \tag{19}$$

This means

$$(x^4-2x^2+1)(F(0)+F(1)x+F(2)x^2+F(3)x^3+\dots) \\ = 5x^2-2x+3.$$

By the comparing the coefficients, we get

$$F(0) = 3,\ F(1) = -2,\ F(2) = 11,\ and\ F(3) = -4 \tag{20}$$

Taking the differential transform for both sides of (19) will give that

$$F(n-4) - 2F(n-2) + F(n) = 0,\ n \geq 4. \tag{21}$$

Since (21) is a difference equation with constant coefficients, if we set $F(n) = \lambda^n$, then the corresponding characteristic equation would be

$$\lambda^{n-4} - 2\lambda^{n-2} + \lambda^n = 0. \tag{22}$$

Solving (22) will give that $\lambda_1 = 1 \ or \ \lambda_2 = -1$ with algebric multiplicities 2 for each of them. Therefore, the general solution is

$$\begin{aligned} F(n) &= c_1(1)^n + c_2 n(1)^n + c_3(-1)^n + c_4 n(-1)^n \\ &= c_1 + c_2 n + c_3(-1)^n + c_4 n(-1)^n. \end{aligned}$$

Now, by using the initial conditions (20), we get that $F(n) = \frac{5}{2}(-1)^n n + \frac{3}{2}n + 2(-1)^n + 1$ is the differential transform of $f(x)$.

*Example 4.5:* The results of this papers can be helpful to find a series approximation for the solution for the initial value problem

$$y'' + \frac{y}{1+x} = x^2; y(0) = 0, y'(0) = 1. \tag{23}$$

Taking the differential tranform for (23) gives

$$(n+2)(n+1)Y(n+2) + (-1)^n \sum_{i=0}^{n} (-1)^i Y(i) = \delta(n-2).$$

Take $n = 0, 1, 2, 3, 4, \cdots$ gives the following system
$2Y(2) + Y(0) = 0,$
$6Y(3) - (Y(0) - Y(1)) = 0,$
$12Y(4) + Y(0) - Y(1) + Y(2) = 1,$
$20Y(5) - (Y(0) - Y(1) + Y(2) + Y(3)) = 0,$
$30Y(5) + Y(0) - Y(1) + Y(2) - Y(3) - Y(4) = 0.$
$\vdots$

Solving this iterative system gives $Y(2) = 0, Y(3) = -\frac{1}{6}, Y(4) = \frac{1}{6}, Y(5) = -\frac{1}{24}$, and $Y(6) = \frac{1}{45}$. Therefore $y(x) = x - \frac{1}{6}x^3 + \frac{1}{6}x^4 - \frac{1}{24}x^5 + \frac{1}{45}x^6 + \cdots$

*Example 4.6:* Using Theorem 3.5, one can conclude that the differential transform of

$$f(x) = \frac{x^{10}}{(1-x)^2},$$

is

$$\begin{aligned} F(n) &= -\sum_{i=0}^{n} \delta(i-10)(i-n-1)_1 \\ &= \begin{cases} n-9, & n \geq 10; \\ 0, & n < 10. \end{cases} \end{aligned}$$

Therefore, the power series of $f$ is

$$f(x) = \frac{x^{10}}{(1-x)^2} = x^{10} + 2x^{11} + 3x^{12} + 4x^{13} + \cdots .$$

*Example 4.7:* By Corollary 3.6, one can deduce that the differential transform of

$$h(x) = \frac{e^{-x}}{(2x+1)^2}$$

is

$$H(n) = \frac{e^{\frac{1}{2}}(-2)^n}{n!} \sum_{j=0}^{1} \begin{pmatrix} n+1-j \\ n \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} (-\frac{1}{2})^j \\ \times\Gamma(n-j+1, \frac{1}{2}).$$

After Simplifications, the differential transform of $h(x)$ is

$$H(n) = \frac{e^{\frac{1}{2}}(-2)^n}{n!}\Big((n+1)\Gamma(n+1, \frac{1}{2}) - \frac{n}{2}\Gamma(n, \frac{1}{2})\Big).$$

## V. Conclusion

The differential transform is an important tool to solve differential equations. One of the tools to be used, specially for quotients, to find the differential transform is the difference operators. Under such assumptions we have shown that the new properties are efficient and thus can be used as alternatives in solving differential equations.

## References


[1] R. AlAhmad, "Laplace transform of the product of two functions," *Italian Journal of Pure and Applied Mathematics*, to be published.

[2] J. K. Zhou, *Differential Transform and its Applications for Electrical Circuits*, Wuhan, China: Huarjung University Press, 1986.

[3] F. Ayaz, "Solutions of the system of differential equations by differential transform method," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 547-567, 2004.

[4] C. L.Chen and L. Sy-Hong, "Application of Taylor transformation to nonlinear predictive control problem," *Applied mathematical modelling*, vol. 20, no. 9, pp. 699-710, 1996.

[5] Z. M. Odibat, "Differential transform method for solving Volterra integral equation with separable kernels," *Mathematical and Computer Modelling*, vol. 48, no. 7-8, pp. 1144-1149, 2008.

[6] Y. Huang, "Explicit multi-soliton solutions for the KdV equation by Darboux transformation," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 3, pp. 135-137, 2013.

[7] H. Song, M. Yi, J. Huang and Y. Pan, "Bernstein polynomials method for a class of generalized variable order fractional differential equations," *IAENG International Journal of Applied Mathematics*, vol. 46, no. 4, pp. 437-444, 2016.

[8] M. C. Agarana and O. O. Ajayi, "Dynamic Modeling and Analysis of Inverted Pendulum using Lagrangian-Differential Transform Method,"in *Lecture Notes in Engineering and Computer Science: World Congress on Engineering and Computer Science 2017*, pp. 1031-1036.

[9] S. Al-Ahmad, M. Mamat, R. AlAhmad, I. M. Sulaiman, P. L. Ghazali and M. A. Mohamed, "On New Properties of Differential Transform via Difference Equations," *International Journal of Engineering & Technology*, vol. 7, no. 28, pp. 321-324, 2018.

[10] A. Arikoglu and I. Ozkol, "Solution of difference equations by using differential transform method," *Applied Mathematics and Computation*, vol. 174, no. 2, pp. 1216-1228, 2006.

[11] E. A. Elmabrouk and F. Abdelwahid, "Useful Formulas for One-dimensional Differential Transform, " *British Journal of Applied Science and Technology*, vol. 18, no. 3, pp. 1-8, 2016.

[12] M. Jang, C. Cheb and Y. Liu, "Two-dimensional differential transform for partial differential equations," *Applied Mathematics and Computation*, vol. 121, no. 2-3, pp. 261-270, 2001.

[13] A. A. Opanuga, E. A. Owoloko, H. I. Okagbue and O. O. Agboola, "Finite difference method and Laplace transform for boundary value problems,"in *Lecture Notes in Engineering and Computer Science: World Congress on Engineering and Computer Science 2017*, pp. 65-69.

[14] A. J. Jerri, *Linear difference equations with discrete transform methods*, Springer Science & Business Media, 2013.

[15] R. AlAhmad and R. Weikard, "On Inverse Problems For Left-Definite Discrete Sturm-Liouville Equations," *Operators and matrices*, vol. 7, no. 1, pp. 35-70, 2013.

[16] R. AlAhmad, "A Hilbert space on left-definite Sturm-Liouville difference equation," *International Journal of Applied Mathematics*, vol. 27, no. 2, pp. 163-170, 2015.

[17] R. AlAhmad, "Products of Incomplete gamma functions Integral representations," *Mathematical Sciences and Applications E-Notes*, vol. 4, no. 2, pp. 47-51, 2016.

[18] R. AlAhmad, "Products of incomplete gamma functions," *Analysis*, vol. 36, no. 3, pp. 199-203, 2016.

[19] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Academic press, 2014.

[20] A. R. Miller and I. S. Moskowitz, "On certain generalized incomplete gamma functions," *Journal of computational and applied mathematics*, vol. 91, no. 2, pp. 179-190, 1998.

[21] N. M. Temme, "Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function," *Mathematics of Computation*, vol. 29, no. 132, pp. 1109-1114, 1975.

[22] P. Agarwal, M. Chand and S. Jain, "Certain integrals involving generalized Mittag-Leffler functions," *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, vol. 85, no .3, pp. 359-371, 2015.

[23] P. Agarwal and J. J. Nieto, "Some fractional integral formulas for the Mittag-Leffler type function with four parameters," *Open Mathematics*, vol. 13, no. 1, pp. 537-546, 2015.

[24] A. Apelblat, *Table of definite and infinite integrals*, Amsterdam: North-Holland, 1983.

[25] A. çetinkaya, "The incomplete second Appell hypergeometric functions," *Applied Mathematics and Computation*, vol. 219, no. 15, pp. 8332-8337, 2013.

[26] J. Choi and P. Agarwal, "Certain integral transforms for the incomplete functions," *Applied Mathematics & Information Sciences*, vol. 9, no. 4, pp. 2161-2167, 2015.

[27] R. Metzler, J. Klafter and J. Jortner, "Hierarchies and logarithmic oscillations in the temporal relaxation patterns of proteins and other complex systems," *Proceedings of the National Academy of Sciences*, vol. 96, no. 20, pp. 11085-11089, 1999.

[28] I. Shavitt, "The Gaussian function in calculations of statistical mechanics and quantum mechanics," *Methods of computational physics*, vol. 2, pp. 1-45, 1963.

[29] I. Shavitt and M. Karplus, "Gaussian-Transform Method for Molecular Integrals. I. Formulation for Energy Integrals," *The Journal of Chemical Physics*, vol. 43, no. 2, pp. 398-414, 1965.

[30] J. Shilpi and P. Agarwal, "A New Class of Integral Relations Involving General Class of Polynomials and I-Functions," *Walailak Journal of Science and Technology*, vol. 12, no. 11, pp. 1009-1018, 2015.

[31] D. Sornette, "Multiplicative processes and power laws," *Physical Review E*, vol. 57, no. 4, pp.4811-4813, 1998.

[32] H. M. Srivastava and P. Agarwal, "Certain Fractional Integral Operators and the Generalized Incomplete Hypergeometric Functions," *Applications & Applied Mathematics*, vol. 8, no. 2, pp. 333-345, 2013.

[33] P. L. Butzer, M. Hauss and M. Schmidt, "Factorial functions and Stirling numbers of fractional orders," *Results in Mathematics*, vol. 16, no. 1-2, pp. 16-48, 1989.

[34] M. H. Protter and C. B. Morrey, "Differentiation under the Integral Sign. Improper Integrals. The Gamma Function," *In Intermediate Calculus. Undergraduate Texts in Mathematics. Springer, New York, NY 1985*, pp. 421-453.